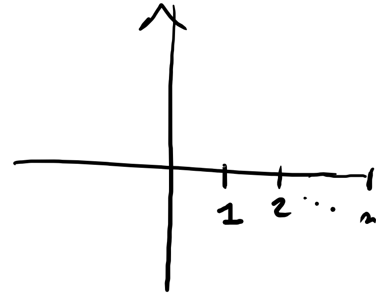
14 Gennaio

$$f(x) = \begin{cases} \int_0^x e^{-2[t]}\,dt & x \geq 0 \\ e^{\frac{1}{x}}(x + x^3) & x < 0 \end{cases}$$



1) $\lim_{x \to -\infty} f(x) = -\infty$

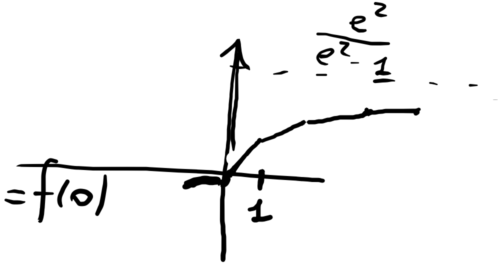
$k = j-1$

$$\lim_{x \to +\infty} f(x) = \int_0^{+\infty} e^{-2[t]}\,dt = \lim_{n \to +\infty} \int_0^{n} e^{-2[t]}\,dt =$$

$$= \lim_{n \to +\infty} \sum_{j=1}^{n} e^{-2(j-1)} = \lim_{n \to +\infty} \sum_{k=0}^{n-1} (e^{-2})^k = \lim_{n \to +\infty} \frac{1 - e^{-2n}}{1 - e^{-2}} = \frac{1}{1 - e^{-2}} = \frac{e^2}{e^2 - 1}$$

2) f è continua per $x \neq 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \int_0^x e^{-2[t]}\,dt = \int_0^0 e^{-2[t]}\,dt = 0 = f(0)$$



$$\lim_{x\to 0^-} e^{\frac{1}{x}}(x+x^3) = \lim_{y\to -\infty} e^{y}\left(\frac{1}{y}+\frac{1}{y^3}\right) = 0$$

$y = \frac{1}{x}$

3) Per $x>0$ $\quad x \notin \mathbb{N}$ $\qquad f'(x) = e^{-2[x]}$

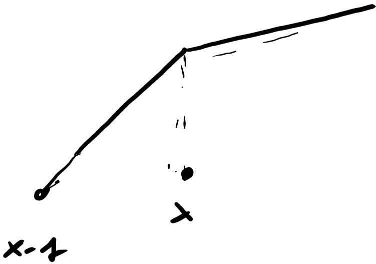
Se $x \in \{0,1,2,\dots\}$ $\qquad f'_d(x) = e^{-2[x^+]} = \lim_{y\to x^+} e^{-2[y]} = e^{-2[x]}$

$= e^{-2x}$

$f_d(0) = 1$

Per $x \in \{1, 2, 3, \dots\}$

$$f'_s(x) = e^{-2[x^-]} = \lim_{y \to x^-} e^{-2[y]} = e^{-2(x-1)}$$



Per $x<0$ $\quad f(x) = e^{\frac{1}{x}}(x+x^3)$

$$f'(x) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)(x+x^3) + e^{\frac{1}{x}}(1+3x^2) =$$

$$= e^{\frac{1}{x}}\left(3x^2 - x + 1 - \frac{1}{x}\right) = \frac{e^{\frac{1}{x}}}{x}\left(3x^3 - x^2 + x - 1\right)$$

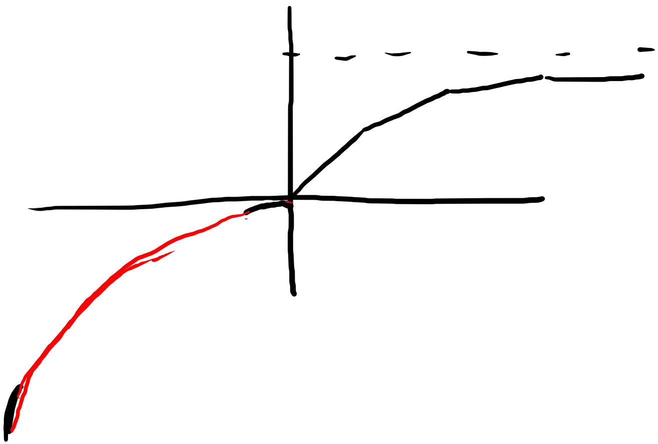
$$f'_s(0) = \lim_{x \to 0^-} \frac{f(x)}{x} \overset{\text{Hop.}}{=} \lim_{x \to 0^-} f'(x) = 0$$

per $x < 0$ $\quad f'(x) = \frac{e^{\frac{1}{x}}}{x} \underbrace{(3x^3 - x^2 + x - 1)}_{p(x)} > 0$

Controlliamo che per $x<0$ si ha $p(x) < -1$

Questo è ovvio, perché per $x<0$ $\quad 3x^3 < 0, \quad -x^2 < 0$

$x < 0 \qquad -1 < 0$



$$f'(x) = e^{\frac{1}{x}}\left(3x^2 - x + 1 - \frac{1}{x}\right)$$

$$f''(x) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\left(3x^2 - x + 1 - \frac{1}{x}\right) + e^{\frac{1}{x}}\left(6x - 1 + \frac{1}{x^2}\right)$$

$$f''(x) = \frac{e^{\frac{1}{x}}}{x^2}\left(-3x^2 + x - 1 + \frac{1}{x}\right) + \frac{e^{\frac{1}{x}}}{x^2}\left(6x^3 - x^2 + 1\right)$$

$$f''(x) = \underset{>0}{\frac{e^{\frac{1}{x}}}{x^2}}\left(\underbrace{6x^3 - 4x^2 + x + \frac{1}{x}}_{<0}\right)$$

$$f(x) = \int_0^{x+1} (1+t)\sin(t^3)\,dt = \underline{\int_0^{1} (1+t)\sin(t^3)}\,dt +$$

$$+ \int_1^{x+1} (1+t)\sin(t^3)\,dt$$

$$\int_1^{x+1} (1+t)\sin(t^3)\,dt =$$

$t = 1+s$

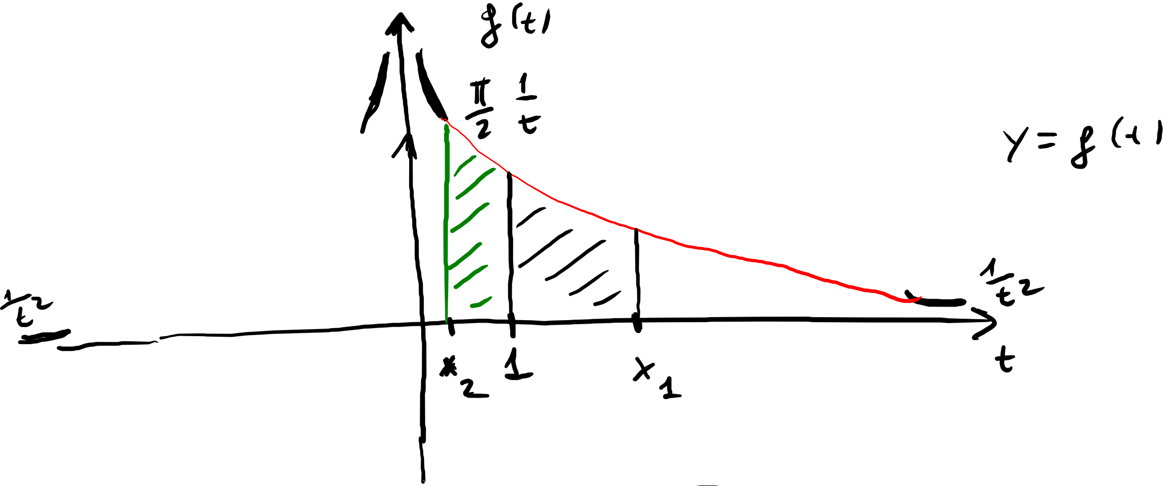
$$= \int_0^{x} (2+s)\sin\left((1+s)^3\right)ds =$$

$$= \int_0^{x} (2+s)\sin\left(1+3s+3s^2+s^3\right)ds$$

$$\sin(x^3+3x^2+3x+1) = \sin(x^3)\cos(3x^2+3x+1) + \cos(x^3)\sin(3x^2+3x+1)$$

$f(\cancel{x}) -$

$$f(x) = \int_1^x \underbrace{\operatorname{arctg}\left(\frac{1}{t}\right)\frac{1}{t}}_{g(t)}\,dt \qquad g(t) = \operatorname{arctg}\left(\frac{1}{t}\right)\frac{1}{t}$$



$\boxed{f(x) \text{ è definito per } x>0}$. Per $x \le 0$ $f(x)$ non è definito

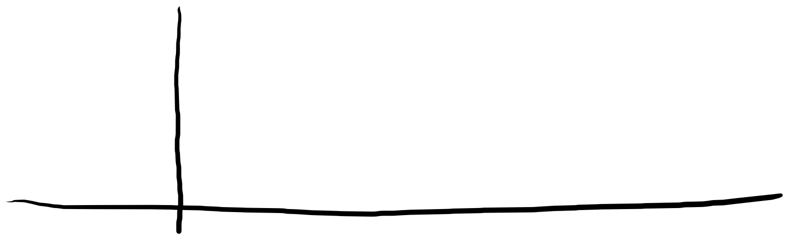
$$f(x) = \int_1^x \operatorname{arctg}\left(\frac{1}{t}\right) \frac{dt}{t} \qquad \lim_{x \to 0^+} f(x) = -\infty$$

$$f(1) = 0 \quad , \quad \lim_{x \to +\infty} \int_1^x \operatorname{arctg}\left(\frac{1}{t}\right) \frac{dt}{t} = L > 0$$

$$\int_1^x \operatorname{arctg}\left(\frac{1}{t}\right) \frac{dt}{t} = \qquad y = \frac{1}{t} \qquad dy = -\frac{1}{t^2}\, dt$$

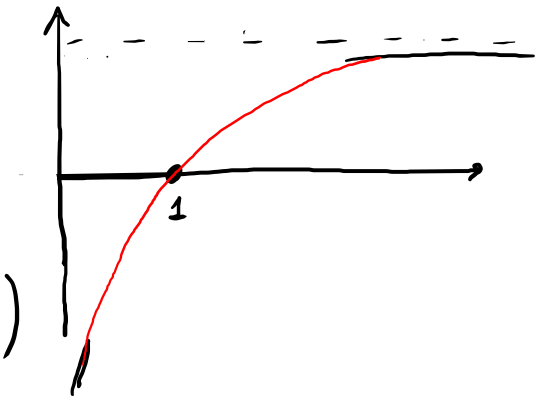
$$\frac{dt}{t} = -t\, dy = -\frac{dy}{y}$$

$$= \int_1^{\frac{1}{x}} \operatorname{arctg}(y) \frac{dy}{y}$$



$$f'(x) = \frac{d}{dx}\int_1^x \operatorname{artg}\left(\frac{1}{t}\right)\frac{dt}{t} =$$

$$= \operatorname{artg}\left(\frac{1}{x}\right)\frac{1}{x} > 0$$



$$f''(x) = -\frac{1}{x^2}\operatorname{artg}\left(\frac{1}{x}\right) + \frac{1}{x}\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2}\operatorname{artg}\left(\frac{1}{x}\right) - \frac{1}{\cancel{x}}\,\frac{1}{1+x^2} < 0$$

$$\left(\operatorname{artg}\left(\frac{1}{x}\right)\right)' = \operatorname{artg}'\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)'$$

$$\cos\left(\mathrm{Re}\left(\frac{z}{1+z}\right)\right) > 0$$

$$\mathrm{Re}\,\frac{z}{1+z} = \mathrm{Re}\,\frac{z\,(1+\bar{z})}{|1+z|^2} = \mathrm{Re}\,\frac{z+|z|^2}{|(x+1)+iy|^2} = \frac{x+x^2+y^2}{(x+1)^2+y^2}$$

$$\cos\left(\frac{x^2+y^2+2\cdot\frac{1}{2}x}{(x+1)^2+y^2}\right) = \cos\left(\frac{(x+\frac{1}{2})^2+y^2-\frac{1}{4}}{(x+1)^2+y^2}\right)$$

Sappiamo $\cos(t) > 0$ per $-\frac{\pi}{2} < t < \frac{\pi}{2}$ $\quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Updownarrow$

$$t \in \bigcup_{k=-\infty}^{+\infty}\left(-\frac{\pi}{2}+2\pi k, \frac{\pi}{2}+2\pi k\right) \quad \Rightarrow \quad < \frac{(x+\frac{1}{2})^2+y^2-\frac{1}{4}}{(x+1)^2+y^2} < \frac{\pi}{2}$$

$$-\frac{\pi}{2} < \frac{(x+\frac{1}{2})^2 + y^2 - \frac{1}{4}}{(x+1)^2 + y^2} < \frac{\pi}{2}$$

$$\left(x+\frac{1}{2}\right)^2 + y^2 - \frac{1}{4} < \frac{\pi}{2}\left((x+1)^2 + y^2\right)$$

$$\left(x+\frac{1}{2}\right)^2 + y^2 - \frac{1}{4} > -\frac{\pi}{2}\left((x+1)^2 + y^2\right)$$

$$x^2 + y^2 + x - \frac{1}{4} < \frac{\pi}{2}\left(x^2 + y^2 + 2x + 1\right)$$

$$\left(\frac{\pi}{2}-1\right)x^2 + \left(\frac{\pi}{2}-1\right)y^2 + (\pi-1)x + \frac{\pi}{2} + \frac{1}{4} > 0$$

$$x^2 + y^2 + 2\,\frac{\pi-1}{\frac{\pi}{2}-1}\,\frac{x}{2} > -\frac{\frac{\pi}{2}-\frac{1}{4}}{\frac{\pi}{2}-1}$$

$$\left(x + \frac{\pi-1}{\frac{\pi}{2}-1}\,\frac{1}{2}\right)^2 + y^2 >$$

$$> \left(\frac{\pi-1}{\frac{\pi}{2}-1}\right)^2 \frac{1}{4} - \frac{1}{2}\,\frac{\pi-\frac{1}{2}}{\frac{\pi}{2}-1}$$

$$= \frac{1}{4\left(\frac{\pi}{2}-1\right)}\left[\frac{(\pi-1)^2}{\frac{\pi}{2}-1} - 2\left(\pi-\frac{1}{2}\right)\right]$$

$$= \frac{1}{4\left(\frac{\pi}{2}-1\right)}\,\frac{\pi^2 - 2\pi + 1 - 2\left(\frac{\pi}{2}-1\right)\left(\pi-\frac{1}{2}\right)}{\frac{\pi}{2}-1}$$

$$\frac{\pi^2-\pi+1-2\left(\frac{\pi}{2}-1\right)\left(\pi-\frac{1}{2}\right)}{4\left(\frac{\pi}{2}-1\right)^2}=\frac{\pi^2-\pi+1-(\pi-2)\left(\pi-\frac{1}{2}\right)}{4\left(\frac{\pi}{2}-1\right)^2}=$$

$$=\frac{\cancel{\pi^2}-\pi+\cancel{1}-\left(\cancel{\pi^2}-2\pi-\frac{1}{2}\pi+\cancel{1}\right)}{4\left(\frac{\pi}{2}-1\right)^2}=\frac{\pi\left(1+\frac{1}{2}\right)}{4\left(\frac{\pi}{2}-1\right)^2}>0$$

$$\left(x+\frac{\pi-1}{\frac{\pi}{2}-1}\;\frac{1}{2}\right)^2+y^2>\frac{\pi\;3}{8\left(\frac{\pi}{2}-1\right)^2}$$

regione esterna al disco di centro $\left(-\frac{\pi-1}{\pi-\frac{1}{2}},\,0\right)$ e raggio

$$\sqrt{\frac{3\pi}{8\left(\frac{\pi}{2}-1\right)^2}}$$

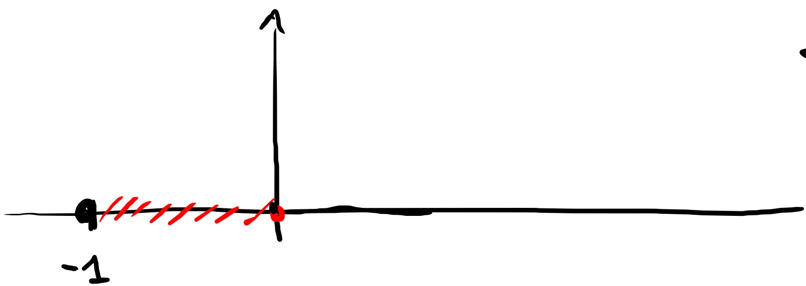
$$f(x)=\begin{cases} \int_0^x \frac{1}{t^2+3t+6}\,dt & \text{per } x \geq 0 \\ \sqrt{x^2+x} & \text{per } x<0 \end{cases}$$

per $x<0$

$$x^2+x \geq 0$$

$$x(x+1) \geq 0 \implies x \leq 0 \text{ e } x+1 \leq 0$$

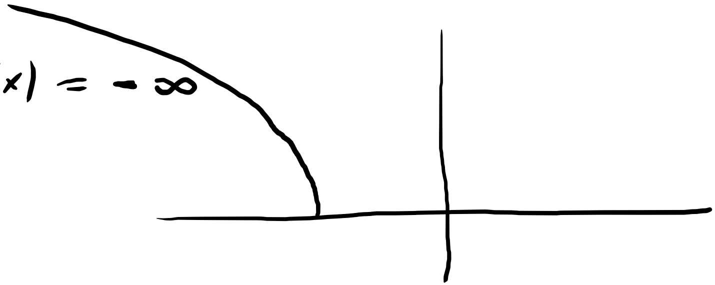
$$\iff \boxed{x \leq -1}$$



per $x \le -1$ $\quad f(x) = \sqrt{x^2+x}$ $\qquad \lim_{x\to -\infty} f(x) = +\infty$

$$f'(x) = \frac{1}{2}\,\frac{1}{\sqrt{x^2+x}}\,(2x+1) < 0$$

inoltre $\lim_{x\to -1^-} f'(x) = -\infty$



$$f''(x) = \frac{1}{\sqrt{x^2+x}} - \frac{1}{4}\,\frac{(2x+1)^2}{(x^2+x)^{\frac{3}{2}}} = \frac{1}{4(x^2+x)^{\frac{3}{2}}}\left(4x^2+4x-(2x+1)^2\right)$$

$$= \frac{1}{4(x^2+x)^{\frac{3}{2}}}\left(\cancel{4x^2+4x} - \cancel{4x^2-4x} - 1\right) = \frac{-1}{4(x^2+x)^{\frac{3}{2}}} < 0$$

Retta asintotica? $\qquad f(x) = \sqrt{x^2+x}$

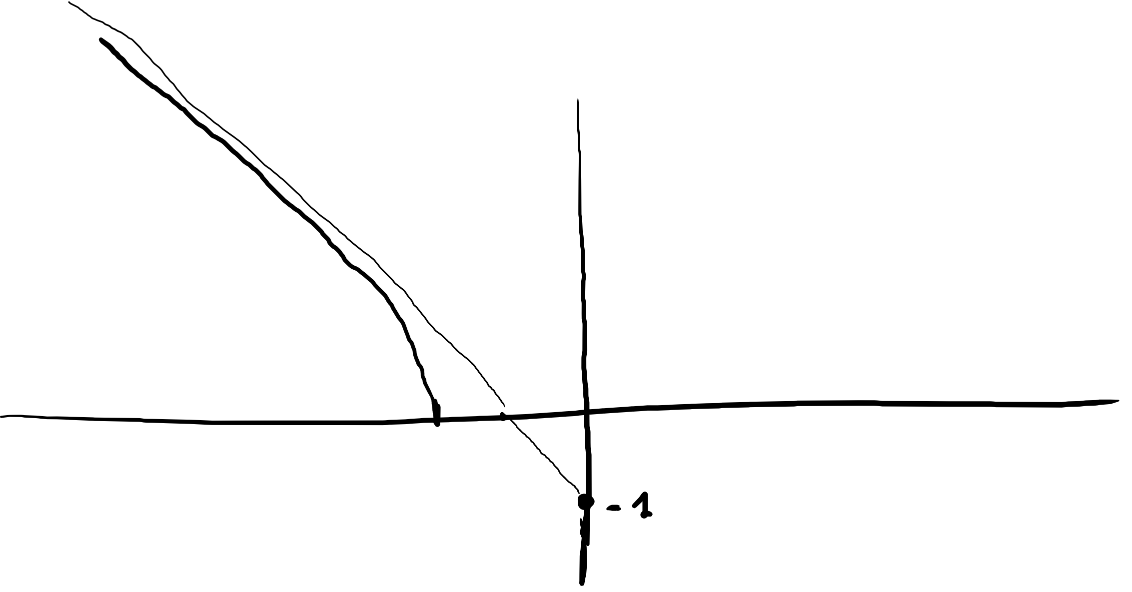
$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+\cancel{x}}}{x} = -1$$

$$\lim_{x\to-\infty} \left(f(x)+x\right) = \lim_{x\to-\infty} \left(\sqrt{x^2+x}+x\right) = -\frac{1}{2}$$

$$= \lim_{x\to-\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}-x} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}-x} = \lim_{x\to-\infty} \frac{x}{-2x}$$

$$= -\frac{1}{2}$$

$\boxed{y = -x - \frac{1}{2}}$ è la retta asintotica a $-\infty$



$$y = -x - \frac{1}{2}$$

per $x \geq 0$ $\qquad f(x) = \int_0^x \frac{1}{t^2+3t+6}\,dt$

$t^2+3t+6=0 \qquad t_\pm = -\frac{3}{2} \pm \frac{\sqrt{9-24}}{2}$ non ci sono radici reali.

$$\int_0^x \frac{1}{t^2+2\frac{3}{2}t+6}\,dt = \int_0^x \frac{1}{\left(t+\frac{3}{2}\right)^2+6-\frac{9}{4}} = \int_0^x \frac{1}{\left(t+\frac{3}{2}\right)^2+\frac{15}{4}}\,dt$$

$y = t+\frac{3}{2}$ $$= \int_{\frac{3}{2}}^{x+\frac{3}{2}} \frac{1}{y^2+\frac{15}{4}}\,dy = \int_{\frac{2}{\sqrt{15}}\frac{3}{2}}^{\frac{2}{\sqrt{15}}\left(x+\frac{3}{2}\right)} \frac{\frac{\sqrt{15}}{2}\,du}{\frac{15}{4}\,(u^2+1)}$$

$y = \frac{\sqrt{15}}{2}u \qquad dy = \frac{\sqrt{15}}{2}\,du$ $$= \int_{\frac{3}{\sqrt{15}}}^{\frac{2}{\sqrt{15}}\left(x+\frac{3}{2}\right)} \frac{du}{u^2+1} \qquad \frac{2}{\sqrt{15}}$$