venerdì 18 dicembre 2020 18:35

FOGLIO 9

① ⓐ $A = \begin{pmatrix} -3 & 0 & 0 \\ 2a & b & a \\ 10 & 0 & 2 \end{pmatrix}$ diagonalizzabile al variare di $a, b \in \mathbb{R}$?

STEP 1: $P_A(\lambda)$?

$\begin{vmatrix} -3-\lambda & 0 & 0 \\ 2a & b-\lambda & a \\ 10 & 0 & 2-\lambda \end{vmatrix} = (-3-\lambda) \begin{vmatrix} b-\lambda & a \\ 0 & 2-\lambda \end{vmatrix} = -(\lambda+3)(\lambda-b)(\lambda-2) \rightsquigarrow$ AUTOVALORI $-3, b, 2$

STEP 2: CASO $b \neq -3$, $b \neq 2$ (TUTTI AUTOVALORI DISTINTI)

$\rightsquigarrow$ autovalori $\lambda_2 = -3, \lambda_2 = 2, \lambda_3 = b$ TUTTI CON $m_A(\lambda_i) = 1$

$\forall i$ $m_A(\lambda_i) \geq m_G(\lambda_i) \geq 1$

$\overset{\downarrow}{\Longrightarrow} m_A(\lambda i) = m_G(\lambda_i) = 1 \ \forall i = 1, 2, 3 \Rightarrow A$ diagonalizzabile ✓.

STEP 3: CASO $b = -3$

$\rightsquigarrow \lambda_1 = -3, \ \lambda_2 = -7. \quad m_A(\lambda_2) = m_G(\lambda_2) = 1$ ✓

$m_A(\lambda_1) = 2. \quad m_G(\lambda_1)$?

$A - (-3I) = \begin{pmatrix} 0 & 0 & 0 \\ 2a & 0 & a \\ 10 & 0 & 5 \end{pmatrix} \Longrightarrow \operatorname{rg} A - (-3I) = \begin{cases} 1 & \text{se } a = 0 \rightsquigarrow m_G(-3) = 3 - 1 = 2 = m_A(-3) \\ 2 & \text{se } a \neq 0 \rightsquigarrow m_G(-3) = 3 - 2 = 1 < m_G(-3) \end{cases}$

$\Longrightarrow \left[A \text{ diagonalizzabile} \iff a = 0 \right]$ (NEL CASO $b = -3$!)

STEP 4: CASO $b = 2$

$m_A(-3) = m_G(-3) = 1, \quad m_A(2) = 2.$

$m_G(2)$?

$A - 2I = \begin{pmatrix} -5 & 0 & 0 \\ 2a & 0 & a \\ 10 & 0 & 0 \end{pmatrix} \Longrightarrow \operatorname{rg}(A - 2I) = \begin{cases} 2 & \text{se } a \neq 0 \\ 1 & \text{se } a = 0 \end{cases}$

$\rightarrow m_G(2) = \begin{cases} 3 - 2 = 1 \ \overset{< m_A(2)}{} & \text{se } a \neq 0 \\ 3 - 1 = 2 \ \underset{= m_A(2)}{} & \text{se } a = 0 \end{cases} \rightarrow \left[A \text{ diag.} \iff a = 0 \right]$ (SE $b = 2$!)

$\rightsquigarrow \left[A \text{ diag.} \iff (b \neq -3, 2 \text{ o } a = 0). \right]$ ✓✓✓.

ⓑ $C = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ c & 0 & 0 & 0 \end{pmatrix}$ diagonalizzabile al variare di $c \in \mathbb{R}$?

STEP 1: $P_C(\lambda)$?

$\begin{vmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & c-\lambda & 0 \\ c & 0 & 0 & \lambda \end{vmatrix} \overset{2^a \text{ RIGA}}{\overset{\downarrow}{=}} (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & c-\lambda & 0 \\ c & 0 & \lambda \end{vmatrix} \overset{2^a \text{ RIGA}}{\overset{\downarrow}{=}} (1-\lambda)(c-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ c & \lambda \end{vmatrix} =$

$= (1-\lambda)(c-\lambda)(-\lambda^2 + \lambda - c) = -(\lambda - 1)(\lambda - c)(\lambda^2 - \lambda + c).$

STEP 2: AUTOVALORI

$\lambda^2 - \lambda + c = 0 \iff \lambda = \dfrac{1 \pm \sqrt{1 - 4c}}{2}.$

$\rightsquigarrow \left[P_C(\lambda) \text{ HA TUTTE LE RADICI IN } \mathbb{R} \iff 1 - 4c \geq 0 \iff c \leq \frac{1}{4} \right].$

$\Rightarrow$ SICURAMENTE PER AVERE C DIAGONALIZZABILE DOBBIAMO AVERE $c \leq \frac{1}{4}$.

$\rightsquigarrow$ AUTOVALORI $1, c, \dfrac{1 + \sqrt{1-4c}}{2}, \dfrac{1 - \sqrt{1-4c}}{2}. \quad \left(c \leq \frac{1}{4}\right)$

SE SONO TUTTI DIVERSI $\Rightarrow C$ è diag. ✓.

CAPIAMO QUALI SONO GLI ALTRI CASI:

- $1 = c$ MAI $\left(c \leq \frac{1}{4}\right)$.
- $1 = \dfrac{1 - \sqrt{1-4c}}{2} \iff 2 = 1 - \sqrt{1-4c} \iff -\sqrt{1-4c} = 1$ MAI $(\sqrt{1-4c} \geq 0)$

- $1 = \frac{1+\sqrt{1-4c}}{2} \xLeftrightarrow{\text{STESSI CALCOLI}} 1 = \sqrt{1-4c} \iff 1-4c=1 \iff c=0.$

- $c = \frac{1\pm\sqrt{1-4c}}{2} \iff 2c-1=\pm\sqrt{1-4c} \implies (2c-1)^2 = 1-4c$

$\implies 4c^2-4c+1 = 1-4c \implies c=0.$

PONENDO $c=0$ in $c = \frac{1\pm\sqrt{1-4c}}{2}$ otteniamo $0 = \frac{1\pm1}{2}$

$\implies$ c è sempre diverso da $\frac{1+\sqrt{1-4c}}{2}$, ed è uguale a $\frac{1-\sqrt{1-4c}}{2}$
$\iff c=0.$

⇝ $\forall\, c \leq 1/4$, $c \neq 0$, ABBIAMO C DIAGONALIZZABILE ✓.

- STEP 3: CASO $c=0$

⇝ ABBIAMO $c=0 = \frac{1-\sqrt{1-4\cdot(0)}}{2}$, $1 = \frac{1+\sqrt{1-4\cdot(0)}}{2}$.

$\implies P_C(\lambda) = -(\lambda-1)\cdot(\lambda)\cdot(\lambda^2-\lambda) = -\lambda^2(\lambda-1)^2$

$\implies$ AUTOVALORI $0, 1$ CON MOLTEPLICITÁ ALGEBRICA 2.

$m_G(0)$?

$$C-0I = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies rg(C-0I)=2 \implies m_G(0)=2 \checkmark$$

$m_G(1)$?

$$C-1I = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \implies rg(C-1I)=3 \implies m_G(1)=1<2=m_A(1).$$

$\implies$ Quindi C non è diagonalizzabile per $c=0$. ✓✓.

② $A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ diagonalizzabile? Trovare S t.c. $S^{-1}AS$ è diagonale.
S è unico?

STEP 1: $P_A(\lambda)$

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ -1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} \underset{\substack{\uparrow \\ 1^a \text{ colonna}}}{=} (1-\lambda)\begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ \lambda & 1 \end{vmatrix} =$$

$= (1-\lambda)(\lambda^2-1) + (\lambda+1) + (1+\lambda) = (1-\lambda)(\lambda+1)(\lambda-1) + 2\cdot(\lambda+1) =$

$= (\lambda+1)((1-\lambda)(\lambda-1)+2) = (\lambda+1)(\lambda-1-\lambda^2+\lambda+2) =$

$= (\lambda+1)(-\lambda^2+2\lambda+1) = -(\lambda+1)(\lambda^2-2\lambda-1).$

- IN $\mathbb{R}$ (E IN $\mathbb{C}$)

$\left(\lambda^2-2\lambda-1 = 0 \iff \frac{2\pm\sqrt{4+4}}{2} = \frac{2\pm2\sqrt{2}}{2} = 1\pm\sqrt{2}. \right)$

$\implies P_A(\lambda) = -(\lambda+1)(\lambda^2-2\lambda-1) = -(\lambda+1)(\lambda-(1+\sqrt{2}))(\lambda-(1-\sqrt{2})).$

⇝ 3 AUTOVALORI DISTINTI $\implies$ A DIAGONALIZZABILE.

- IN $\mathbb{Z}_2$

$$P_A(\lambda) = (\lambda+1)(\lambda^2-2\lambda-2) = (\lambda+1)(\lambda^2-1)$$

$$\simeq (\lambda+1)^2(\lambda-1) \underset{\substack{\uparrow \\ 1=-1}}{=} (\lambda-1)^3.$$

$\longrightarrow$ UN AUTOVALORE CON $m_A(1) = 3.$

$m_G(1)$?

$$A - I = A + I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow rg(A-I) = 2$$

$\rightarrow m_G(1) = 3-2 = 1 < 3. \Rightarrow$ A NON DIAGONALIZZABILE. X

MA! $P_A(\lambda) = (\lambda-1)^3$ HA GRADO 3 E 3 RADICI IN $\mathbb{Z}_2 \Rightarrow$ A É TRIANGOLARIZZABILE ✓.

$\mathbb{Z}_3$?

$$P_A(\lambda) = (\lambda+1)\underbrace{(\lambda^2-2\lambda-1)}_{\text{HA RADICI IN } \mathbb{Z}_3\text{?}}$$

$q(\lambda) = \lambda^2 - 2\lambda - 1.$

$q(0) = -1 \neq 0 \qquad q(1) = 1-2-1 = -2 \neq 0$

$q(2) = 2^2 - 2\cdot 2 - 1 = 2\cdot 2 - 2\cdot 2 - 1 = -1 \neq 0$

$\Rightarrow P_A(\lambda)$ NON HA $\underset{\substack{\shortparallel \\ deg(P_A)}}{3}$ RADICI IN $\mathbb{Z}_3 \Rightarrow$ A NON TRIANGOLARIZZABILE X

$\Downarrow$

A NON DIAGONALIZZABILE X.

$\mathbb{Z}_5$?

COME IN $\mathbb{Z}_3$ DOBBIAMO TROVARE RADICI DI $q(\lambda)$.

$q(0) = -1 \neq 0 \qquad q(1) = -2 \neq 0 \qquad q(2) = -1 \neq 0$

$q(3) = 9 - 6 - 1 = 2 \neq 0 \qquad q(4) = 16 - 8 - 1 = 7 \equiv 2 \neq 0.$

$\rightsquigarrow$ q NON HA RADICI IN $\mathbb{Z}_5 \Rightarrow P_A$ NON HA 3 RADICI. $\Rightarrow$ NO TRIANG X
NO DIAG. X

③ $A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ DIAGONALIZZABILE? TROVA S t.c. SAS^{-1} DIAGONALE.
S É UNICA?

$$P_A(\lambda) = \begin{vmatrix} 1-\lambda & & \cdot & \\ 0 & 2-\lambda & & \vdots \\ 0 & 0 & 3-\lambda & \vdots \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} \underset{\substack{\uparrow \\ \text{FORMA} \\ \text{TRIANGOLARE}}}{=} (1-\lambda)(2-\lambda)(3-\lambda)^2$$

$\rightarrow$ AUTOVALORI 1, 2, 3. $\quad m_A(1) = m_G(1) = 1, \quad m_A(2) = m_G(2) = 1$

$m_A(3) = 2$. $\quad \underline{m_G(3) = 2?}$ $\left(\Longleftrightarrow A \text{ DIAGONALIZZABILE } \checkmark \right)$

$$A - 3I = \begin{pmatrix} -2 & 1 & 0 & 4 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow rg\,(A - 3I) = 2 \Longrightarrow m_G(3) = 4 - 2 = 2 \checkmark.$$

IN PARTICOLARE, $\exists S$ t.c. $S^{-1} A S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

COME TROVARE S?

CERCHIAMO $\mathcal{B}$ BASE DI AUTOVETTORI DI A $\left(\mathcal{B} = \{v_1, v_2, v_3, v_4\} \right)$, t.c. $\left\{ \begin{array}{l} A v_1 = v_1 \\ A v_2 = 2 v_2 \\ A v_3 = 3 v_3 \\ A v_4 = 3 v_4 \end{array} \right)$

$$\Longrightarrow M^{\mathcal{B}}_{\mathcal{B}}(L_A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$M^{\mathcal{B}}_{\mathcal{B}}(L_A) = \left(M^{C}_{\mathcal{B}} \right)^{-1} \cdot \underbrace{M^{C}_{C}(L_A)}_{= A} \cdot M^{C}_{\mathcal{B}},$$ CON C LA BASE CANONICA

$$\Longrightarrow S = M^{C}_{\mathcal{B}}.$$

UNICITÀ? PRENDENDO PER ESEMPIO $\mathcal{B}' = \{7v_1, v_2, v_3, v_4\} \longrightarrow \left\{ \begin{array}{l} A(7v_1) = 1 \cdot (7v_1) \\ A(v_2) = 2 v_2 \\ A(v_3) = 3 v_3 \\ A(v_4) = 3 v_4 \end{array} \right.$

$$\Longrightarrow M^{\mathcal{B}'}_{\mathcal{B}'}(L_A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\overset{\shortparallel}{\left(M^{C}_{\mathcal{B}'} \right)^{-1} A \cdot \underbrace{M^{C}_{\mathcal{B}'}}_{= S'}} \Longrightarrow S^{-1} A S = (S')^{-1} A S',$$

$$S \neq S'.$$

$\leadsto$ TROVIAMO $\mathcal{B}$.

- $$A - I = \begin{pmatrix} 0 & 1 & 0 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$ 1ª COLONNA NULLA $\Rightarrow (A - I)\, e_1 = \underline{0}$.

$$\Longrightarrow A e_1 = e_1 \Longrightarrow v_1 = e_1. \quad \left(e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \right)$$

- $$A - 2I = \begin{pmatrix} -1 & 1 & 0 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leadsto (A - 2I) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Longleftrightarrow \left\{ \begin{array}{l} -x_1 + x_2 + 4x_4 = 0 \\ 3x_3 + x_4 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{array} \right.$$

$$\Longleftrightarrow x_1 = x_2;\ x_3 = x_4 = 0. \longrightarrow \underbrace{Ker\,(A - 2I)}_{\text{AUTOSPAZIO DI } 2} = \langle \underbrace{(1, 1, 0, 0)}_{v_2} \rangle$$

- $$A - 3I = \begin{pmatrix} -2 & 1 & 0 & 4 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \left[(A - 3I) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Longleftrightarrow \left\{ \begin{array}{l} -2x_1 + x_2 + 4x_4 = 0 \\ -x_2 + 3x_3 + x_4 = 0 \end{array} \right. \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{l} x_1 = \dfrac{x_2 + 4x_4}{2} = \dfrac{3}{2} x_3 + \dfrac{5}{2} x_4 \\ x_2 = 3x_3 + x_4 \end{array} \right] \Longrightarrow Ker\,(A - 3I) = \left\{ \left(\frac{3}{2} x_3 + \frac{5}{2} x_4, 3x_3 + x_4, x_3, x_4 \right) \right\}$$

$$\Longrightarrow Ker\,(A - 3I) = \langle \underbrace{\left(\tfrac{3}{2}, 3, 1, 0 \right)}_{v_3}, \underbrace{\left(\tfrac{5}{2}, 1, 0, 1 \right)}_{v_4} \rangle.$$

$$\Longrightarrow \mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 5/2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow$$

$$\Longrightarrow S = M^{C}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 3/2 & 5/2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \checkmark\checkmark\checkmark$$

④

$V = \mathbb{R}[t]_{\leq 2}$ $\quad \left\{ \begin{array}{l} T : V \longrightarrow V \\ \quad p(x) \longmapsto p(x+2) \end{array} \right.$

$\mathcal{B} = \{1, t, t^2 - \frac{2}{3}\}$. $M^{\mathcal{B}}_{\mathcal{B}}(T)$? T NON DIAGONALIZZABILE?

$T(1) = 1$, $\quad T(t) = t+2$, $\quad T(t^2 - \frac{2}{3}) = (t+2)^2 - \frac{2}{3} = t^2 - \frac{2}{6} + 2t + 2$

$$\Rightarrow M^{\mathcal{B}}_{\mathcal{B}}(t) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

T diagonalizzabile?

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} \underset{\substack{\uparrow \\ \text{PRIMA} \\ \text{COLONNA}}}{=} (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3.$$

$\Rightarrow$ AUTOVALORE 1, $m_A(1) = 3$.

$m_G(1)$?

$$rg(A - I) = rg\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 2 \Rightarrow m_G(1) = 3 - 2 = 1.$$

$\Rightarrow m_G(1) < m_A(1) \Longrightarrow T$ NON DIAGONALIZZABILE. ✓✓✓.

AUTOSPAZIO DI 1?

$$T(a + bt + c(t - \tfrac{2}{3})) = a + bt + c(t - \tfrac{2}{3})$$

$$\Longleftrightarrow$$

$$(A - I) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.$$

$$Ker(A - I) = Ker\left(\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) = \left(\begin{array}{l} \text{SOLUZIONI DI} \\ \left\{ \begin{array}{l} b + c = 0 \\ 2c = 0 \end{array} \right. \end{array}\right) = \{(a, 0, 0) \mid a \in \mathbb{R}\}$$

$\Rightarrow$ AUTOSPAZIO DI T RELATIVO A 1 $= \langle 1 \rangle = \{$POLINOMI COSTANTI$\}$. ✓✓✓.