POSILLIPO

① diagonalizzare $A=\begin{pmatrix}1&3\\-1&1\end{pmatrix}$, $B=\begin{pmatrix}0&1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$ in $\mathbb{R}, \mathbb{C}, \mathbb{Z}_3, \mathbb{Z}_5$.

A?

- $P_A(\lambda)=\det(A-\lambda I)=\begin{vmatrix}1-\lambda&3\\-1&1-\lambda\end{vmatrix}=(1-\lambda)^2+3=\lambda^2-2\lambda+1+3=\lambda^2-2\lambda+4.$

in $\mathbb{R}$ e $\mathbb{C}$

$\Rightarrow P_A(\lambda)=0 \iff \lambda=\frac{2\pm\sqrt{4-16}}{2}=\frac{2\pm\sqrt{-12}}{2}=1\pm i\sqrt{3}$

$\Rightarrow$ in $\mathbb{R}$ non ha radici $\Rightarrow$ A è diagonalizzabile.

in $\mathbb{C}$ ha 2 radici distinte $\Rightarrow$ è diagonalizzabile e simile a $\begin{pmatrix}1+i\sqrt{3}&0\\0&1-i\sqrt{3}\end{pmatrix}$

$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)=S^{-1}AS$, con $S=M_{\mathcal{B}}^{\mathcal{B}}$, $\mathcal{B}$ la base canonica e $\mathcal{B}$ una base di autovettori. (in generale $A=M_{\mathcal{C}}^{\mathcal{C}}(L_A)$)

$\rightarrow$ B?

autovalore $1+i\sqrt{3}$

$A-(1+i\sqrt{3})I=\begin{pmatrix}-i\sqrt{3}&1\\-1&-i\sqrt{3}\end{pmatrix}\xrightarrow[R_2\to R_2-i\sqrt{3}R_1]{}\begin{pmatrix}-i\sqrt{3}&1\\0&0\end{pmatrix}$

$\Rightarrow \ker(A-(1+i\sqrt{3})I)=\{(x_1,x_2)\in\mathbb{R}^2 \mid -x_1+i\sqrt{3}x_2=0\}=$
$=\{(-i\sqrt{3}x_2,x_2) \mid x_2\in\mathbb{R}\}=\underbrace{\langle(i\sqrt{3},1)\rangle}_{\text{autovettore associato a } 1+i\sqrt{3}}$

Autovalore $1-i\sqrt{3}$:

$A-(1-i\sqrt{3})I=\begin{pmatrix}i\sqrt{3}&1\\-1&i\sqrt{3}\end{pmatrix}\xrightarrow{R_2\to R_2+i\sqrt{3}R_1}\begin{pmatrix}0&0\\-1&i\sqrt{3}\end{pmatrix}$

$\Rightarrow \ker(A-(1-i\sqrt{3})I)=\{(x_1,x_2)\in\mathbb{R}^2 \mid -x_1+i\sqrt{3}x_2=0\}=\langle(i\sqrt{3},1)\rangle$

$\Longrightarrow B=\left\{\begin{pmatrix}-i\sqrt{3}\\1\end{pmatrix},\begin{pmatrix}i\sqrt{3}\\1\end{pmatrix}\right\} \Rightarrow S=\begin{pmatrix}-i\sqrt{3}&i\sqrt{3}\\1&1\end{pmatrix}$ ✓✓✓

- in $\mathbb{Z}_3$:

$P_A(\lambda)=\lambda^2-2\lambda+4=\lambda^2 \Rightarrow$ autovalore 0, con molt. algebrica 2.

molt. geom?

$rg(A)=rg\left(\begin{pmatrix}-1&1\\-1&1\end{pmatrix}\right)\overset{\text{Gauss}}{=}rg\begin{pmatrix}1&-1\\0&0\end{pmatrix}=1 \Rightarrow m_g(0)=2-1=1.$

$\Rightarrow m_g(0)<m_a(0) \Rightarrow$ A non diagonalizzabile.

- in $\mathbb{Z}_5$:

$P_A(\lambda)=\lambda^2-2\lambda+4=\lambda^2-7\lambda-1$, poi $\forall\lambda=0,1,2,3,4 \neq P_A(\lambda)\neq 0$

$\Rightarrow$ A non è diagonalizzabile.

B?

$P_B(\lambda)=\begin{vmatrix}-\lambda&1&-1\\-1&2-\lambda&-1\\-1&-1&2-\lambda\end{vmatrix}\overset{\text{Sarrus}}{=}\begin{vmatrix}-\lambda&1&-1\\-1&2-\lambda&-1\\0&-1+\lambda&1-\lambda\end{vmatrix}\overset{C_3\to C_3+C_2}{=}\begin{vmatrix}-\lambda&1&-2\\-1&2-\lambda&-3\\0&-1&0\end{vmatrix}=$

$=(\lambda-1)\begin{vmatrix}-\lambda&-2\\-1&\lambda-3\end{vmatrix}=(\lambda-1)\cdot(-\lambda^2+3\lambda-2)=-(\lambda-1)(\lambda^2-3\lambda+2)=$

$=(\lambda-1)\cdot(\lambda-2)\cdot(\lambda-1)=(\lambda-1)^2\cdot(\lambda-2)$

$\Rightarrow$ in tutti i campi autovalori 1 e 2,

$m_a(2)=2$, $m_a(1)=m_a(2)=1$.

$m_a(1)=2$?

$rg(B-I)=rg\begin{pmatrix}-1&1&-1\\-1&1&-1\\-1&-1&1\end{pmatrix}=1$ in tutti i campi

$\Longrightarrow$ B diagonalizzabile in tutti i campi.

BASE DI AUTOVETTORI?

autovalore 1

$\ker(B-I)=\ker\left(\begin{pmatrix}-1&1&-1\\-1&1&-1\\-1&-1&1\end{pmatrix}\right)=\{(x_1,x_2,x_3) \mid x_1-x_2+x_3=0\}=$
$=\{(x_1,x_2,x_3) \mid x_1=-x_2+x_3\}=\{(-x_2+x_3,x_2,x_3) \mid x_2,x_3\in\mathbb{K}\}=$
$=\langle(-1,1,0),(1,0,1)\rangle.$

autovalore 2

$\ker(B-2I)=\ker\begin{pmatrix}-2&1&-1\\-1&0&-1\\-1&-1&0\end{pmatrix}=\{(x_1,x_2,x_3)\in\mathbb{K}^3 \mid \begin{cases}-2x_1+x_2-x_3=0\\-x_1-x_3=0\\-x_1-x_2=0\end{cases}\}=$

$=\{(x_1,x_2,x_3)\in\mathbb{K}^3 \mid -x_1=x_3=-x_2=0\}=\{(x_1,x_2,x_3)=x_1\cdot\mathbb{K}\}=\langle(1,0,1)\rangle$

$\Rightarrow \ker(B-2I)=\langle(1,1,1)\rangle \quad \forall\mathbb{K}$ ✓

$\Longrightarrow B=\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ base di $\mathbb{K}^3$ di autovettori di B $\forall\mathbb{K}$

$\Longrightarrow \forall\mathbb{K} \quad \begin{pmatrix}1&0&1\\0&1&1\\1&0&1\end{pmatrix}^{-1}B\begin{pmatrix}1&0&1\\0&1&1\\1&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$ ✓✓✓

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② $A=\begin{pmatrix}-6&7&-5\\-9&4&-2\\10&-7&8\end{pmatrix}$, $B=\begin{pmatrix}1&5\\-2&8\end{pmatrix}$ matrici reali.

Dire quale diagonalizzabile su $\mathbb{R}$, quale diagonalizzabile su $\mathbb{C}$, quale diagonalizzabile su $\mathbb{Q}$.

A?

$P_A(\lambda)=\begin{vmatrix}-6-\lambda&7&-5\\-9&4-\lambda&-2\\10&-7&8-\lambda\end{vmatrix}\overset{C_1\to C_1-2C_3}{=}\begin{vmatrix}4-\lambda&2&-1\\0&4-\lambda&-2\\2\lambda-6&3&8-\lambda\end{vmatrix}=$

$=(4-\lambda)\begin{vmatrix}4-\lambda&-2\\3&8-\lambda\end{vmatrix}+(2\lambda-6)\begin{vmatrix}2&-5\\4-\lambda&-2\end{vmatrix}=$

$=(4-\lambda)\cdot(\lambda^2-12\lambda+22+6)+(2\lambda-6)\cdot(\underbrace{-4+20}_{=16}-5\lambda)=$

$=(4-\lambda)^3-8(4-\lambda)+32\lambda-96=(4-\lambda)^3+24\lambda-96=$

$=-\lambda^3+6\lambda^2-24\lambda+56$

NON PRATICO CALCOLARE $\Longrightarrow$ P_A HA UNA DOPPIA RADICE E DUE COMPLESSE TutTe coniugate

$\Longrightarrow$ 2 autovalori con $m_a=m_g=2$. (radici multiple in campo, 5 sempre semplice in campo)

$B=\begin{pmatrix}1&5\\-2&8\end{pmatrix}$

$P_B(\lambda)=\begin{vmatrix}1-\lambda&5\\-2&8-\lambda\end{vmatrix}=\lambda^2-\lambda-10=0 \iff \lambda=\frac{1\pm\sqrt{1+40}}{2}$

$\Rightarrow$ DUE autovalori $\lambda_1=\frac{1+\sqrt{41}}{2}$, $\lambda_2=\frac{1-\sqrt{41}}{2}$; entrambi con $m_a=m_g=1$. (campo minimo, qualche cosa deve essere sempre semplice)

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③ determinare $a,b,c,d,e,f\in\mathbb{R}$, sapendo che $(1,1,1),(1,0,-1),(1,-1,0)\in\mathbb{R}^3$ sono autovettori di $A=\begin{pmatrix}1&1&1\\a&b&c\\d&e&f\end{pmatrix}$.

$\begin{pmatrix}1&1&1\\a&b&c\\d&e&f\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1&1&1\\a&b&c\\d&e&f\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}=\lambda_2\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\begin{pmatrix}1&1&1\\a&b&c\\d&e&f\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\lambda_3\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

$\begin{cases}3=\lambda_1\\a+b+c=\lambda_1=3\\d+e+f=\lambda_1=3\end{cases}$ $\begin{cases}0=\lambda_2\\a-c=0\\d-f=-\lambda_2=0\end{cases}$ $\begin{cases}0=\lambda_3\\a-b=-\lambda_3=0\\d-e=0\end{cases}$

$\Longrightarrow \begin{cases}a+b+c=3\\d+e+f=3\\a-c=0\\d-f=0\\a-b=0\\d-e=0\end{cases} \Longrightarrow \begin{cases}a=b=c=1\\d=e=f=1\end{cases}$ ✓✓✓

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④ $A\in M_n(\mathbb{K})$, $S\in GL_n(\mathbb{K})$, $B=SAS^{-1} \Rightarrow B^k=SA^kS^{-1}$?

NO DIAGONALE

$k=1 \rightarrow B=SAS^{-1}$ ✓.

$k=2 \rightarrow$?

$B^k=B\cdot B^{k-1}\underset{\text{IPOTESI}}{=}(SAS^{-1})(SA^{k-1}S^{-1})\underset{\text{ASSOCIATIVITÀ}}{=}(SA)\underbrace{(S^{-1}S)}_{=I}(A^{k-1}S^{-1})=SA^{k-1}S^{-1}=SA^kS^{-1}$ ✓✓✓.

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⑤ $A=\begin{pmatrix}-2&4&-4\\-3&4-r&-4\\3&3&-4-r\end{pmatrix}$. $r\overset{S}{=}?$ (USARE L'ESCLUSIONE TRA GLI DIVISORI)

DIAGONALIZZANDO A $\Rightarrow A^5=S^{-1}D^5S$, con D la matrice diagonalizzante.

- $P_A(\lambda)=\begin{vmatrix}-2-\lambda&4&-4\\-3&4-r&-4\\3&3&-4-r-\lambda\end{vmatrix}\underset{R_3\to R_3-R_2}{=}\begin{vmatrix}-2-\lambda&4&-4\\3&4-r&-4\\0&-1+\lambda&-r-\lambda\end{vmatrix}\underset{\text{SVILUPPO A CAMPANA}}{=}(2-\lambda)\begin{vmatrix}4-r&-4\\-1+\lambda&-r-\lambda\end{vmatrix}-3\begin{vmatrix}4&-4\\-1+\lambda&-r-\lambda\end{vmatrix}=$

$=(2-\lambda)\underbrace{(\lambda^2+\lambda-2+r+\lambda+4)}_{=\lambda^2+2\lambda+2}-3\underbrace{(-r-4-4+\lambda+\lambda)}_{=0}=-(\lambda-2)(\lambda^2+2\lambda+2)=$

$=-(\lambda-2)(\lambda+1)(\lambda+2) \Longrightarrow$ 3 autovalori distinti $\Rightarrow$ A diagonalizzabile, $D=\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-2\end{pmatrix}$

5?

- $A-2I=\begin{pmatrix}0&4&-4\\3&-1&-2\\3&3&-6\end{pmatrix}\Longrightarrow \ker(A-2I)=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{cases}\text{soluzioni di}\\4y-4z=0\\3x-y-2z=0\\3x+3y-6z=0\end{cases}\right\}\overset{\text{non servono, ridondanti}}{=}\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{cases}\text{soluzioni di}\\y-z=0\\3x-y-2z=0\end{cases}\right\}=$

$=\{(x,y,z)\in\mathbb{R}^3 \mid x=y=z\}=\langle(1,1,1)\rangle$

- $A+I=\begin{pmatrix}3&4&-4\\3&3&-2\\3&3&-3\end{pmatrix}\xrightarrow[R_3\to R_3-R_2]{R_2\to R_2-R_1}\begin{pmatrix}3&4&-4\\0&-1&2\\0&-1&1\end{pmatrix}$

$\Longrightarrow \ker(A+I)=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{cases}\text{soluzioni di}\\3x+4y-4z=0\\-y+2z=0\end{cases}\right\}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{cases}\text{soluzioni di}\\3x=0\\y=2z\end{cases}\right\}=\langle(0,2,1)\rangle.$

- $A+2I=\begin{pmatrix}4&4&-4\\3&4&-2\\3&3&-2\end{pmatrix}\xrightarrow{R_2\to R_2-3R_1}\begin{pmatrix}-2&-2&0\\3&7&-2\\3&3&-2\end{pmatrix}\xrightarrow[R_3\to R_3-3R_1]{}\begin{pmatrix}-2&-2&0\\0&1&-2\\0&3&-2\end{pmatrix}$

$\Longrightarrow \ker(A+2I)=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{cases}\text{soluzioni di}\\-2x-2y=0\\y-2z=0\\3y-2z=0\end{cases}\right\}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{cases}-2x=0\\y=0\\z=0\end{cases}\right\}=\langle(1,-1,0)\rangle.$

$\Longrightarrow S=\begin{pmatrix}1&0&1\\1&1&-1\\0&1&1\end{pmatrix}$. $\Longrightarrow S^{-2}=\begin{pmatrix}1&1&-1\\-1&-1&2\\0&-1&1\end{pmatrix}$.

$A^5=\begin{pmatrix}1&1&-1\\-1&-1&2\\0&-1&1\end{pmatrix}\begin{pmatrix}2^5&0&0\\0&-1^5&0\\0&0&-2^5\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&-1\\0&1&1\end{pmatrix}=\begin{pmatrix}1&1&-1\\-1&-1&2\\0&-1&1\end{pmatrix}\begin{pmatrix}32&&\\&-1&\\&&-32\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&-1\\1&1&0\end{pmatrix}=$

$=\begin{pmatrix}32&-1&32\\-32&1&-64\\0&1&-32\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&-1\\1&1&0\end{pmatrix}=\begin{pmatrix}63&31&31\\-95&-63&-33\\-31&-31&-1\end{pmatrix}$. ✓✓✓