Mathematics Class

December 18, 2020

Remark. Some of the following exercises are extracted from written exams of the past years.

Exercise 1. Solve the following equations or inequalities:

a) $3^x \cdot 3^{2x-1} = \dfrac{3^2}{\sqrt{3}}$

b) $\log_2(x+4) - 1 = \dfrac{1}{2}$

c) $\dfrac{(|x|-7)\log(-x)}{x^2+1} = 0$

d) $\dfrac{e^{4|x|-6}-2}{\sqrt{x}} = 0$

e) $\dfrac{|x|+1}{\log(x-3)} < 0$

f) $\log_{\frac{1}{3}}(4-9x^2) \geq -1$

g) $\log|x-1| < \log x + \log 2$

h) $\sqrt{\log_{\frac{1}{2}}(x-1)} \geq 0$

i) $\log_2 x + \log_8 x < 1$

j) $\log_3|3-2x| \geq -\log_{\frac{1}{3}}(1+x^2)$

k) $2^{|x+2|} \leq 32 \cdot 2^{2x}$

l) $3^{\frac{x^2-1}{|x+2|}} < 9$

Solutions

a) $\frac{5}{6}$

b) $2(\sqrt{2}-2)$

c) $\{-7,-1\}$

d) $\frac{3}{2} + \frac{\log 2}{4}$

e) $]3,4[$

f) $]-\frac{2}{3},-\frac{1}{3}] \cup [\frac{1}{3},\frac{2}{3}[$

g) $]\frac{1}{3},1[\cup]1,+\infty[$

h) $]1,2]$

i) $]0,\sqrt[4]{8}[$

j) $[-(1+\sqrt{3}),\sqrt{3}-1]$

k) $[-\frac{7}{3},+\infty[$

l) $]1-\sqrt{6},1+\sqrt{6}[$

Exercise 2. Determine the domain of the following functions:

a) $f(x) = \log_4(x^2+1)$

b) $f(x) = \log\left(\dfrac{x^2+4}{x}\right)$

c) $f(x) = \sqrt{\dfrac{e^{x+2}}{x^3-x}}$

d) $f(x) = \log(1-\log x)$

e) $f(x) = \log_{\frac{1}{2}}\left(\dfrac{x}{\sqrt{x}-2}\right)$

f) $f(x) = \dfrac{e^{-x}}{\cos(2x)}$

g) $f(x) = e^{\frac{x(x+2)}{(x+1)^3}}$

h) $f(x) = \dfrac{1}{2^{x-4}-7}$

i) $f(x) = \dfrac{4+\cos x}{(3-e^{2x})^4}$

Solutions

a) $\mathbb{R}$

b) $]0,+\infty[$

c) $]-1,0[\cup]1,+\infty[$

d) $]0,e[$

e) $]4,+\infty[$

f) $\mathbb{R} \setminus \{\frac{\pi}{4} + \frac{k\pi}{2} : k \in \mathbb{Z}\}$

g) $]-\infty,-1[\cup]-1,+\infty[$

h) $]-\infty,4+\log_2 7[\cup]4+\log_2 7,+\infty[$

i) $]-\infty,\log\sqrt{3}[\cup]\log\sqrt{3},+\infty[$

Exercise 3. The following limits are indeterminate forms. Compute them by applying suitable relevant limits.

a) $\lim\limits_{x \to -\infty} \dfrac{x^5 + 2x^3 + 1}{x^2 + 7x + 4}$

b) $\lim\limits_{x \to -\infty} \dfrac{x^2 + \sin x}{x^2 + 2x - 5}$

c) $\lim\limits_{x \to +\infty} \dfrac{x^2 + 1}{5^x}$

d) $\lim\limits_{x \to +\infty} (x^3 + 3x)2^{-x}$

e) $\lim\limits_{x \to +\infty} (x^2 + 3^x - \log_3 x)3^{-x}$

f) $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{2x}}$

g) $\lim\limits_{x \to 0} \dfrac{\sin(5x)}{7x}$

h) $\lim\limits_{x \to 0} \dfrac{\sqrt{4 - x + 3x^2} - 2 + \sin(2x)}{3x}$

i) $\lim\limits_{x \to 0} \dfrac{\sin(x^2 + x)}{x}$

j) $\lim\limits_{x \to 0^+} \dfrac{1 - \cos \sqrt{x}}{x}$

k) $\lim\limits_{x \to 0^+} \sin x \log_6 x$

l) $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{\log(1 + x)}$

m) $\lim\limits_{x \to 1} \dfrac{\sin(\sqrt{x} - 1)}{x - 1}$

n) $\lim\limits_{x \to 0} \dfrac{x \sin x}{\cos x - 1}$

o) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{\sqrt{1 - \cos x}}$

p) $\lim\limits_{x \to 0} \dfrac{e^{x \sin x} - 1}{1 - \cos x}$

q) $\lim\limits_{x \to 0} \dfrac{e^{\sin^2 x} - 1}{1 - \sqrt{x^2 + 1}}$

r) $\lim\limits_{x \to 0} \dfrac{\log(\cos x)}{x^2}$

s) $\lim\limits_{x \to -\infty} \log\left(1 - \dfrac{1}{x}\right)^x$

t) $\lim\limits_{x \to +\infty} \dfrac{\sqrt{x^2 - 3} - 4^x}{7x + 1}$

u) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \tan x}{\sin x - \cos x}$

v) $\lim\limits_{x \to 1} \dfrac{e^{\cos(x-1)} - e}{1 - \cos^2(x - 1)}$

w) $\lim\limits_{x \to 0} \dfrac{e^{e^x} - e}{\log(x + 1)}$

x) $\lim\limits_{x \to +\infty} \dfrac{e^{\frac{1}{x}} - 1}{\sin\left(\frac{2}{x}\right)}$

Solutions

$a)\ -\infty$	$b)\ 1$	$c)\ 0$
$d)\ 0$	$e)\ 1$	$f)\ \sqrt{e}$
$g)\ \dfrac{5}{7}$	$h)\ \dfrac{7}{12}$	$i)\ 1$
$j)\ \dfrac{1}{2}$	$k)\ 0$	$l)\ 3$
$m)\ \dfrac{1}{2}$	$n)\ -2$	$o)$ it does not exist
$p)\ 2$	$q)\ -2$	$r)\ -\dfrac{1}{2}$
$s)\ -1$	$t)\ -\infty$	$u)\ -\sqrt{2}$
$v)\ -\dfrac{1}{2}$	$w)\ e$	$x)\ \dfrac{1}{2}$