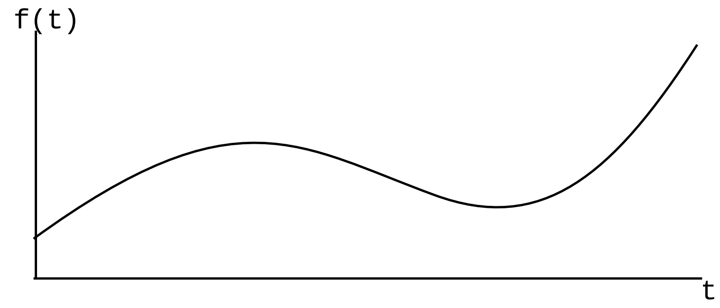
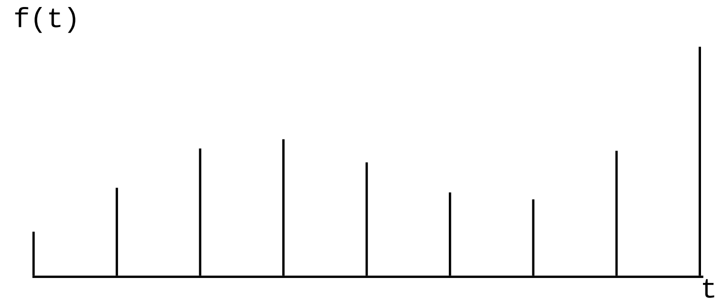
# Sampling

Continuous




Discrete



# Sampling

Sampling a continuous function $f$ to produce a discrete function $\hat{f}$

$$\hat{f}[n] = f(n\Delta t)$$

is just multiplying it by a comb:

$$\hat{f} = f \ \mathrm{comb}_h$$

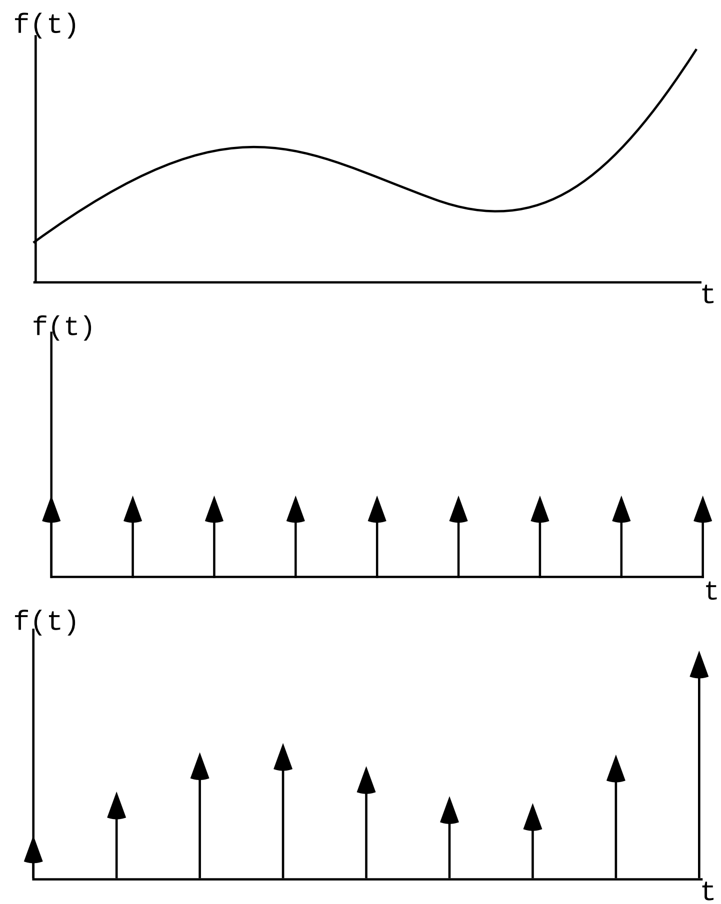
where $h = \Delta t$

# Sampling In The Spatial Domain - Graphical Example

Continuous

Sampling Comb

Discrete



## Sampling In The Frequency Domain

Sampling (multiplication by a $\mathrm{comb}_h$)

$$\hat{f} = f \ \mathrm{comb}_h$$

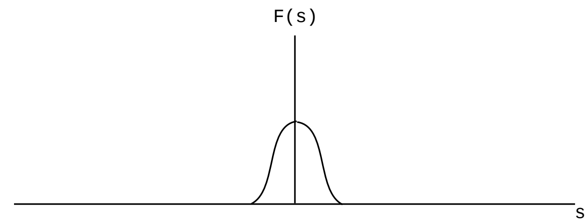
is convolution in the frequency domain with the transform of a comb:

$$\hat{F} = F * \ \mathrm{comb}_{1/h}$$

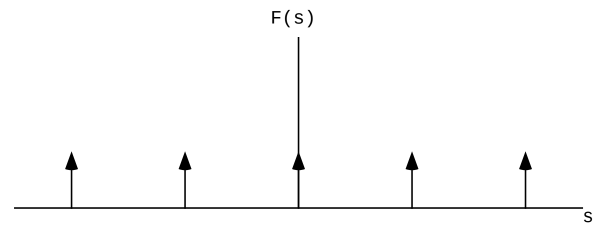
Convolution of a function and a comb causes a copy of the function to "stick" to each tooth of the comb, *and all of them add together*.

# Sampling In The Frequency Domain - Graphical Example

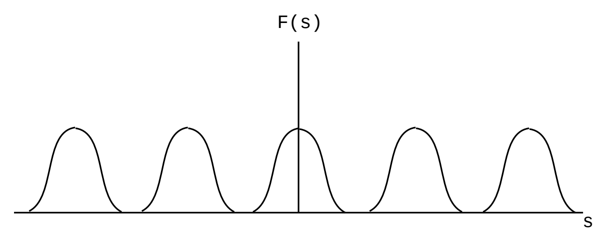
Spectrum




Comb’s Spectrum




Spectrum of the Discrete Signal



# Reconstruction

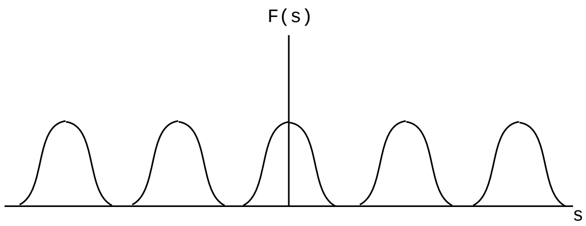
In theory, we can reconstruct the original *continuous* function by removing all of the extraneous copies of its spectrum created by the sampling process:

$$F(s) = \hat{F}(s)\ \mathrm{rect}_{1/h}(s)$$

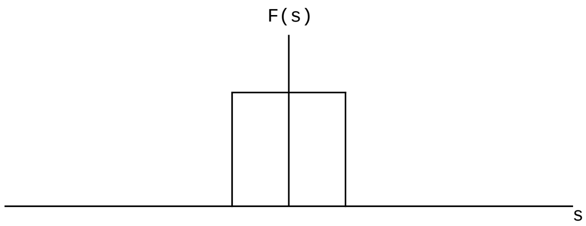
In other words, keep everything in the frequency domain between $[-1/2h, 1/2h]$ and throw the rest away.

# Reconstruction - Graphical Example

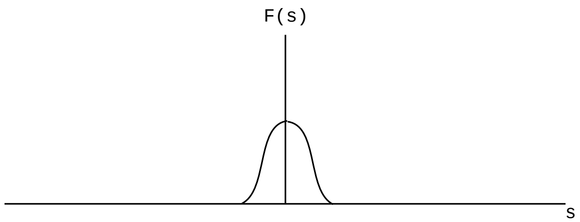
Spectrum of the Discrete Signal




Rectangular Filter




Reconstructed Signal Spectrum



## The Sampling Theorem

We can only do this reconstruction *if the duplicated copies do not overlap*.

They do not overlap if:

1. The signal is bandlimited, and
2. The highest frequency in the signal is less than $1/2h$

In other words, the sampling rate $1/h$ must be twice the frequency of the highest frequency in the image.

This is called the *Nyquist rate*.

# Aliasing

What if the duplicated copies in the frequency domain *do* overlap?

High frequency parts of the signal (those higher than $1/2h$) intrude into other copies.

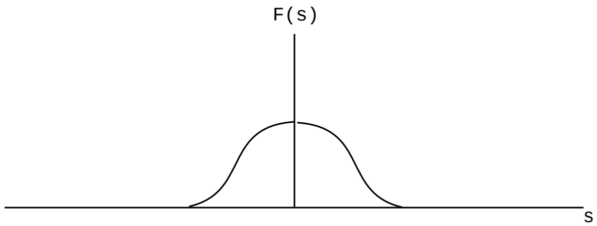
The higher the frequency, the *lower* the point of overlap in the adjacent copy.

This high-frequency masquerading as low frequencies is called *aliasing*.

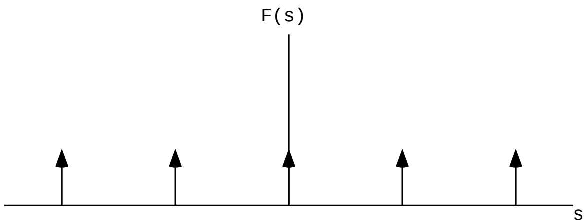
False low-frequency patterns called *Moiré patterns*.

# Aliasing - Graphical Example

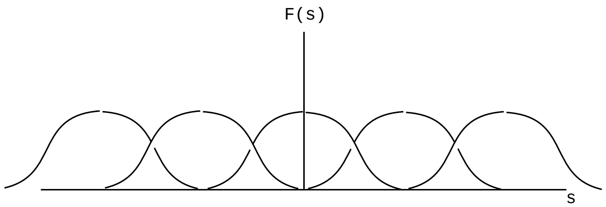
Spectrum



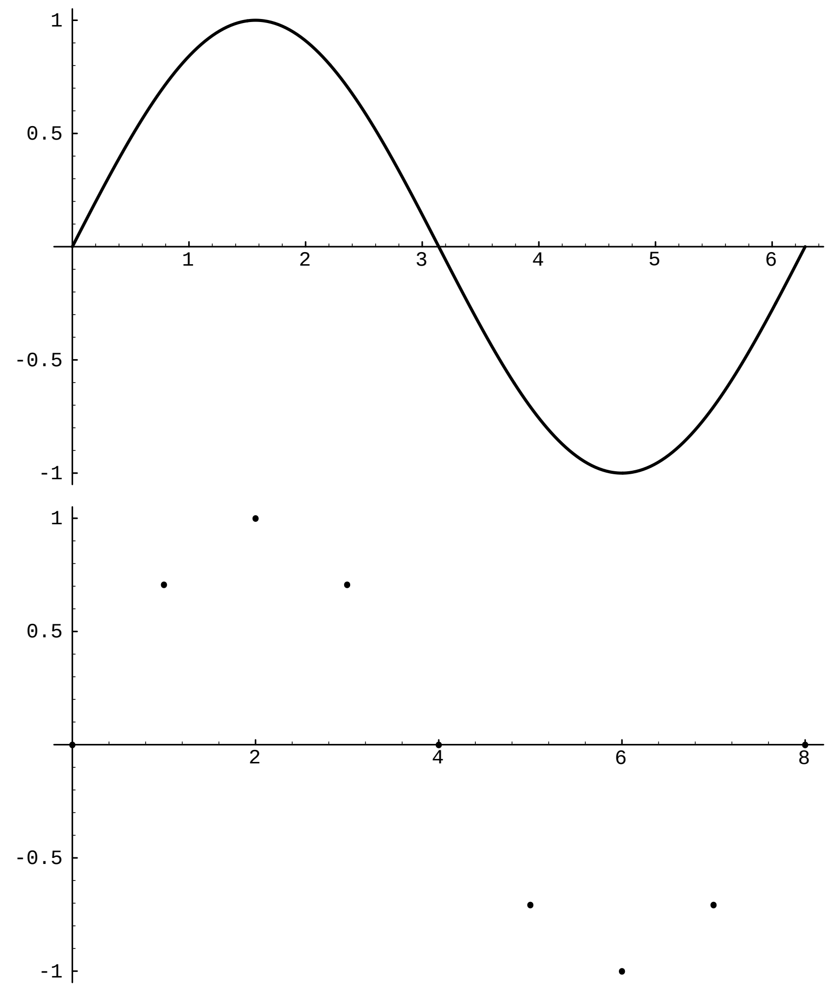

Comb’s Spectrum



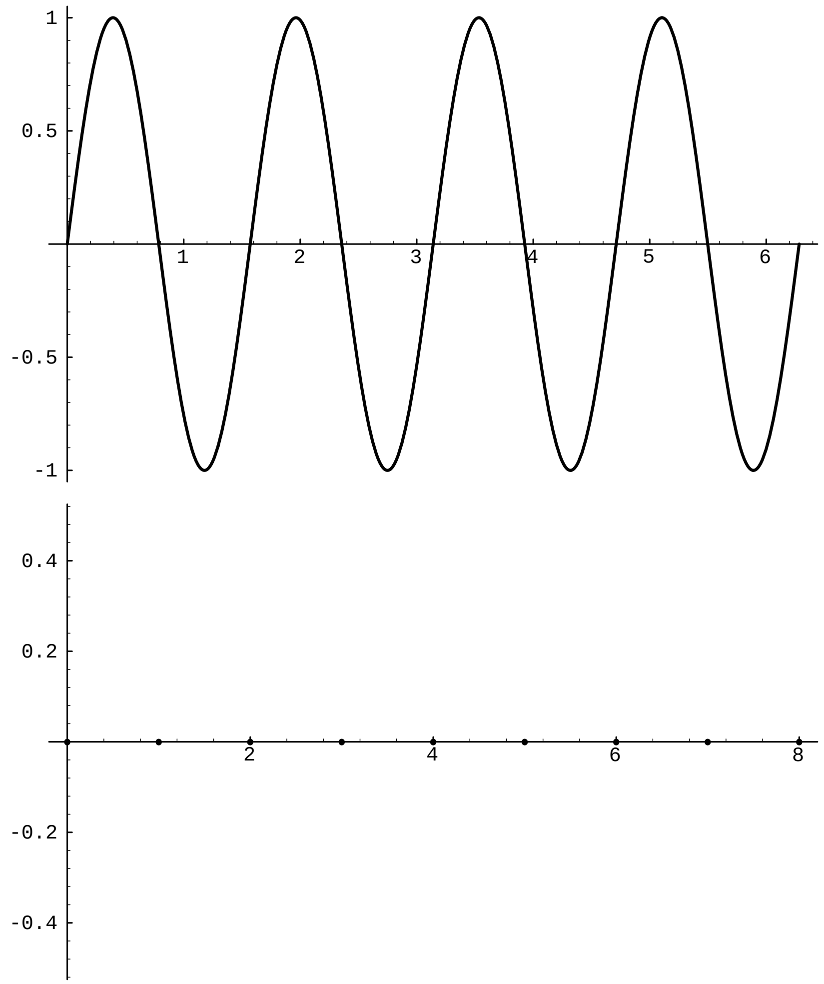

Spectrum of the Discrete Signal



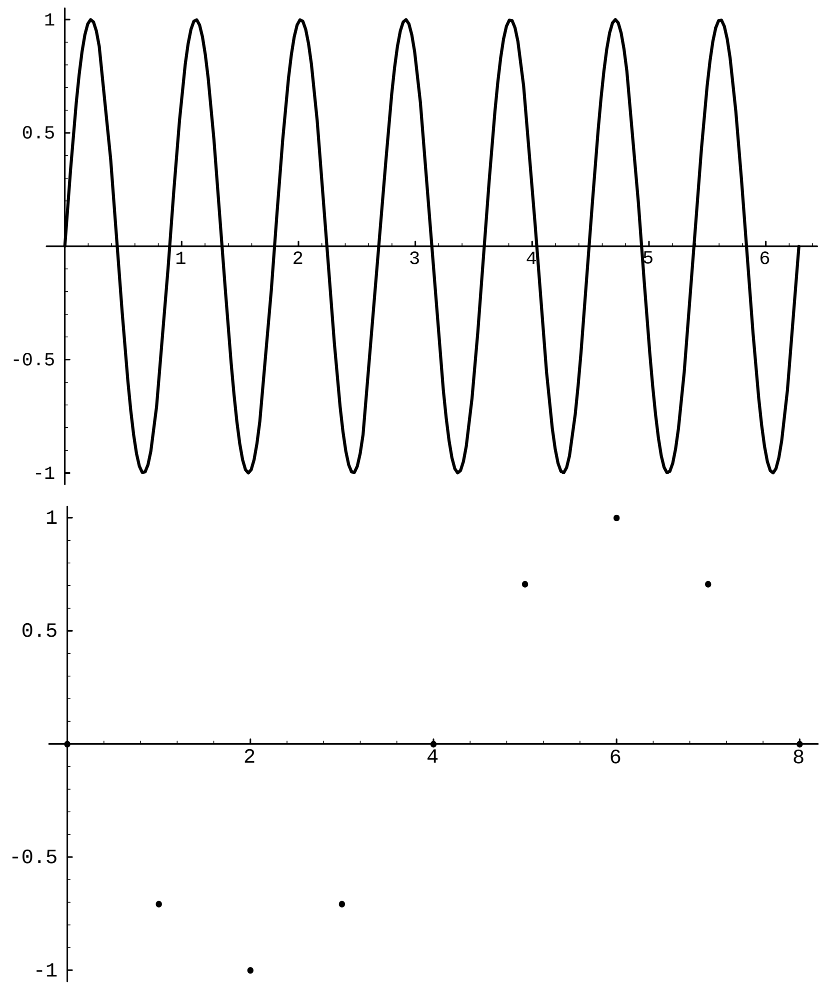
# Sampling - Above the Nyquist Rate



# Sampling - At the Nyquist Rate

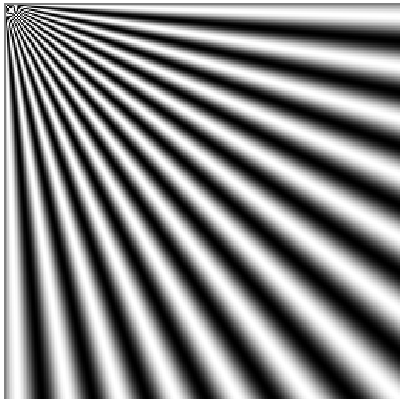


# Sampling - Below the Nyquist Rate

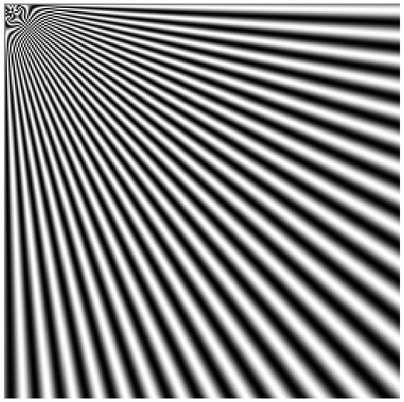


# Moiré patterns

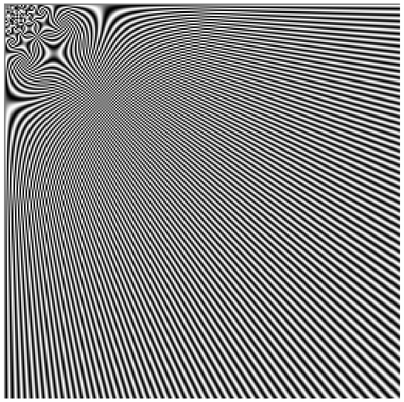
sine.10.im



sine.50.im



sine.100.im



sine.400.im

# Preventing Aliasing

You have two choices:

1. Increase your sampling
2. Decrease the highest frequency in the signal *before sampling*.

# Reconstruction - Revisited

Reconstruction was

$$F(s) = \hat{F}(s)\ \mathrm{rect}_{1/h}(s)$$

But in the time/spatial domain this is equivalent to

$$f(t) = \hat{f}(t) *\ \mathrm{sinc}(2\pi t/h)$$

So, convolve your discretely-sampled (non-aliased) image with a sinc function and you can reconstruct the original continuous one!

# Imperfect Reconstruction

Problem: you can’t do it—the sinc function has infinite extent.

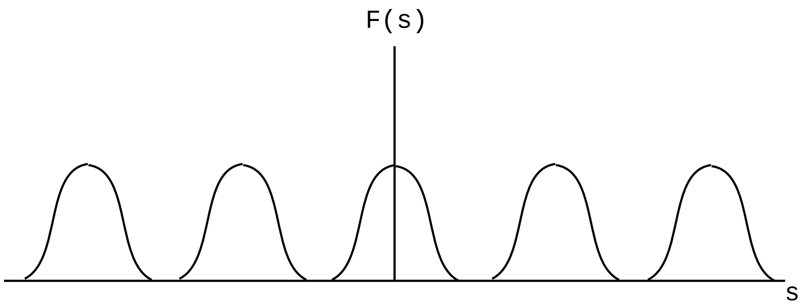
The best you can do is to come close.

By not perfectly clipping in the frequency domain, the duplicate copies now look like false *high* frequencies.

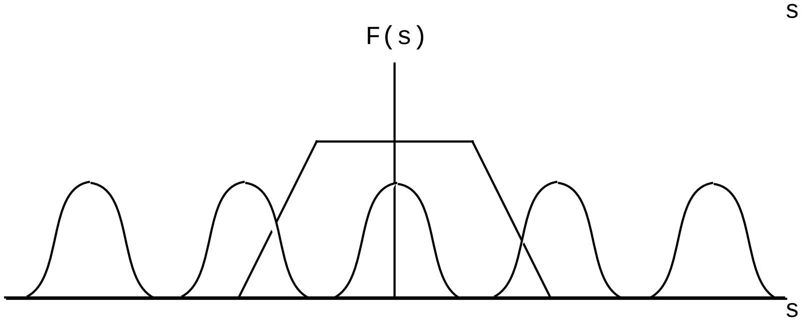
“Jaggies” in graphics: false high frequencies caused by poor reconstruction.

# Imperfect Reconstruction - Graphical Example

Spectrum of the Discrete Signal




Imperfect Reconstruction



# Correcting Imperfect Reconstruction

1. Sample well above the Nyquist rate.

2. Low-pass filter *after* reconstruction.

# Typical Sampling/Processing/Reconstruction Pipeline

1. Low-pass filter to reduce aliasing
2. Sample
3. Do something with the digitized signal
4. Reconstruct
5. Low-pass filter to correct for imperfect reconstruction

# The Discrete Frequency Domain

If sampling in the time/spatial domain is multiplication by a comb, so is sampling (discretizing) the frequency domain.

Multiplication by a comb in one domain is convolution with a comb of inverse spacing in the other.

Discrete time/spatial samples = replicated copies of the signal's spectrum appear in the frequency domain.

Discrete frequencies = replicated copies of the *signal itself* appear in the time/spatial domain.

## The Discrete Frequency Domain

If a signal is $N$ time samples long, and we discretize the frequency domain at $1/N$ intervals (like the DFT), we reproduce the signal every $N$ samples *in the time domain*.

The Discrete Fourier Transform of a truncated (finite-domain) signal is the Continuous Fourier Transform of the same *periodic* signal.

# Frequency Resolution

An $N$-element signal is accurate in the frequency domain only to $1/N$.

To be more accurate in the spatial domain, *sample more frequently*.

To be more accurate in the frequency domain, *sample longer*.