3/13 Trovare $S \in SO(2)$ t.c. ${}^tSAS = D$ diagonale con $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\boxed{{}^tS = S^{-1}}$$

${}^t\bar{S}AS$ (congruente compl.)

${}^tSAS$ (congruente) su $\mathbb{C}$

Autovalori di A $\quad P_A = \det(A - x I_2) = \begin{vmatrix} 1-x & 2 \\ 2 & 1-x \end{vmatrix} =$

$$= (1-x)^2 - 4 = (1-x+2)(1-x-2) = (3-x)(-1-x) = (x+1)(x-3)$$

$x_1 = -1, \quad x_2 = 3 \qquad \boxed{P_A(x) = 0}$

$m_a(-1) = 1, \quad m_a(3) = 1 \quad \Rightarrow \quad m_g(-1) = m_g(3) = 1$

$\boxed{x_1 = -1}$ $\qquad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x + y = 0$

$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \|v_1\| = 1$

$\boxed{\lambda_2 = 3}$ $\quad \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -x + y = 0 \qquad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \|v_2\| = 1$

(v_1, v_2) base ortonormale di $\mathbb{R}^2$

$$\frac{1}{2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 \qquad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$S^{-1} = {}^tS = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\boxed{{}^tS A S = D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$\boxed{4/13}$ Trovare $T \in SO(3)$ t.c. ${}^tT\,B\,T = D$ diag. con $B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$$\boxed{{}^tT = T^{-1}}$$

Autovalori

$$P_B = \det \begin{pmatrix} -1-x & 1 & 1 \\ 1 & -1-x & 1 \\ 1 & 1 & -1-x \end{pmatrix} = -(1+x) \begin{vmatrix} -1-x & 1 \\ 1 & -1-x \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 1 & -1-x \end{vmatrix} +$$

$$+ \cancel{\begin{vmatrix} 1 & -1-x \\ 1 & 1 \end{vmatrix}} =$$

$$= -(1+x)\left(\underline{(1+x)^2 - 1}\right) - 2(-1-x-1) =$$

$$= -(1+x)\left((2+x)x\right) + 2\,\underline{(2+x)} = (2+x)\left(-x-x^2+2\right) =$$

$$= -(x+2)\left(x^2+x-2\right) = -(x+2)(x-1)(x+2) = \underline{-(x+2)^2(x-1)}$$

$$P_B = -(x+2)^2(x-1)$$

$$\boxed{1 \le m_g(\lambda) \le m_a(\lambda)}$$

$$\lambda_1 = -2 \quad , \quad \lambda_2 = 1$$

$$\underline{m_a(-2) = 2} \quad , \quad m_a(1) = 1$$

$$m_g(1) = 1$$

$$m_g(-2) = 2$$

$$\boxed{\lambda_1 = -2}$$

$$\underline{(B + 2I_3)} X = 0 \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x + y + z = 0 \qquad \begin{cases} x = -u - t \\ y = u \\ z = t \end{cases} \quad u, t \in \mathbb{R}$$

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad (u,t) = (1,0) \quad , \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (u,t) = (0,1)$$

$$\underline{V_{-2}} = \mathrm{Span}(v_1, v_2)$$

Gram - Schmidt $\quad v_1, v_2$ $\qquad \|t_1\|^2 = 2 \qquad \langle v_1, v_2 \rangle = 1$

$$t_1 = v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$t_2 = v_2 - \frac{\langle v_2, t_1 \rangle}{\langle t_1, t_1 \rangle} t_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$V_{-2} = \mathrm{Span}(t_1, t_2)$$

$$\|t_2\|^2 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$\begin{cases} u_1 = \dfrac{t_1}{\|t_1\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ u_2 = \dfrac{t_2}{\|t_2\|} = \sqrt{\dfrac{2}{3}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = -\dfrac{1}{2}\sqrt{\dfrac{2}{3}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \end{cases}$$

$V_{-2} = \mathrm{Span}(u_1, u_2)$ base ortonormale di V_{-2}.

$\boxed{\lambda_2 = 1}$ $\quad (B - I_3) X = 0$

$$\begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{cases} -y + z = 0 \\ x - 2y + z = 0 \end{cases} \qquad \begin{cases} z = u \\ y = u \\ x = u \end{cases} \qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$

$a < 0$

$$\det = \underbrace{\frac{1}{\sqrt{2}} \left(-\frac{1}{2} \sqrt{\frac{2}{3}} \right) \frac{1}{\sqrt{3}}}_{\substack{\shortparallel \\ a}} \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = a \left(-1 \cdot 3 - 1 \cdot 3 \right) \geq 0$$

$$\boxed{T = S}$$

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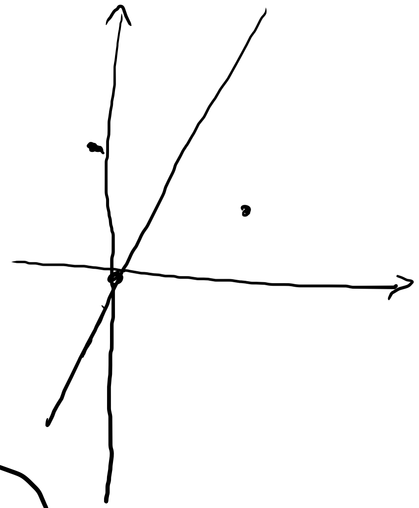
$$S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}$$

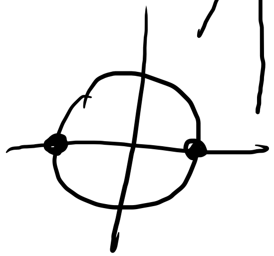
$$S_\theta \in O(2) - SO(2) \qquad \det S_\theta = -1$$



$${}^tS_\theta = S_\theta$$

Se $\sin\theta = 0 \iff \theta = k\pi \quad , k \in \mathbb{Z}$

$$\Rightarrow S_{k\pi} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}$$



Assumiamo $\theta \neq k\pi \ , \forall k \in \mathbb{Z}$.

1) Autovalori

$$p_{S_\theta} = \det \begin{pmatrix} \cos\theta - x & \sin\theta \\ \sin\theta & -\cos\theta - x \end{pmatrix} = (x - \cos\theta)(x + \cos\theta) - \sin^2\theta =$$

$$= x^2 - \cos^2\theta - \sin^2\theta = x^2 - 1$$

$\lambda_1 = 1, \quad \lambda_2 = -1$

$m_a(1) = m_a(-1) = 1 = m_g(1) = m_g(-1)$. $\quad \boxed{(\sin\theta \neq 0)}$

$$(\lambda_1 = 1) \quad (S_\theta - I_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\cos\theta - 1)\,x + \sin\theta\, y = 0$$

$$\begin{cases} x = u \\ y = \dfrac{1-\cos\theta}{\sin\theta}\, u \end{cases} \qquad u = \sin\theta \qquad v_1 = \begin{pmatrix} \sin\theta \\ 1-\cos\theta \end{pmatrix}$$

$$\|v_1\|^2 = \sin^2\theta + (1-\cos\theta)^2 = \sin^2\theta + 1 - 2\cos\theta + \cos^2\theta = 2(1-\cos\theta)$$

$$> 0$$

$$(\sin\theta \neq 0)$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2(1-\cos\theta)}} \begin{pmatrix} \sin\theta \\ 1-\cos\theta \end{pmatrix}$$

$(\lambda_2 = -1)$ $\quad (S_\theta + I_2)X = 0 \quad \begin{pmatrix} \cos\theta + 1 & \sin\theta \\ \sin\theta & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\sin\theta \neq 0 \rightsquigarrow (\cos\theta + 1)x + \sin\theta\, y = 0$

$\begin{cases} x = u \\ y = -\dfrac{\cos\theta + 1}{\sin\theta} u \end{cases} \quad u \in \mathbb{R} \qquad u = \sin\theta \rightsquigarrow v_2 = \begin{pmatrix} \sin\theta \\ -(1+\cos\theta) \end{pmatrix}$

$\|v_2\|^2 = \sin^2\theta + (1+\cos\theta)^2 = \underline{\sin^2\theta} + 1 + 2\cos\theta + \underline{\cos^2\theta} =$

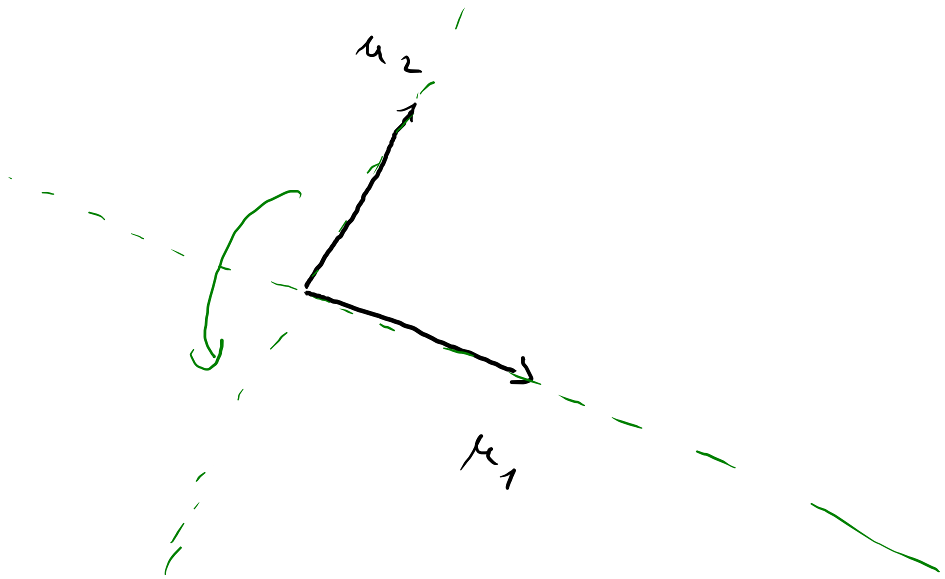
$= 2(1+\cos\theta) > 0 \quad (\sin\theta \neq 0)$

$u_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{1}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix} \sin\theta \\ -(1+\cos\theta) \end{pmatrix}$

(u_1, u_2) base ortonormale di $\mathbb{R}^2$ diagonalizzante.

$$S_\theta u_1 = u_1$$

$$S_\theta u_2 = -u_2$$



S_θ riflessione lineare di $\mathbb{R}^2$ rispetto a $r = \mathrm{span}(u_1)$

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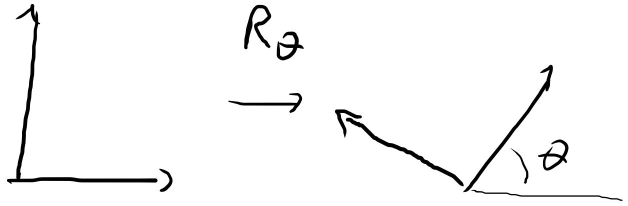
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ matrice di rotazione di angolo $\theta \in \mathbb{R}$



$$R_\theta \in SO(2) \Rightarrow R_\theta \in SU(2)$$

(teorema spettrale unitario)

Se $\sin\theta = 0$ $\quad R_{k\pi} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$

$\theta = k\pi, k \in \mathbb{Z}$

Possiamo assumere $\sin\theta \neq 0$

Autovalori

$$p_{R_\theta} = \det\left(R_\theta - x I_2\right) = \begin{vmatrix} \cos\theta - x & -\sin\theta \\ \sin\theta & \cos\theta - x \end{vmatrix} =$$

$$= (\cos\theta - x)^2 + \sin^2\theta = 1 - 2x\cos\theta + x^2$$

$$x^2 - 2\cos\theta\, x + 1 = 0$$

$$x = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm \sqrt{-(1-\cos^2\theta)} = \underline{\underline{\cos\theta \pm i\sin\theta}}$$

$$= e^{\pm i\theta} \in \mathbb{C} - \mathbb{R} \text{ se } \sin\theta \neq 0$$

$m_a(\lambda_1) = m_a(\lambda_2) = 1$

$\Rightarrow m_g(\lambda_1) = m_g(\lambda_2) = 1$

$(\sin\theta \neq 0)$

<u>Autovettori</u> $(\sin\theta \neq 0)$

$\lambda_1 = \cos\theta + i\sin\theta$

$$(R_\theta - \lambda_1 I_2)X = 0 \qquad \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\sin\theta\, x - i\sin\theta\, y = 0 \Rightarrow x - iy = 0$$

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \rightsquigarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\lambda_2 = \cos\theta - i\sin\theta$

$$(R_\theta - \lambda_2 I_2)\, X = 0 \qquad \begin{pmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$x + iy = 0 \qquad v_2 = \begin{pmatrix} 1 \\ +i \end{pmatrix} \qquad \|v_2\| = \sqrt{2} \qquad u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

(u_1, u_2) base ortonormale di $\mathbb{C}^2$ che diagonalizza R_θ