15 gennaio

Verificare che non esistono costanti $A_1, \dots, A_n$ t.c.

$$R(x) = \frac{1+x+\cdots+x^n}{(x-1)\cdots\cdots(x-n)} = \frac{A_1}{x-1} + \cdots\cdots + \frac{A_n}{x-n} \quad (1)$$

Il fatto che (1) non è possibile segue dall'essere i gradi di numeratore e denominatore in $R(x)$ entrambi uguali ad n. Infatti

$$\lim_{x\to+\infty} \frac{1+x+\cdots+x^n}{(x-1)(x-2)\cdots(x-n)} = \lim_{x\to+\infty} \frac{x^n}{x^n} = \lim_{x\to+\infty} 1 = 1$$

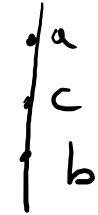
mentre

$$\lim_{x\to+\infty}\left(\frac{A_1}{x-x_1}+\cdots+\frac{A_n}{x-x_n}\right)=0 \quad \text{per ogni scelta di}$$

$A_1,\dots,A_n$. Quindi l'uguaglianza (1) non può essere vera per alcuna scelta di $A_1,\dots,A_n$.

Se $f(x) \le g(x)$ $\forall x \in X$ $\sup X = +\infty$ $\Longrightarrow$ $X \subseteq \mathbb{R}$

$$a = \lim_{x \to +\infty} f(x) \le \lim_{x \to +\infty} g(x) = b.$$



Per assurdo sia $a > b$ e sia $b < c < a$

$\lim_{x \to +\infty} f(x) = a \Longrightarrow \exists\, M_1 \in \mathbb{R}$ t.c. $x > M_1$ e $x \in X \Longrightarrow f(x) > c$

$\lim_{x \to +\infty} g(x) = b \Longrightarrow \exists\, M_2 \in \mathbb{R}$ t.c. $x > M_2$ e $x \in X \Longrightarrow g(x) < c.$

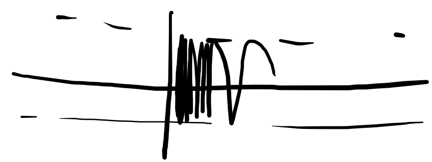
Allora per $x > \max\{M_1, M_2\} \Longrightarrow$ $f(x) > c > g(x) \ge f(x)$

$f(x) > f(x)$ assurdo.

$$F(x) = \int_0^x \sin\left(\frac{1}{t}\right) dt \implies F'(0) = 0$$

Notiamo $F'(x) = \sin\left(\frac{1}{x}\right)$ $\forall x \neq 0$ dal teor fond. del calcolo. Inoltre $F \in C^0(\mathbb{R})$

Supponiamo che $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$ e' primitivabile in $\mathbb{R}$, cioè esiste G t.c. $G'(x) = f(x)$ $\forall x \in \mathbb{R}$.



Se considero $F-G$, risulta che

$$(F-G)'(x) = F'(x) - G'(x) = 0 \quad \forall\, x \neq 0$$

Inoltre $F-G \in C^0(\mathbb{R})$

Il teor di Lagrange in $[0; +\infty)$ implica

che $F(x)-G(x)$ è costante in $[0, +\infty)$.

$$\Rightarrow \quad F(x) - G(x) = F(0) - G(0) \quad \forall\, x \geq 0.$$

Il teor di Lagrange in $(-\infty, 0]$ implica che $F-G$ è cost in

$(-\infty, 0] \Rightarrow F(x) - G(x) = F(0) - G(0) \quad \forall\, x \leq 0$

Conclusioni

$$F(x) - G(x) = F(0) - G(0) \qquad \forall\, x \in \mathbb{R}$$

$$\Rightarrow \quad F(x) = G(x) + F(0) - G(0)$$

$$\Rightarrow \quad F'(0) = G'(0) = 0$$

$$f(x) = [x]^3 \, e^{-[x]} \left(\operatorname{th}(2[x]) + 1 \right) \quad \text{in } \mathbb{R}$$

dimostrare che ha punti di min/max assoluto in $\mathbb{R}$.

Qui, $\displaystyle\lim_{x \to \infty} f(x) = 0$

$$\operatorname{th}(y) + 1 = \frac{e^y - e^{-y}}{e^y + e^{-y}} + 1 = \frac{e^y - \cancel{e^{-y}} + e^y + \cancel{e^{-y}}}{e^y + e^{-y}} = \frac{2e^y}{e^y + e^{-y}} =$$

$$= \frac{2e^y}{e^{-y}(1 + e^{2y})} = 2e^{2y} \frac{1}{1 + e^{2y}} = 2e^{2y}(1 + o(1))$$

$$f(x) = [x]^3 \, e^{-[x]} \, 2e^{4[x]}(1 + o(1)) = 2[x]^3 e^{3[x]}(1 + o(1))$$

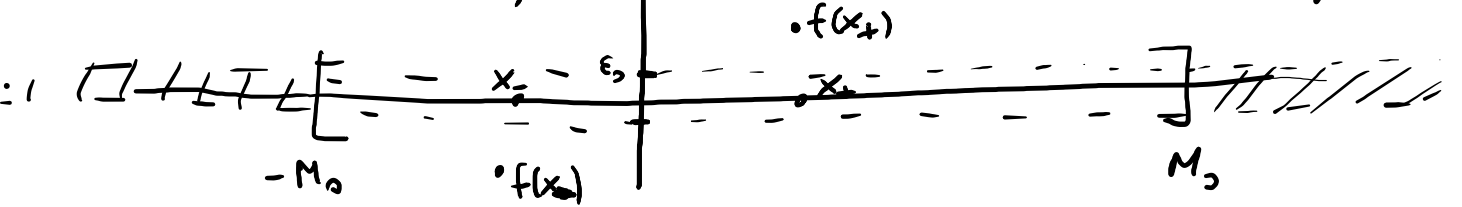
Esistono punti $x_- < 0 < x_+$ t.c.

$f(x_-) < 0$ e $f(x_+) > 0$.

Sia $0 < \varepsilon_0 < \min\{|f(x_-)|, f(x_+)\}$. Allora

$\lim_{x\to\infty} f(x) = 0 \Rightarrow \exists\, M_0 \in \mathbb{N}$ t.c. $|x| > M_0$ ho $|f(x)| < \varepsilon_0$

Notare che $x_-, x_+ \in [-M_0, M_0]$ perchè ad esempio



fosse $x_+ \notin [-M_0, M_0] \implies x_+ > M_0 \implies f(x_p) < \varepsilon_0 \leq f(x_+)$

$\implies f(x_+) < f(x_+)$ assurdo $\implies x_+ \in [-M_0, M_0]$.

Si dimostra $x_- \in [-M_0, M_0]$ in modo analogo.

Osserviamo che $f([-M_0, M_0])$ è un insieme formato da un numero finito di elementi, perché è l'insieme

$$\#\{f(n) : -M_0 \leq n \leq M_0 \text{ e } n \in \mathbb{Z}\} \leq \#\{n \in \mathbb{Z} : |n| \leq M_0\} = 2M_0 + 1$$

$\implies \exists \; \begin{matrix} y_{max} \\ y_{min} \end{matrix} \in f([-M_0, M_0])$ valore $\begin{matrix} \text{max} \\ \text{min} \end{matrix}$

$\Rightarrow \quad x_M, x_m \in [-M_0, M_0]$ con $f(x_m) = y_{min} \quad f(x_M) = y_{max}$.

$$\Rightarrow \quad \boxed{f(x_m) \le f(x) \le f(x_M) \quad \forall\, x \in [-M_0, M_0]}$$

In particolare $f(x_m) \le f(x_-) < f(x_+) \le f(x_M)$

Più precisamente $f(x_m) \le f(x_-) \le -\varepsilon_0 < 0 < \varepsilon_0 \le f(x_+) \le f(x_M)$

Siccome per $x \notin [-M_0, M_0] \Leftrightarrow |x| > M_0 \Rightarrow -\varepsilon_0 < f(x) < \varepsilon_0$

allora $x \notin [-M_0, M_0] \Rightarrow f(x_m) \le -\varepsilon_0 < f(x) < \varepsilon_0 \le f(x_M)$

$$\Rightarrow \quad \boxed{f(x_m) \le f(x) < f(x_M) \quad \forall\, x \notin [-M_0, M_0]}$$

conclusione

$$f(x_m) \le f(x) \le f(x_M) \qquad \forall x \in \mathbb{R}$$

x_m p.t. min ass.

x_M max.

$$f(x) = \begin{cases} 1 & \text{se} \quad x \in \mathbb{Z} \cup (\mathbb{Q} \cap [0,1]) \\ 0 & \text{altrove} \end{cases}$$

$$\overline{\int_0^{10}} f(x)\,dx, \quad \underline{\int_0^{10}} f(x)\,dx$$

$$[0,10] = [0,1] \cup [1,10]$$

In $[1,10]$ $\qquad f(x) = \begin{cases} 0 & \text{se} \quad x \notin \mathbb{Z} \\ 1 & \text{se} \quad x \in \mathbb{Z} \end{cases}$

$\Longrightarrow f \in L[1,10]$ con $\int_1^{10} f(x)\,dx = 0$

$$\overline{\int_0^{10}} f(x)\,dx = \overline{\int_0^{1}} f(x)\,dx \left(+ \overline{\int_1^{10}} f(x)\,dx \right) = 0$$

$$\underline{\int_0^{10}} f(0)\,dx = \underline{\int_0^{1}} f(x)\,dx \left(+ \underline{\int_1^{10}} f(x)\,dx \right)$$

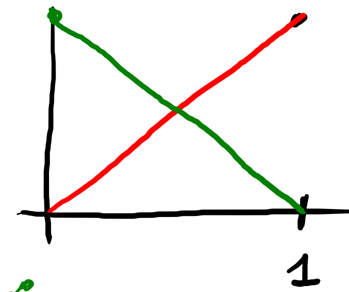
In $[0,1]$

$$f(x) = \begin{cases} 1 & \text{se} \quad x \in (\mathbb{Q} \cap [0,1]) \\ 0 & \text{altrove} \end{cases}$$

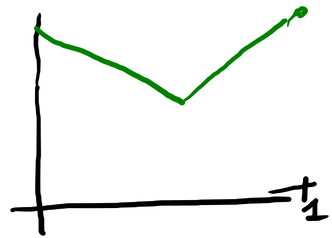
$\Rightarrow \quad f = D \quad$ in $[0,1]$ $\Rightarrow$ $\overline{\int_0^1} f\,dx = \overline{\int_0^1} D\,dx = 1$

$$\underline{\int_0^1} f = \underline{\int_0^1} D\,dx = 0$$

$$f(x) = \begin{cases} 1-x & x \in Q \\ x & x \in \mathbb{R} \setminus Q \end{cases}$$

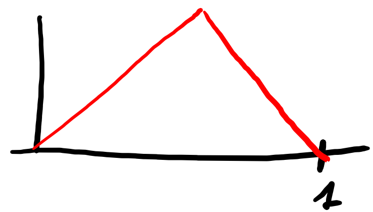


$$\overline{\int_0^1} f(x)\,dx \quad , \quad \underline{\int_0^1} f(x)\,dx$$



$$\max\{x, 1-x\}$$

$$\min\{x, 1-x\}$$



$$\underbrace{\min\{x, 1-x\}}_{m(x)} \le f(x) \le \overbrace{\max\{x, 1-x\}}^{M(x)}$$

Se considera

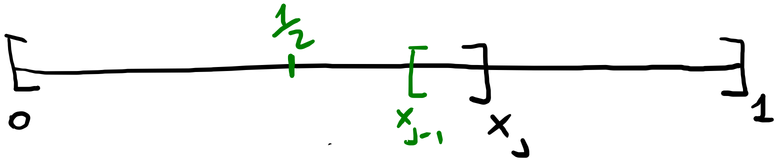
$\Delta \quad x_0 = 0 < x_1 < \cdots < x_n = 1$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup f([x_{j-1}, x_j]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup M([x_{j-1}, x_j])$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf f([x_{j-1}, x_j]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf m([x_{j-1}, x_j])$$

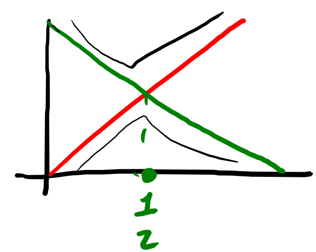
$$\overline{\int_0^1} f = \int_0^1 M \, dx$$

$$\underline{\int_0^1} f = \int_0^1 m \, dx$$



$$f(x) = \begin{cases} 1-x & \text{per } x \in \mathbb{Q} \\ x & \text{per } x \notin \mathbb{Q} \end{cases}$$

$$\sup f([x_{j-1}, x_j]) = \sup M([x_{j-1}, x_j])$$



$\frac{1}{2} \leq x_{j-1} \Rightarrow$ che per $x \in [x_{j-1}, x_j]$ ho

$$x \geq 1 - x$$

$$\sup f([x_{j-1}, x_j]) = \sup \{ f(x) : x \in [x_{j-1}, x_j] \} \geq$$

$$\geq \sup \{ x : x \in [x_{j-1}, x_j] \cap (\mathbb{R} \setminus \mathbb{Q}) \} = x_j = \sup \{ M(x) : x \in [x_{j-1}, x_j] \}$$

Dimostro che la disuguaglianza è una uguaglianza

Se per assurdo fosse $\sup f([x_{j-1}, x_j]) > x_j$

allora $\sup f([x_{j-1}, x_j]) = \sup\{f(x) : x \in [x_{j-1}, x_j] \cap \mathbb{Q}\}$

$= \sup\{1 - x : x \in [x_{j-1}, x_j] \cancel{\cap \mathbb{Q}}\} = 1 - x_{j-1} \le x_{j-1} < x_j$ assurdo

$$\int_1^{+\infty} \frac{\sin[x]}{[x]}\, dx$$

$$\int_1^{+\infty} \frac{\sin[x]}{[x]}\,dx =$$

$$\frac{\sin x}{x}$$

$$\frac{\sin[x]}{[x]} = \frac{\sin(x+[x]-x)}{[x]} = \frac{\sin x \cos([x]-x)}{[x]} + \frac{\cos x}{[x]}\sin([x]-x)$$

$$\frac{\sin x}{[x]}\cos([x]-x) = \frac{\sin x}{x}\cos(x-[x]) + \sin x \cos(x-[x])\left(\frac{1}{[x]} - \frac{1}{x}\right)$$

$$\frac{1}{[x]} - \frac{1}{x} = \frac{x-[x]}{[x]x}$$ è integrabile in $[1,+\infty)$ | ass. integrabile

$$\boxed{\int_1^{+\infty} \frac{\sin x}{x} \cos(x-[x])}$$

$$\cos(x-[x])$$

$$\int_1^{+\infty} \sin([x])$$

$$\lim_{n\to+\infty} \sum_{j=1}^{n} \frac{1}{j^p} = \lim_{n\to+\infty} \int_1^n \frac{1}{[x]^p}\,dx$$

$$\boxed{\sum_{j=1}^{\infty} \frac{1}{j^p}}$$