// 12 Gennaio

$$\int_0^{\infty} \underbrace{\sin\left(x+\frac{1}{x}\right)\, \mathrm{th}(x)\, \frac{1}{1+x}}_{f(x)}\, dx$$

$f(x) \in L_{loc}([0,+\infty))$

$$f(x) = \left(\sin(x)\cos\left(\frac{1}{x}\right) + \cos(x)\sin\left(\frac{1}{x}\right)\right) \mathrm{th}(x)\, \frac{1}{1+x}$$

infatti $f \in C^0([0,+\infty))$ se pongo $f(0)=0$

$$\int_0^{+\infty} \sin\left(x+\frac{1}{x}\right) \operatorname{th}(x) \frac{1}{1+x}\, dx = \boxed{\int_0^1 f(x)\, dx} +$$

$$+ \boxed{\int_1^{+\infty} \sin\left(x+\frac{1}{x}\right) \operatorname{th}(x) \frac{1}{1+x}}$$

$$\operatorname{th}(x) = 1 + \operatorname{th}(x) - 1 = 1 + 2e^{-2x}(1+o(1))$$

$$= 1 + o(x^{-n})$$

$$f(x) = \frac{\sin\left(x+\frac{1}{x}\right)}{1+x} + \frac{\sin\left(x+\frac{1}{x}\right)}{1+x}\, o(x^{-1})$$

$$* = \left| \frac{\sin\left(x+\frac{1}{x}\right)}{1+x} \, o(x^{-1}) \right| \quad \text{è integrabile in } [1,+\infty)$$

per confronto asintotico

$$0 = \lim_{x\to+\infty} \frac{\left| \frac{\sin\left(x+\frac{1}{x}\right)}{1+x} \, o(x^{-1}) \right|}{x^{-2}} =$$

$$= \lim_{x\to+\infty} \frac{\left|\sin\left(x+\frac{1}{x}\right) o(1)\right| \cancel{x^{-1}} \, \cancel{x^{-1}} \, \underbrace{\frac{x}{1+x}}_{\to 1}}{\cancel{x^{-2}}} = 0$$

$$x^{-2} \in L[1,+\infty) \implies * \in L[1,+\infty)$$

$$\frac{\sin\left(x+\frac{1}{x}\right)}{1+x} = \frac{\sin\left(x+\frac{1}{x}\right)}{x} + \sin\left(x+\frac{1}{x}\right)\left(\frac{1}{1+x}-\frac{1}{x}\right)$$

$$= \frac{\sin\left(x+\frac{1}{x}\right)}{x} + \sin\left(x+\frac{1}{x}\right)\frac{\cancel{x}-(1+\cancel{x})}{x(1+x)}$$

$$= \frac{\sin\left(x+\frac{1}{x}\right)}{x} - \sin\left(x+\frac{1}{x}\right)\frac{1}{x(1+x)}$$

è an. integrabile

$$\left|\sin\left(x+\frac{1}{x}\right)\right|\frac{1}{x(1+x)} \leq \frac{1}{x(1+x)} < \frac{1}{x^2}$$

$$\frac{\sin\left(x+\frac{1}{x}\right)}{x} = \frac{\sin(x)\cos\left(\frac{1}{x}\right)}{x} + \frac{\cos(x)\sin\left(\frac{1}{x}\right)}{x}$$

$$= \boxed{\frac{\sin x}{x}} + \frac{\sin(x)}{x}\underbrace{\left(\cos\left(\frac{1}{x}\right)-1\right)}_{o\left(\frac{1}{x}\right)} + \cos x\,\frac{\overbrace{\sin\left(\frac{1}{x}\right)}^{\frac{1}{x}\left(1+o(1)\right)}}{x}$$

ass. conv.

$$\lim_{y\to 0}\frac{1-\cos(y)}{y^2} = \frac{1}{2}$$

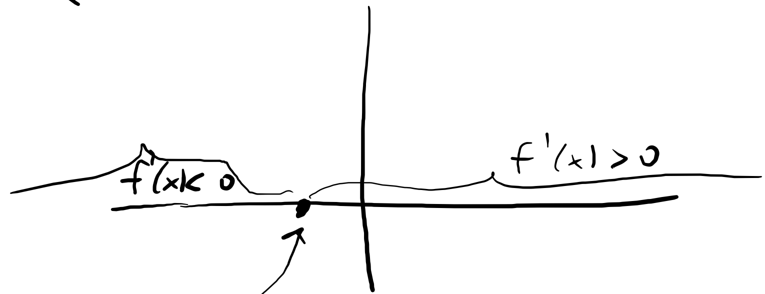
$$\lim_{y\to 0}\frac{\sin y}{y} = 1$$

$f(x) = (x^6 + 3x + 1)\cancel{e^{-x^2}}$ stabilire se ha degli zeri.

$$\lim_{x \to \infty} f(x) = +\infty$$

$$f'(x) = 6x^5 + 3 = 0$$

$$x^5 = -\frac{1}{2} \qquad x = -\frac{1}{\sqrt[5]{2}}$$



$$f\left(-2^{-\frac{1}{5}}\right) = 2^{-\frac{6}{5}} - 3 \cdot 2^{-\frac{1}{5}} + 1 < 0$$

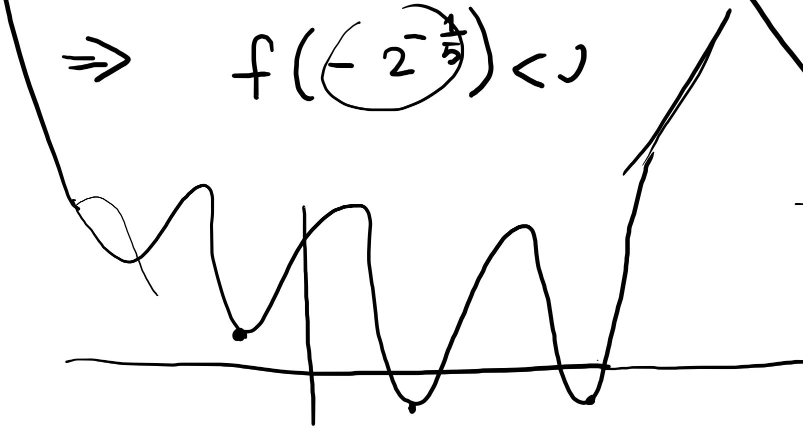
$$3 \cdot 2^{-\frac{1}{5}} > 2^{-\frac{6}{5}} + 1$$

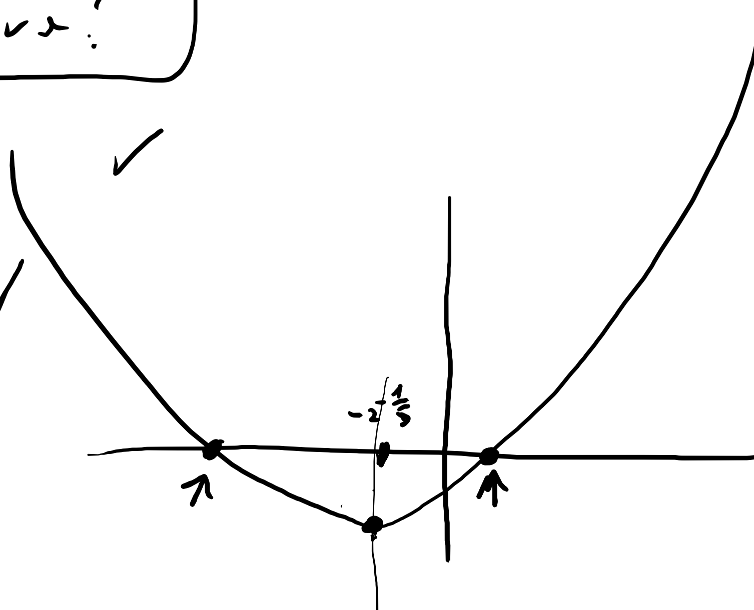
$$3 > 2^{-1} + 2^{\frac{1}{5}}$$

$\boxed{3 > 2^{-1} + 2^{\frac{1}{5}} \quad \text{vero?}}$

$$\frac{1}{2} + 2^{\frac{1}{5}} < \frac{1}{2} + 2 < 3 \quad \checkmark$$

$$\Rightarrow \quad f\left(-2^{-\frac{1}{5}}\right) < 0$$





$$\int_1^x \sin\left(t^2+\frac{\pi}{4}\right) dt = \int_1^x \left(\sin(t^2) \overbrace{\cos\left(\frac{\pi}{4}\right)}^{\frac{1}{\sqrt{2}}} + \cos(t^2) \overbrace{\sin\left(\frac{\pi}{4}\right)}^{\frac{1}{\sqrt{2}}}\right) dt$$

$$= \frac{1}{\sqrt{2}} \int_1^x \left(\sin(t^2)+\cos(t^2)\right) dt =$$

$$= -\frac{1}{\sqrt{2}} \int_0^1 \left(\sin(t^2)+\cos(t^2)\right) dt + \frac{1}{\sqrt{2}} \int_0^x \left(\sin(t^2)+\cos(t^2)\right) dt$$

$$P_{4n+3}(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} \frac{x^{4j+3}}{4j+3}$$

$$+ \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} \frac{x^{4j+1}}{4j+1}$$

P_{4n+1}

$$\sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} \frac{x^{4j+3}}{4j+3} + o\left(x^{4n+3}\right)$$

$$+ \sum_{j=0}^{n+1} \frac{(-1)^j}{(2j)!} \frac{x^{4j+1}}{4j+1} + o\left(x^{4n+5}\right)$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} \frac{x^{4j+3}}{4j+3} +$$

$$+ \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} \frac{x^{4j+1}}{4j+1} + \underbrace{\frac{(-1)^{n+1}}{(2n+2)!} \frac{x^{4n+5}}{4n+5} + o\left(x^{4n+5}\right) + o\left(x^{4n+3}\right)}_{o\left(x^{4n+3}\right)}$$

$$\int_0^x \sin(t^2)\,dt + \int_0^x \cos(t^2)\,dt = \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!}\,\frac{x^{4j+3}}{4j+3} + o(x^{4n+3})$$

$$+ \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!}\,\frac{x^{4j+1}}{4j+1} + o(x^{4n+1}) =$$

$$= \underbrace{\sum_{j=0}^{n-1} \frac{(-1)^j}{(2j+1)!}\,\frac{x^{4j+3}}{4j+3} + \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!}\,\frac{x^{4j+1}}{4j+1}}_{P_{4n+1}(x)} + \underbrace{\frac{(-1)^n}{(2n+1)!}\,\frac{x^{4n+3}}{4n+3} + o(x^{4n+3}) + o(x^{4n+1})}_{= o(x^{4n+1})}$$

$$\int_1^x \sin(t^2)\,dt = \int_1^x \left(\sum_{j=0}^{n} (-1)^j \frac{t^{4j+2}}{(2j+1)!}\,dt + o(t^{4n+2}) \right) dt =$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} \int_1^x t^{4j+2} + \int_1^x o(t^{4n+2})\,dt$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} \frac{x^{4j+3}-1}{4j+3} + \int_1^x o(t^{4n+2})\,dt$$

$$\int_0^x o(t^{n})\,dt = o(x^{n+1})$$

$$\int_1^0 o(t^{4n+2})\,dt$$

$f(x) = x^3 + x$ $\qquad \lim_{x \to +\infty, -\infty} f(x) = \pm \infty \Rightarrow f$ suriettiva

$f'(x) = 3x^2 + 1 \geq 1 \Rightarrow f$ strett crescente $\Rightarrow$ iniettiva.

$\Rightarrow f$ biettiva $\mathbb{R} \to \mathbb{R}$

Sia g la funzione inversa.

Trovare un $p \in \mathbb{R}$ t.c. $\lim_{x \to +\infty} \frac{g(x)}{x^p} = 1$

$f(x) = x^3 + x$

$$f(x) = x^3 + x \qquad \lim_{y\to+\infty} g(y) = +\infty$$

$$1 = \lim_{y\to+\infty} \frac{g(y)}{y^p} = \lim_{x\to+\infty} \frac{x}{(x^3+x)^p} = \lim_{x\to+\infty} \frac{x}{\left(x^3(1+x^{-2})\right)^p} =$$

$$y = x^3 + x$$

$$x = g(y)$$

$$= \lim_{x\to+\infty} \frac{x}{x^{3p}\underbrace{(1+x^{-2})^p}_{\to 1}}$$

$$= \lim_{x\to+\infty} x^{\overset{0}{\widetilde{1-3p}}} = 1$$

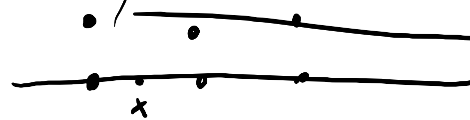
quando $1-3p=0$ $\quad \boxed{p = \frac{1}{3}}$

$f:[a,b]\to\mathbb{R}$, $f(x)=0$ per ogni $x\neq x_1,\dots,x_n$

Dimostrare che

$$F(x)=\int_a^x f(t)\,dt=0 \qquad \forall x.$$

$F\in C^0([a,b])$. Voglio applicare il teorema fondamentale del calcolo. Notiamo che nei punti $\boxed{x\neq x_1,\dots,x_n}$



$$\lim_{y\to x} f(y)=f(x)=0$$

$$\Longrightarrow \quad F'(x)=f(x)=0$$

Anche nei punti $x_1, \dots, x_n$ esiste

$$F'_d(x_j) = f(x_j^+) = \lim_{x \to x_j^+} f(x) = \lim_{x \to x_j^+} 0 = 0$$

$$F'_s(x_j) = f(x_j^-) = \lim_{x \to x_j^-} f(x) = \lim_{x \to x_j^-} 0$$