FOGLIO 11

① $A: \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 4 & 6 & 0 \end{pmatrix}$ rappresentano un $\mathbb{R}$-sp.vett. $\mathbb{R}^3_{[2]}$?

Ma che $\mathbb{R}$ domanda è? Struttura tra: $A \in \mathbb{R}^{3\times 3}$ ...

$\mathbb{R}_A(\alpha) = \begin{pmatrix} 1-\alpha & 0 & 2 \\ 0 & 1-\alpha & 1 \\ 4 & 6 & -\alpha \end{pmatrix}$; $\begin{vmatrix} 1-\alpha & 0 & 2 \\ 0 & 1-\alpha & 1 \\ 4 & 6 & -\alpha \end{vmatrix} \overset{\text{Sarrus}}{=} (1-\alpha)(1-\alpha)(-\alpha) + 0 + 24 + 8(1-\alpha) - 6(1-\alpha) - 0 =$

$= (1-\alpha)[\alpha(\alpha-1) + 24 + 8 - 6] = (1-\alpha)(\alpha^2-\alpha+26) =$

$= -\alpha^3 + \alpha^2 - 26\alpha + \alpha^2 - \alpha + 26 = -(\alpha^3 - 2\alpha^2 + 27\alpha - 26)$

$= -\alpha^3 + 2\alpha^2 - 27\alpha + 26 = -(\alpha-1)(\alpha^2+\alpha-26)$, $\left(\frac{\Delta}{\text{ridotto}}\right)$

$\mathbb{R}$

$\mathbb{R}_A(\alpha) = -(\alpha-1)(\alpha^2+\alpha-26)$ ha 3 radici $\Rightarrow$ $A \in \mathbb{R}^{3\times 3}$ è $\mathbb{R}$-DIAGONALIZZABILE, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$

Vediamo se $\mathbb{R}^3 = \mathbb{R}^3_{[2]}$

- autospazio $\lambda_1 = 1$: $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 4 & 6 & -1 \end{pmatrix}$
  $A - \mathrm{id} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 4 & 6 & -1 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 4 & 6 & -1 \end{pmatrix}$
  $\longrightarrow \begin{cases} z=0 \\ 4x+6y-z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} z=0 \\ x=-\frac{3}{2}y \end{cases}$ $\quad \mathrm{Ad}(\mathrm{id}) = \left\{\left(-\frac{3}{2}y, y, 0\right) \middle| y \in \mathbb{R}\right\}$

- autospazio $\lambda_2 = 5$: $\begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 1 \\ 4 & 6 & -5 \end{pmatrix}$
  $A - 5\mathrm{id} = \begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 1 \\ 4 & 6 & -5 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{} \begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 1 \\ 0 & 6 & -3 \end{pmatrix}$
  $\longrightarrow \begin{cases} -4x+2z=0 \\ -4y+z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} z=4y \\ x=2y \end{cases}$ $\quad \mathrm{Ad}(5) = \left\{\left(2y, y, 4y\right) \middle| y \in \mathbb{R}\right\}$

- autospazio $\lambda_3 = -6$: $\begin{pmatrix} 7 & 0 & 2 \\ 0 & 7 & 1 \\ 4 & 6 & 6 \end{pmatrix}$
  $A + 6\mathrm{id} = \begin{pmatrix} 7 & 0 & 2 \\ 0 & 7 & 1 \\ 4 & 6 & 6 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{pmatrix} 7 & 0 & 2 \\ 0 & 7 & 1 \\ -3 & 6 & 4 \end{pmatrix}$
  $\longrightarrow \begin{cases} 7x+2z=0 \\ 7y+z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} z=-7y \\ x=\frac{7}{2}y \end{cases}$ $\quad \mathrm{Ad}(-6) = \left\{\left(\frac{7}{2}y, y, -7y\right) \middle| y \in \mathbb{R}\right\}$

$\Rightarrow$ è possibile base diagonalizzante di autovettori:

$V_{\mathbb{R}} = \{(-\frac{3}{2}, 1, 0), (2, 0, 4), (\frac{7}{2}, 1, -7)\}$ $\quad v_1 = (-3, 2, 0)$, $v_2 = (4, 0, 8)$, $v_3 = (7, 2, -14)$

$\Rightarrow$ $B = \{v_1, v_2, v_3\}$ base di autov.

$\overset{\text{teorema}}{\Rightarrow}$ $A = M^B_B(\mathbb{R}_A) \Rightarrow A = M^B_B(\mathbb{R}_A) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -6 \end{pmatrix} M^B_B$, con C la loro somma

$\Rightarrow$ $S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$, $\overset{\text{teorema}}{\Rightarrow}$ $S = (S^{-1})^{-1} = \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$\mathbb{Z}_2$

$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$; $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$

$P_A(\alpha) = -(\lambda+\alpha)(\lambda+\alpha)(\lambda-\alpha) = -\lambda(\lambda+\alpha)^2$

$\Rightarrow$ autovalori $\begin{cases} 0 & \text{con } m_a(0)=1 \\ \alpha & \text{con } m_a(\alpha)=2 \end{cases} \Rightarrow \begin{matrix} m_a(0)+m_a(\alpha)=3 \\ \dim(\mathbb{Z}_2^3)=3 \end{matrix}$

vediamo anche molteplicità di A con $A\mathbb{e}_1 = \mathbb{e}_1$, $A\mathbb{e}_2 = \alpha\mathbb{e}_2 \Rightarrow m_a(1)=1$ ✓

inoltre, $A\mathbb{e}_3 = A\mathbb{e}_1 + \alpha\mathbb{e}_2 \Rightarrow A(\mathbb{e}_1 - \mathbb{e}_3) = \mathbb{0}$

$\Rightarrow$ A è diagonalizzabile e simile a $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$;

trova base di autovettori: $B = \{\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_1 - \mathbb{e}_3\}$

$\Rightarrow$ $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$; $\quad S^{-1} = M^{\mathcal{E}}_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ $\Rightarrow$ C è la loro somma.

$\overset{\text{conti}}{\Rightarrow}$ $S = M^B_{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$.

$\mathbb{Z}_3$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $P_A(\lambda) = -(\alpha+\lambda)\left(\lambda^2\right)(\alpha-\lambda) = -\lambda^2$

$\Rightarrow$ unica zero doppio 0 con $m_a(0)=3 \Rightarrow$ $\left[\ A \text{ diagonalizzabile} \iff A = 0\ \right]$.

$\Rightarrow$ A non è diagonalizzabile. ///

② $A: \begin{pmatrix} 2 & 1 & -1 \\ 3 & -1 & 0 \end{pmatrix}$ calcolare A^n $\forall n \in \mathbb{Z}$ $\left(\overset{\text{induzione}}{\text{(facoltativo)}}\right)$

$P_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 3 & -1-\lambda & 0 \end{vmatrix} = -(2-\lambda)(1+\lambda) + 3 = -2 - \lambda + \lambda^2 + 3 = \lambda^2 + \lambda + 1$

$\Rightarrow$ autovalori $\begin{cases} \alpha & \text{con } m_a(\alpha)=1 \\ \beta & \text{con } m_a(\beta)=1 \end{cases} \Rightarrow m_a(\alpha)+m_a(\beta)=2$

$\Rightarrow$ A è diagonalizzabile e simile a $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

$\overset{\text{teorema}}{\Rightarrow}$ $A = M^{\mathcal{E}}_{\mathcal{E}}(\mathbb{R}_A) = M^{\mathcal{E}}_B \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} M^B_{\mathcal{E}}$ $\quad \overset{\text{conti}}{\Rightarrow}$ $M^{\mathcal{E}}_B = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}$, $M^B_{\mathcal{E}} = \begin{pmatrix} \beta & -1 \\ -\alpha & 1 \end{pmatrix} \frac{1}{\beta - \alpha}$

$\Rightarrow$ $A^n = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} \begin{pmatrix} \beta & -1 \\ -\alpha & 1 \end{pmatrix} \frac{1}{\beta - \alpha} =$

$= \begin{pmatrix} \beta - \alpha^{n+1} & -1 + \alpha^n \\ \alpha\beta - \alpha\beta^n & -\alpha + \beta^n \end{pmatrix} \frac{1}{\beta - \alpha} = \begin{pmatrix} \frac{\beta - \alpha^{n+1}}{\beta - \alpha} & \frac{-1 + \alpha^n}{\beta - \alpha} \\ \alpha\beta \frac{1 - \beta^{n-1}}{\beta - \alpha} & \frac{-\alpha + \beta^n}{\beta - \alpha} \end{pmatrix}$. ///

③ $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ diagonalizzabile su $\mathbb{R}, \mathbb{C}, \mathbb{Z}_2, \mathbb{Z}_3$?

Vediamo con "trucco a mano" e successiva $\mathbb{F}$-diagonalizzabilità condizionata:

$\mathbb{R}_B(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} \overset{\text{Laplace}}{=} \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix} =$

$= (1-\lambda)\left(-\lambda + \lambda^2 + 0\right) = (1-\lambda)\lambda(\lambda-1) = \lambda(\lambda-1)^2$

$\Rightarrow$ $\alpha = 0$ $\mathbb{R}_B(\alpha) = \alpha(\alpha-1)^2 = 0$ con $\alpha = 0, 1$ $\Rightarrow$ $\begin{matrix} \text{possibile per } \mathbb{R} \\ \text{e } \mathbb{Z}_2 \end{matrix}$

$\alpha = 0$: $B\mathbb{e}_2 = \mathbb{0}$ $\Rightarrow$ $m_a(0) \geq 1$ $\Rightarrow$ $\begin{matrix} \text{possibile per } \mathbb{Z}_3 \\ \text{e } \mathbb{C} \end{matrix}$

$\alpha = 1$: $B\mathbb{e}_1 = \mathbb{e}_1$ $\Rightarrow$ $m_a(1) \geq 1$ $\Rightarrow$ $\begin{matrix} \text{possibile per } \mathbb{Z}_3 \\ \text{e } \mathbb{C} \end{matrix}$

$\Rightarrow$ $\mathbb{R}$ non è diagonalizzabile, $\mathbb{C}$ è algebricamente chiuso, $\mathbb{Z}_2$ non è diagonalizzabile, $\mathbb{Z}_3$ è diagonalizzabile.

$\Rightarrow$ B è $\mathbb{Z}_3$-diagonalizzabile, $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $S = M^{\mathcal{E}}_B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S^{-1} = M^B_{\mathcal{E}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ ///

ESERCIZIO

Base su autovettori?

- $\mathrm{Ad}(\mathrm{id}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ogni vettore va in $\mathbb{0}$

$B - \mathrm{id} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{} \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$\longrightarrow \begin{cases} x+2y=0 \\ -y=0 \\ y+z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$ $\quad \mathrm{Ad}(1) = \left\{(0,0,0) \middle| \begin{matrix} \text{non ci} \\ \text{interessa} \end{matrix}\right\}$

- $\mathrm{Ad}(1)$

$B - 1\mathrm{id} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{matrix} x=-1-z \\ y=0 \\ z=z \end{matrix}$

$\longrightarrow \begin{cases} x+y+z=-1 \\ 0=0 \\ 0=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} x=-1-z \\ y=0 \\ z=z \end{cases}$ $\quad \mathrm{Ad}(1) = \left\{(-1-z, 0, z) \middle| \begin{matrix} z \in \mathbb{R} \\ \text{qui serve} \end{matrix}\right\}$

- $\mathrm{Ad}(-1)$

$B + 1\mathrm{id} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{matrix} x=-z \\ y=0 \\ z=z \end{matrix}$

$\longrightarrow \begin{cases} x+y+z=0 \\ y=0 \\ 0=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} x=-z \\ y=0 \\ z=z \end{cases}$ $\quad \mathrm{Ad}(-1) = \left\{(-z, 0, z) \middle| \begin{matrix} z \in \mathbb{R} \\ \text{serve} \end{matrix}\right\}$

ESERCIZIO

$\mathbb{R}$-spazio vettoriale $\mathbb{R}^{2\times 2}_{[2]}$, endom. $B: S^{-1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) S = S^2\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$

Chiaramente stesso Δ visto ($B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ e $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$)

ESERCIZIO

Autovalori: $1, 2, -2$. Base di autovettori?

- $\mathrm{Ad}(1)$

$B - \mathrm{id} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{matrix} x=-y+z \\ y=y \\ z=z \end{matrix}$

$\longrightarrow \begin{cases} x+y-z=0 \\ 0=0 \\ 0=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} x=-y+z \\ y=y \\ z=z \end{cases}$ $\quad \mathrm{Ad}(1) = \left\{(-y+z, y, z) \middle| \begin{matrix} y, z \in \mathbb{R} \\ \text{non serve} \end{matrix}\right\}$

- $\mathrm{Ad}(2)$

$B - 2\mathrm{id} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 2 & -3 \end{pmatrix} \xrightarrow[R_2 \to R_2 + R_1]{} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 2 & -3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{pmatrix}$

$\longrightarrow \begin{cases} -x+y-z=0 \\ 2y-3z=0 \\ -2z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$ $\quad \mathrm{Ad}(2) = \left\{(0,0,0) \middle| \begin{matrix} \text{non ci} \\ \text{interessa} \end{matrix}\right\}$

- $\mathrm{Ad}(-2)$

$B + 2\mathrm{id} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -1 \end{pmatrix}$

$\longrightarrow \begin{cases} z=0 \\ x+3y-z=0 \\ 3x+y-z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} z=0 \\ x=-y \\ 3x+y-z=0 \end{cases}$ $\quad \mathrm{Ad}(-2) = \left\{(-y, y, 0) \middle| \begin{matrix} y \in \mathbb{R} \\ \text{serve} \end{matrix}\right\}$

④ $b: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, con $b((x,y,z),(x',y',z')) = 2xx' + 3yy' + 5zz' + xy' + yx' + xz' + zx' + yz' + zy'$

b bilineare simmetrica? Matrice di b rispetto a $\mathcal{E}$? Segnatura b, $\mathrm{rad}(b)$?

b bilineare ok $\Rightarrow$ $\begin{cases} b((x,y),(x',y')) = xx' + yy' & \text{quadrp non canonica: Sono} \\ b((x,y),(x',y')) = -x'x & \text{coefficienti VALIDI} \\ & \text{(visto con un sol. Binary} \\ & \text{con Matrice Diagonale non nulla)} \end{cases}$

- Scrivendo la matrice $M_{\mathcal{E}}(b)$, b presa su una M come

$M = \begin{pmatrix} b(\mathbb{e}_1,\mathbb{e}_1) & b(\mathbb{e}_1,\mathbb{e}_2) & b(\mathbb{e}_1,\mathbb{e}_3) \\ b(\mathbb{e}_2,\mathbb{e}_1) & b(\mathbb{e}_2,\mathbb{e}_2) & b(\mathbb{e}_2,\mathbb{e}_3) \\ b(\mathbb{e}_3,\mathbb{e}_1) & b(\mathbb{e}_3,\mathbb{e}_2) & b(\mathbb{e}_3,\mathbb{e}_3) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$ è SIMMETRICA ✓.

- $\mathrm{rango}(b) = \mathrm{rango}(M) = 3$. $\overset{\text{quindi}}{\text{invertibile}}$: $b(\mathbb{e}_3,\mathbb{e}_3) = (\mathbb{e}_3 \mathbb{e}_3 \mathbb{e}_3) \cdot M \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- Autovalori di M?

$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 5-\lambda \end{vmatrix} \overset{\text{Laplace}}{=} -\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1 \end{vmatrix} + (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} =$

$= -\lambda + \lambda^2 + (1-\lambda)\left(\lambda^2 - 5\lambda + 5\right) - 1 = \lambda^2 + \lambda^3 - 5\lambda^2 + 5\lambda - \lambda^2 + 5\lambda - 2 =$

$= -\lambda^3 + 9\lambda - 2$ $\quad \overset{\text{Ruffini}}{\Rightarrow}$ $\lambda = -3$ è zero $\Rightarrow$ $-\lambda^3 + 9\lambda - 2 = (\lambda+3)(-\lambda^2 + 3\lambda - 1)$

$\Rightarrow$ autovalori $\begin{cases} -3 & \\ \frac{3+\sqrt{5}}{2} & \\ \frac{3-\sqrt{5}}{2} & \end{cases}$ $\quad \begin{matrix} \text{segnatura} \\ (3,0,0) \end{matrix}$

⑤ C forma bilineare su $\mathbb{R}^3$ con matrice associata alla base canonica $C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C simmetrica? Non serve? $\overset{\text{perché}}{\text{sempre}}$ Matrice associata a $\mathcal{B} = \left\{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$?

$\mathrm{car}(\mathcal{B}(v_1,v_2))$? Base ortonormale per $\mathbb{E}^3$?

- C simmetrica $\Rightarrow$ C semplice ✓.

- Autovalori di C?

$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \overset{\text{Laplace}}{=} (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left(\lambda^2 - 4\lambda + 3\right) = (1-\lambda)(\lambda-3)(\lambda-1)$

$\Rightarrow$ autovalori strettamente positivi $\Rightarrow$ C definita positiva ✓.

$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 1$

- C associata a $\mathcal{B}$ è $C_{\mathcal{B}} = \mathrm{car}(\mathcal{B}(v_i,v_j))_{ij=1,2,3}$

in regione $C(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x^2 + y^2 + z^2 - 2xy$

$\Rightarrow$ $C(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- $\mathrm{car}\left(\mathcal{B}(v_1,v_2)\right) = \dfrac{|\langle v_1, v_2 \rangle_C|}{\|v_1\|_C \cdot \|v_2\|_C} = \dfrac{|\langle v_1, v_2 \rangle_C|}{\sqrt{\langle v_1, v_1 \rangle_C}\sqrt{\langle v_2, v_2 \rangle_C}} = \dfrac{|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} C \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}|}{\sqrt{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}\sqrt{\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} C \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}} = \dfrac{|\begin{pmatrix} 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}|}{\sqrt{2}\sqrt{4}} = \dfrac{4}{2\sqrt{2}} = \dfrac{2}{\sqrt{2}}$.

- $\mathcal{E}^C_2 = \left\{ v \in \mathbb{R}^3 \middle| C\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \mathbb{0} \right\} = \left\{ v = (x,y,z) \in \mathbb{R}^3 \middle| 2x^2 + y^2 + z^2 - 2xy = 0 \right\} = \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-y)^2 + y^2 + z^2 = 0 \right\}$

$\Rightarrow$ $\mathcal{E}^C_2 = \mathcal{L}\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right) \cap \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)^{\perp_{\mathbb{E}^3}}$ $\quad \overset{\text{Gram-Schmidt}}{\text{già fatto}}$ $\rightarrow$ $\mathcal{B}$ è anche ortonormale.

$\Rightarrow$ $f^C_{\mathcal{E}} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$, $\quad f_1 = \dfrac{(1,0,0)}{\sqrt{\langle (1,0,0), (1,0,0) \rangle_C}} = \dfrac{(1,0,0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, 0\right)$

$f_2 = \dfrac{(0,1,0) - \langle (0,1,0), f_1 \rangle_C f_1}{\|(0,1,0) - \langle (0,1,0), f_1 \rangle_C f_1\|_C} = \dfrac{(0,1,0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, 0\right)}{\|\left(-\frac{1}{2}, 1, 0\right)\|_C} = \dfrac{\left(-\frac{1}{2}, 1, 0\right)}{\sqrt{\frac{1}{2} + 2 + 2 - 1}} = \dfrac{\left(-\frac{1}{2}, 1, 0\right)}{\sqrt{\frac{7}{2}}} = \left(-\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, 0\right)$

$f_3 = \dfrac{(0,0,1) - \langle (0,0,1), f_1 \rangle_C f_1 - \langle (0,0,1), f_2 \rangle_C f_2}{\|(0,0,1) - \langle (0,0,1), f_1 \rangle_C f_1 - \langle (0,0,1), f_2 \rangle_C f_2\|_C} = (0,0,1) - 3\sqrt{2}\left(\frac{1}{\sqrt{2}}, 0, 0\right) - \frac{1}{\sqrt{7}}\left(-\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, 0\right) = \dots = \frac{f_2}{\|f_2\|}$

$\Rightarrow$ $\left\{ f_1, f_2^C \right\}$ base ortonormale di $\mathcal{E}^C_2$

⑥ Determinare una base ortonormale di $\mathbb{R}^3$ rispetto alla quale è diagonale.

$\left\{ f^C_{\mathcal{E}}, f^C_{\mathcal{E}} \right\}$ base ort di $\mathcal{E}^C_2$ $\Rightarrow$ $\left\{ f_1, f_2, f_3 \right\}$ base ortonormale di $\mathbb{R}^3$

$M_{\mathcal{B}}(C) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \overset{\text{ovvio}}{\Rightarrow}$ $\left\{ f_1, f_2, f_3 \right\}$ base ort di $\mathbb{R}^3$ ///

⑦ $W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$. Determinare basi ort ogni $W \subseteq W^{\perp}$.

Troviamo $W^{\perp}$ usando prodotto scalare standard. $\left(\begin{matrix} \Psi((x_1,y_1,z_1),(x_2,y_2,z_2)) \\ = \text{quello} \cdot \text{visto} \cdot \text{quando} \cdot \text{abbiamo} \cdot \text{un} \cdot \text{prodotto} \cdot \text{scalare} \cdot \text{su} \cdot \mathbb{R}^3 \end{matrix} \right)$

$W^{\perp} = \left\{ v \in \mathbb{R}^3 \middle| \forall w \in W, \langle v, w \rangle_{\text{st}} = 0 \right\} = \left\{ v \in \mathbb{R}^3 \middle| \begin{matrix} \langle v, w_1 \rangle_{\text{st}} = 0, \\ \langle v, w_2 \rangle_{\text{st}} = 0 \end{matrix} \right\}$

$v \in \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \overset{\text{ortog.}}{\perp} \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \overset{\text{ortog.}}{\perp} \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle^{\perp_{\text{st}}} \overset{\text{def}}{=} \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \cdot \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} x+y=0 \\ x+z=0 \end{matrix} \right\} \overset{\text{risolvo}}{=} \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$\Rightarrow$ $\begin{cases} x+y=0 \\ x+z=0 \\ -x+y+z=0 \end{cases} \xrightarrow{\text{risolvo}} \begin{cases} x=-y \\ x=-z \\ -x+y+z=0 \end{cases}$ $\quad W^{\perp} = \left\langle (-1,1,1), (1,-1,0), (1,0,-1) \right\rangle$.