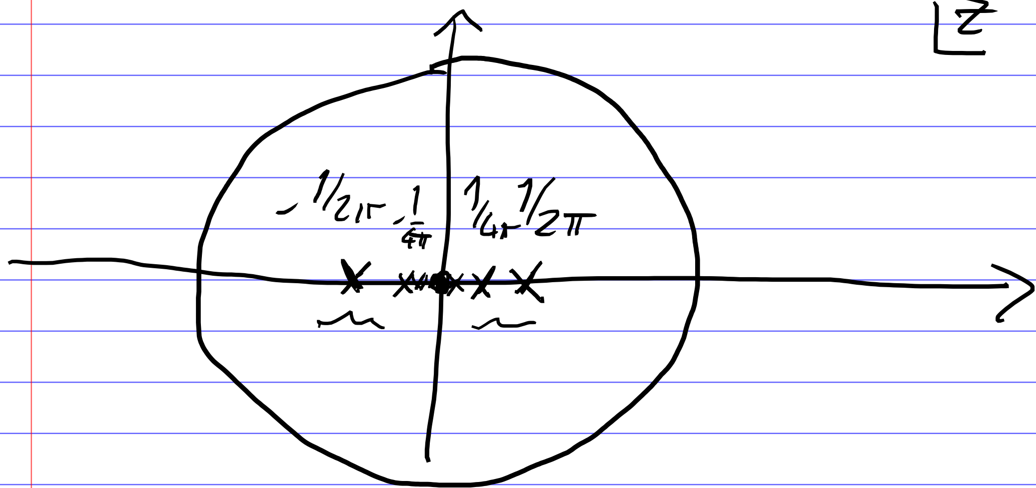
6. $$\int_{\gamma(0,1)} dz \, \frac{z+1}{z^4\underbrace{\left(1-\cos\frac{1}{z}\right)}}$$

<u>Singolarità</u>: * $\cos\frac{1}{z}$ singolarità essenziale in $z=0$

$\Rightarrow z=0$ singolarità essenziale (?) ✗

* $\left[\cos\frac{1}{z}=1\right] \rightsquigarrow \frac{1}{z_K}=2\pi K, \; K\in\mathbb{Z}$

$z_K=\frac{1}{2\pi K}$ infiniti poli di ordine 1 nei punti z_K



z_k è una successione di punti che si accumulano nell'origine

$\Rightarrow$ $z=0$ NON è una singolarità isolata

punto di accumulazione di poli

Teorema esterno dei residui:

$$\int = -2\pi i \underbrace{\mathrm{Res}_{\frac{z+1}{z^4\left(1-\cos\frac{1}{z}\right)}}(z=\infty)}$$

$\mathrm{Res}_f(z=\infty)$: dobbiamo espandere f per grandi z in potenze di $\frac{1}{z}$, e prendere il coefficiente di $\left(\frac{1}{z}\right)^1$.

Equivalentemente: $g(w) = f\left(\frac{1}{w}\right)$
e possiamo espandere g in $w=0$ e prendere coefficiente di w^1.

Non confondersi: $\boxed{\mathrm{Res}_f(z=\infty) \neq \mathrm{Res}_g(w=0)}$

$\frac{1}{z}$ per z grande $\neq$ $\frac{1}{w}$ per w piccolo

w

$$\frac{z+1}{z^4\left(1-\cos\frac{1}{z}\right)} = \frac{z+1}{z^4\left(1-\left(1-\frac{1}{2}\frac{1}{z^2}+O\left(\frac{1}{z^4}\right)\right)\right)}$$

$$= \frac{z+1}{z^4\left(\frac{1}{2}\frac{1}{z^2}+O\left(\frac{1}{z^4}\right)\right)} = \frac{z+1}{\frac{1}{2}z^2\left(1+O\left(\frac{1}{z^2}\right)\right)}$$

$$= \frac{2(z+1)}{z^2}\left(1+O\left(\frac{1}{z^2}\right)\right) = \left(\frac{2}{z}+\frac{2}{z^2}\right)\left(1+O\left(\frac{1}{z^2}\right)\right)$$

$$= \frac{2}{z}\left(1+\frac{1}{z}+O\left(\frac{1}{z^2}\right)\right) \qquad \mathrm{Res}_f(z=\infty) = -2$$

$$\Longrightarrow \int = +4\pi i .$$

Se $f(z) \xrightarrow[z\to\infty]{} 0$ come in questo caso

possiamo calcolare il residuo:

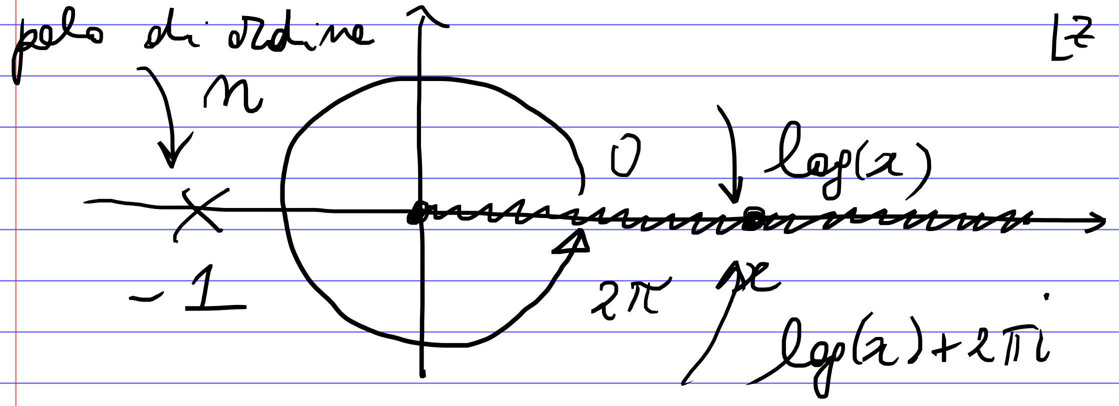
$$\mathrm{Res}_f(z=\infty) = -\lim_{z\to\infty} \left(z f(z) \right)$$

12. $\displaystyle\int_0^\infty dx \frac{x^\alpha}{(1+x)^n}$ n intero positivo

$\alpha \in \mathbb{R}$, $\boxed{-1 < \alpha < n-1}$

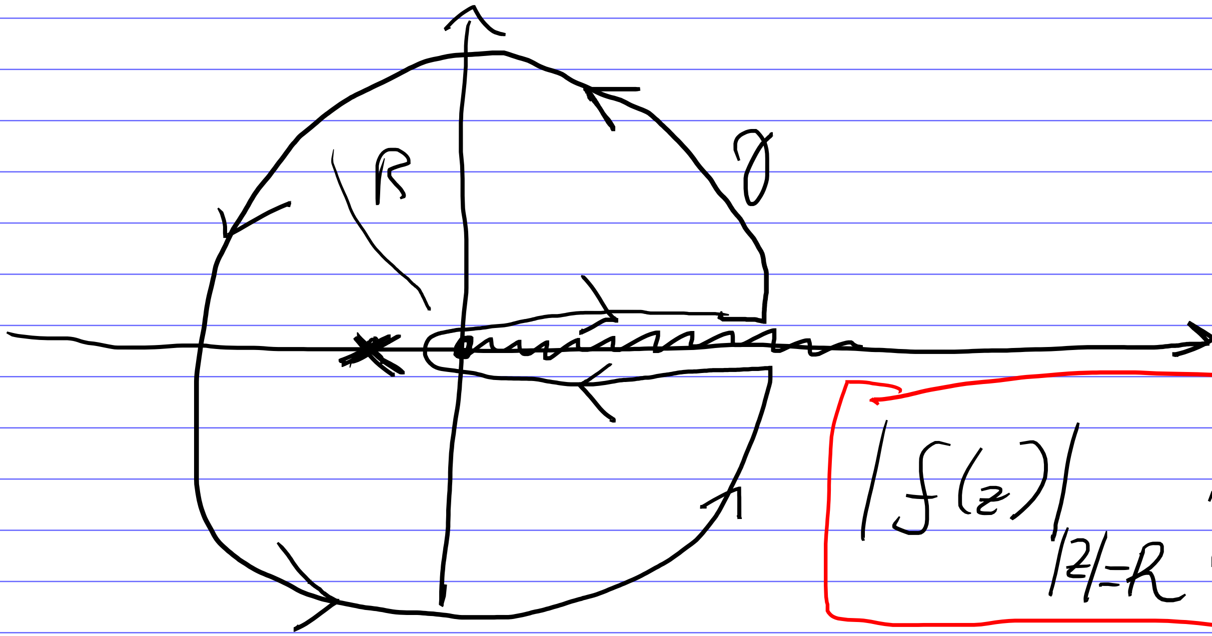
$$f(z) = \frac{z^\alpha}{(1+z)^n} = \frac{e^{\alpha \log z}}{(1+z)^n}$$

$z = 0$ punto di diramazione



$$f(z) \text{ sopra il taglio: } \quad \frac{e^{\alpha \log(x)}}{(1+x)^n} = \frac{x^\alpha}{(1+x)^n}$$

$$f(z) \text{ sotto il taglio: } \quad \frac{e^{\alpha(\log(x)+2\pi i)}}{(1+x)^n} = e^{2\pi i \alpha} \times \frac{x^\alpha}{(1+x)^n}$$



$$\boxed{\left|f(z)\right|_{|z|=R} \underset{R\gg 1}{\sim} R^{\alpha-n} < \frac{1}{R}}$$

$$\int_\gamma dz\, f(z) = 2\pi i\, \mathrm{Res}_f(z=-1)$$

$$\llcorner = \left[\int_{\gamma_R} dz + \int_0^R dx + \int_R^0 dx\right] f(z)$$

$$= \int_0^\infty dx\, f^{\text{sopra}}(x) + \int_\infty^0 dx\, f^{\text{sotto}}(x)$$

$$= \left(1 - e^{2\pi i \alpha}\right) \underbrace{\left[\int_0^\infty dx \frac{x^\alpha}{(1+x)^n}\right]}_{I}$$

cambio di estremi di integrazione

$$\left(1 - e^{2\pi i \alpha}\right) I = 2\pi i \, \mathrm{Res}_f (z=-1)$$

$$= 2\pi i \frac{1}{(n-1)!} \left(\frac{d^{n-1}}{dz^{n-1}} z^\alpha \right) \Bigg|_{z=-1}$$

$$z = -1 + \delta$$

$$z^{\alpha} = (-1+\delta)^{\alpha} = e^{i\pi\alpha}(1-\delta)^{\alpha}.$$

$$(-1)^{\alpha} = e^{i\pi\alpha}$$

$$\frac{d^{n-1}}{dz^{n-1}} z^{\alpha}\Big|_{z=-1} = \underline{\underline{e^{i\pi\alpha}}} \left[\frac{d^{n-1}}{d\delta^{n-1}} (1-\delta)^{\alpha} \Big|_{\delta=0} \right]$$

$$(1-\delta)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^{k} \delta^{k}, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

$$\left[\frac{1}{(n-1)!} \frac{d^{n-1}}{d\delta^{n-1}} (1-\delta)^{\alpha} \Big|_{\delta=0} = \binom{\alpha}{n-1} (-1)^{n-1} \right]$$

$$\left(1 - e^{2\pi i \alpha}\right) I = 2\pi i \, e^{\pi i \alpha} \binom{\alpha}{n-1} (-1)^{n-1}$$

$$I = \frac{2\pi i \, e^{\pi i \alpha}}{1 - e^{2\pi i \alpha}} \binom{\alpha}{n-1} (-1)^{n-1}$$

$$= \frac{\pi (2i)}{e^{-\pi i \alpha} - e^{\pi i \alpha}} \binom{\alpha}{n-1} (-1)^{n-1} \longrightarrow -\frac{1}{\sin(\pi\alpha)}$$

$$= \frac{\pi}{\sin(\pi\alpha)} (-1)^{n} \binom{\alpha}{n-1}$$

$n=1$: $\boxed{-\dfrac{\pi}{\sin(\pi\alpha)}}$

$n=2$: $\boxed{\dfrac{\pi\alpha}{\sin(\pi\alpha)}}$

7. $$\frac{\partial}{\partial t}F(t,x) = D\frac{\partial^2}{\partial x^2}F(t,x)$$

$$\left[\hat{F}(t,k) = \int dx\, F(t,x)\, e^{ikx}\right]$$

$$\frac{\partial}{\partial t}\hat{F}(t,k) = -Dk^2\hat{F}(t,k)$$

$$\frac{\partial}{\partial x} \rightsquigarrow -ik$$

$$= -k^2 D\int dx\, F e^{ikx}$$

$$\int dx\, \frac{\partial}{\partial t}F(t,x)\, e^{ikx} = \int dx\, D\frac{\partial^2 F}{\partial x^2}e^{ikx}$$

$$\partial_t \hat{F}(t,k) = -Dk^2 \hat{F}(t,k) \quad *$$

$$F(t=0, x) = f(x)$$

$$\Rightarrow \quad \hat{F}(t=0, k) = \hat{f}(k) \quad *$$

$$\boxed{\hat{F}(t,k) = e^{-Dk^2 t}\,\hat{f}(k)}$$

$$\hat{F}(t,k) = \underbrace{e^{-Dk^2 t}}\ \underbrace{\hat{f}(k)}$$

$$\begin{cases} F(t,x) = \int dx' \boxed{G(t,x-x')} f(x') \\ \hat{G}(t,k) = e^{-Dk^2 t} \end{cases}$$

$$G(t,x) = \int \frac{dk}{2\pi} e^{-Dk^2 t} e^{-ikx}$$

$$\int \frac{dk}{2\pi}\, e^{-Dt\underbrace{\left(k^2 + \frac{ikx}{Dt}\right)}_{\left[\left(k + \frac{ix}{2Dt}\right)^2 + \frac{x^2}{4D^2t^2}\right]}}$$

$$= \underbrace{\left(\int \frac{dk}{2\pi}\, e^{-Dt\left(k + \frac{ix}{2Dt}\right)^2}\right)}_{\frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}} \;=\; \frac{1}{2\sqrt{\pi D t}}} e^{-\frac{x^2}{4Dt}}$$

$$= \left[\frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}\right]$$

8./9. $\dfrac{1}{\sqrt{|x|}}$

$\in L^1(\mathbb{R})$? : NO $\quad \displaystyle\int dx \, \frac{1}{\sqrt{|x|}}$ diverge a ∞

$\in L^2(\mathbb{R})$? : NO $\quad \displaystyle\int dx \, \frac{1}{|x|}$ diverge sia in $x=0$, in $x=\infty$

$\varphi \in \mathcal{S}(\mathbb{R})$ funzione a decrescenza rapida

$$\int dx \, \frac{1}{\sqrt{|x|}} \varphi(x) < \infty$$

$T_{\frac{1}{\sqrt{|x|}}}$ è una distribuzione ben definita

Possiamo definire $\mathcal{F}$ nel senso delle distribuzioni:

$$\left[\mathcal{F}\left(\frac{1}{\sqrt{|x|}}\right)(k) = \frac{C}{\sqrt{|k|}}\right] \rightsquigarrow \frac{C^2}{\sqrt{|y|}} = \frac{2\pi}{\sqrt{|y|}}$$

$C = ?$ $\qquad \mathcal{F}^2 = 2\pi R$

$$\Rightarrow C^2 = 2\pi, \; C = \sqrt{2\pi}$$

Per calcolare $\mathcal{F}$ nel senso della distrib. troviamo famiglie di funzioni in L^1 che converge alla nostra distrib. (nel senso delle distribuzioni) e commutiamo $\mathcal{F}$ con il limite. Vale perché $\mathcal{F}$ è continua rispetto al limite nel senso delle distribuzioni.

$$\frac{1}{\sqrt{|x|}} = \lim_{\varepsilon \to 0^+} \underbrace{e^{-\varepsilon |x|} \frac{1}{\sqrt{|x|}}}_{L^1(\mathbb{R})}$$

$$\mathcal{F}\left(\frac{1}{\sqrt{|x|}}\right)(k) = \lim_{\varepsilon \to 0^+} \boxed{\mathcal{F}\left(e^{-\varepsilon |x|} \frac{1}{\sqrt{|x|}}\right)(k)}$$

$$\int_{-\infty}^{+\infty} dx \, e^{-\varepsilon|x|} \frac{1}{\sqrt{|x|}} e^{ikx}$$

$$= \int_0^{\infty} dx \, e^{-(\varepsilon - ik)x} \frac{1}{\sqrt{x}} + \int_{-\infty}^{0} dx \, e^{(\varepsilon + ik)x} \frac{1}{\sqrt{-x}}$$

$x = s^2$, $ds = \frac{dx}{2\sqrt{x}}$ $\qquad$ $x = -s^2$, $ds = -\frac{dx}{2\sqrt{-x}}$

$$= 2\int_0^{\infty} ds \, e^{-(\varepsilon - ik)s^2} + 2\int_0^{\infty} ds \, e^{-(\varepsilon + ik)s^2}$$

$$dx = 2s\,ds = 2\sqrt{|x|}\,ds$$

$$= \frac{\sqrt{\pi}}{\sqrt{\varepsilon - ik}} + \frac{\sqrt{\pi}}{\sqrt{\varepsilon + ik}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{k + i\varepsilon}} \frac{1}{\sqrt{-i}} + \frac{\sqrt{\pi}}{\sqrt{k - i\varepsilon}} \frac{1}{\sqrt{i}}$$

$$\left[\sqrt{-i} = e^{-i\frac{\pi}{4}} \,,\; \sqrt{i} = e^{i\frac{\pi}{4}} \right]$$

$$= \sqrt{\frac{\pi}{k}} \left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}} \right)$$

$\varepsilon \to 0$

$$= \sqrt{\frac{\pi}{k}}\, 2 \cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{k}} \sqrt{2} = \sqrt{\frac{2\pi}{k}} \checkmark$$

$$2\int_0^\infty dx \, \frac{1}{\sqrt{|x|}} \left(\frac{e^{ikx} + e^{-ikx}}{2} \right)$$

$$\int_0^\infty dx \, \frac{1}{\sqrt{|x|}} \cos(kx)$$