

# ANOMALIES & PATH INTEGRAL

Consideriamo  $\mathcal{L} = \bar{\Psi} (i\mathcal{D} - m) \Psi$       $\mathcal{D} = \gamma^\mu (\partial_\mu + iA_\mu)$

$\uparrow$   
 fermioni  
 di Dirac in  
 sp. R

$\uparrow$   
 abeliano  
 non-ab. ( $A_\mu = A_\mu^a t_a$ )

$\nearrow$   
 in dimensioni d  
 PARI

Passiamo all'Euclidea (P.I. è meglio definito)

$$x^0 = -ix^4 \quad \partial_0 = i\partial_d \quad \gamma_0 = -i\gamma_E^d \quad \gamma_n^i = \gamma_E^i$$

$$A_0 = iA_d \quad i = 1, \dots, d-1$$

$$\{\gamma_\mu^\mu, \gamma_\nu^\nu\} = 2\eta^{\mu\nu} \rightarrow \{\gamma_E^\mu, \gamma_E^\nu\} = -2\delta^{\mu\nu}$$

$$\mathcal{D}_M = \gamma_M^\mu \partial_\mu = \gamma_0^0 \partial_0 + \gamma_n^i \partial_i = \gamma_E^d \partial_d + \gamma_E^i \partial_i = \mathcal{D}_E$$

$$(\gamma_M^5)^2 = \mathbb{1} \quad \{\gamma_M^5, \gamma_M^\mu\} = 0 \rightarrow \gamma_M^5 = - (i)^{\frac{d}{2}+1} \gamma^0 \gamma^1 \dots \gamma^{d-1}$$

Controlliamo  $(\gamma_n^5)^2 = (-1)^{\frac{d}{2}+1} \overbrace{\gamma^0 \gamma^1 \dots \gamma^{d-1}}^{d-1} \overbrace{\gamma^0 \gamma^1 \dots \gamma^{d-1}}^{(i^0)^2=1} =$

$$= (-1)^{\frac{d}{2}+1} (-1)^{d-1} \gamma^1 \dots \gamma^{d-1} \gamma^1 \dots \gamma^{d-1} =$$

$$= (-1)^{\frac{d}{2}+1} (-1)^{d-1} \cancel{(-1)^{d-1}} (-1)^{\sum_{k=1}^{d-1} k} \overbrace{(\gamma_E^i)^2}^{d-1} = (-1)^{\frac{d}{2}+1} (-1)^{\frac{(d-1)d}{2}} =$$

$$= (-1)^{\frac{d}{2}+1 + \frac{d^2-d}{2} + \frac{d^2-3d}{2} + 1} = (-1)^{\frac{d^2-d}{2}} = (-1)^{\frac{d^2}{2}} = 1$$

$d$  pari  $\Rightarrow d^2 \in 4\mathbb{Z}$

$$\gamma_E^5 = - (i)^{\frac{d}{2}+1} (-i) \gamma_E^d \gamma_E^1 \dots \gamma_E^{d-1} = - i^{d/2} (-1)^{d-1} \gamma_E^1 \dots \gamma_E^d =$$

$$= (-i)^{3d/2} \gamma_E^1 \dots \gamma_E^d$$

$\uparrow$   
 $\frac{1}{i} \cdot 2d-2$

Consistentemente con  $(\gamma_E^\mu)^2 = -1$  possiamo prendere  $\gamma_E^\mu$  ANTI-HERM.

$$\Rightarrow \mathcal{D}^\dagger = (\gamma^\mu)^\dagger (-\partial_\mu - iA_\mu^\dagger) = \gamma^\mu (\partial_\mu + iA_\mu) = \mathcal{D}$$

$\Rightarrow \mathcal{D}$  è HERMITIANO!  $\leftarrow$  qto vale in Euclideo (non in Mink.)

$$iS_M = i \int d^d x \bar{\Psi} (i \not{D}_n - m) \Psi = i(-i) \int d^d x_{\bar{E}} \bar{\Psi} (i \not{D}_{\bar{E}} - m) \Psi =$$

$$= - \int d^d x_{\bar{E}} \bar{\Psi} (-i \not{D}_{\bar{E}} + m) \Psi \equiv -S_E$$

Consideriamo il funzionale generatore (def in Eud.)

$$Z[A] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{\int dx \bar{\Psi} (i \not{D} - m) \Psi}$$

$\downarrow$   $\det (i \not{D} - m)$

$\swarrow$   $L$  è invariante sotto transf. assiel. <sup>(globali)</sup> se  $m=0$

Troviamo le WI relative alla simm. (globale) assiale:

consideriamo transf. assiali locali:

$$\Psi \mapsto \Psi' = e^{i\beta(x)\gamma_5} \Psi \quad \bar{\Psi} \mapsto \bar{\Psi}' = \bar{\Psi} e^{i\beta(x)\gamma_5} \quad (*) \quad \beta = \beta^a t_a$$

$$\bar{\Psi} (i \not{D} - m) \Psi \mapsto \bar{\Psi}' (i \not{D} - m) \Psi' = \bar{\Psi} (i \not{D} - m) \Psi - 2im\beta \bar{\Psi} \gamma^5 \Psi$$

$\uparrow$  transf. infinitesime

$$- (\partial_\mu \beta) \underbrace{\bar{\Psi} \gamma^\mu \gamma^5 \Psi}_{j_A^\mu}$$

$$S' = S - \int dx \beta(x) \left[ \partial_\mu j_A^\mu - 2imP(x) \right]$$

$$Z[A] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{-S[\Psi, \bar{\Psi}, A]}$$

cambiamo nome delle variab. integr.  $\rightarrow$

$$= \int \mathcal{D}\Psi' \mathcal{D}\bar{\Psi}' e^{-S[\Psi', \bar{\Psi}', A]}$$

cambiamo coordinate usando (\*)  $\rightarrow$

$$= \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} J[\beta, A] e^{-S[\Psi, \bar{\Psi}, A]} e^{\int dx \beta(x) (\partial_\mu j_A^\mu - 2imP)}$$

$\nearrow$  tipicamente lo Jacobiano è assunto essere  $= 1$  (misura invariante) e se  $P$  è uno stesso WI compatibile con ciò che uno si aspetta classicamente.

Ma in questo caso (trans. esibile)  $J \neq 1$  e questo porta a una VIOLAZIONE delle WI, cioè a un'ANOMALIA.

Riscriviamo lo Jacobiano come (vedi dopo)

$$J[\beta, A] = e^{-\int dx \beta(x) \mathcal{A}[A](x)}$$

Prendendo  $\beta \ll 1$ :

$$\begin{aligned} \underline{Z[A]} &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S} \left( 1 + \int dx \beta(x) [\partial_\mu j_A^\mu - 2imP - \mathcal{A}[A]] \right) \\ &= \underline{Z[A]} + \int dx \beta(x) \langle \partial_\mu j_A^\mu - 2imP - \mathcal{A}[A] \rangle_A \end{aligned}$$

$$\Rightarrow \partial_\mu \langle j_A^\mu \rangle_A = 2im \langle P \rangle + \mathcal{A}[A] \quad \begin{array}{l} \text{ANOMALOUS} \\ \text{A WI} \end{array}$$

Metodo di Fujikawa per calcolare lo Jacobiano:

Espandiamo  $\psi$  e  $\bar{\psi}$  in autofunz. di  $\mathcal{D}$

$$\psi(x) = \sum_n a_n \varphi_n(x) = \sum_n a_n \langle x|n\rangle$$

$$\bar{\psi}(x) = \sum_m \varphi_m^\dagger(x) \bar{b}_m = \sum_m \langle m|x\rangle \bar{b}_m$$

$a_n, \bar{b}_m$  sono variabili di Grassmann

$\varphi_n$  sono spinori di Dirac in rap. R del grupo G

$\mathcal{D}$  è HERM.  $\rightarrow$  autovalori REALI

$$\mathcal{D}\varphi_n = \lambda_n \varphi_n \quad \lambda_n \in \mathbb{R}$$

Il set di autofunz. è ortonormale e completo:

$$\int dx \varphi_m^\dagger(x) \varphi_n(x) = \int dx \langle m|x\rangle \langle x|n\rangle = \langle m|n\rangle = \delta_{mn}$$

Prendiamo la misura:

$$D\psi D\bar{\psi} = \prod_n da_n \prod_m d\bar{b}_m \left( \underbrace{\det \langle m|x \rangle \cdot \det \langle x|n \rangle}_{\det \langle m|n \rangle = 1} \right)^{-1}$$

↑  
numeri  
di Grassmann

$$= \prod_n da_n \prod_m d\bar{b}_m$$

Azione:

$$\int dx \bar{\psi} (i\mathcal{D} - m) \psi = \sum_n (i\lambda_n - m) \bar{b}_n a_n$$

$$\begin{aligned} \hookrightarrow \det(i\mathcal{D} - m) &= \int D\psi D\bar{\psi} e^{\int dx \bar{\psi} (i\mathcal{D} - m) \psi} = \int \prod_n da_n d\bar{b}_n e^{\sum_n (i\lambda_n - m) \bar{b}_n a_n} \\ &= \prod_n (i\lambda_n - m) \end{aligned}$$

Vediamo come trasformano  $a_n$  e  $\bar{b}_m$  sotto trasf. esat.

$$\begin{aligned} \psi'(x) &= \sum_n a'_n \psi_n(x) \\ &= (1 + i\beta(x) \gamma_5) \sum_m a_m \psi_m(x) \end{aligned}$$

$$\begin{aligned} \bar{\psi}'(x) &= \dots \\ &= \dots \end{aligned}$$

⇓ usiamo o.n. della base  $\{\psi_n\}$

$$a'_n = \sum_m C_{nm} a_m$$

con  $C_{nm} = \delta_{nm} + i \int dx \beta(x) \psi_n^\dagger \gamma_5 \psi_m$

$$\bar{b}'_m = \sum_k C_{km} \bar{b}_k$$

(Variabili di Grassmann)

$$\prod_n da'_n = (\det C)^{-1} \prod_n da_n$$

$$\prod_m d\bar{b}'_m = (\det C)^{-1} \prod_m d\bar{b}_m$$

$$\Rightarrow D\psi' D\bar{\psi}' = (\det C)^{-2} D\psi D\bar{\psi} \Rightarrow J[\beta, A] = (\det C)^{-2}$$

Ora  $\det C = e^{\text{Tr} \ln C}$  e  $\ln(1+\xi) = \xi + O(\xi^2)$

$$J[\beta, A] = e^{-2\text{Tr} \ln C} = e^{-2\text{Tr} \ln (\delta_{nm} + i \int dx \beta(x) \psi_n^\dagger(x) \gamma_5 \psi_m(x))} \simeq$$

$$\simeq e^{-2\text{Tr} i \int dx \beta(x) \psi_n^\dagger(x) \gamma_5 \psi_m(x)}$$

$$= e^{-2i \int dx \beta(x) \underbrace{\sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x)}_{\text{Mal definita:}}}$$

Mal definita:

$$\sum_n \psi_n^\dagger \gamma_5 \psi_n = \text{tr}_{\text{Spin. ind.}} (\gamma_5 \underbrace{\sum_n \psi_n^\dagger \psi_n}_{\delta(x-x)}) \sim \text{tr} \gamma_5 \cdot \delta(0) = 0 \cdot \infty$$

→ necessità di REGOLARIZZARE

Idea di Fujikawa: usare un CUTOFF GAUSSIANO

$$\begin{aligned} \sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x) &= \lim_{M \rightarrow \infty} \sum_n \psi_n^\dagger(x) \gamma_5 e^{-\lambda_n^2/M^2} \psi_n(x) \\ &= \lim_{M \rightarrow \infty} \sum_n \psi_n^\dagger(x) \gamma_5 e^{-D^2/M^2} \psi_n(x) \end{aligned}$$

Si può generalizzare la regolarizzazione:

$$\sum_n \psi_n^\dagger \gamma_5 \psi_n \equiv \lim_{M \rightarrow \infty} \sum_n \psi_n^\dagger \gamma_5 f(D^2/M^2) \psi_n$$

con  $f(y)$  t.c.  $f^{(n)}(y) \xrightarrow{y \rightarrow \infty} 0$   $\forall n = 0, 1, 2, \dots$   
e  $f(0) = 1$

Calcoliamo la somma regolarizzata allo sp. di Fourier :

$$\varphi_n(x) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{\varphi}_n(k)$$

$$\begin{aligned} v^T A w &= \text{tr } v^T A w = \\ &= \text{tr } A w v^T \end{aligned}$$

$$\left\| \sum_n \varphi_n^+ \gamma_5 \varphi_n \right\| = \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \sum_n \tilde{\varphi}_n^+(q) e^{-iqx} \gamma_5 f\left(\frac{\not{D}^2}{\Lambda^2}\right) e^{ikx} \tilde{\varphi}_n(k)$$

$$= \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{-iqx} \text{tr } \gamma_5 f\left(\frac{\not{D}^2}{\Lambda^2}\right) \underbrace{\sum_n \tilde{\varphi}_n(k) \tilde{\varphi}_n^+(q)}_{(2\pi)^d \delta(k-q)} e^{ikx}$$

↑  
traccia su indici  
spinoriali e di rep. R

$$= \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} e^{-ikx} \text{tr} \left[ \gamma_5 f\left(\frac{\not{D}^2}{\Lambda^2}\right) \right] e^{ikx}$$

$$\begin{aligned} \not{D}^2 &= \gamma^\mu \gamma^\nu D_\mu D_\nu = \left( \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} + \frac{1}{2} [ \gamma^\mu, \gamma^\nu ] \right) D_\mu D_\nu = \\ &= -D^2 + \frac{i}{4} [ \gamma^\mu, \gamma^\nu ] F_{\mu\nu} \end{aligned}$$

Inoltre  $e^{-ikx} F(\not{\partial}) e^{ikx} = F(\not{\partial} + ik_\mu)$

$$\left\| \sum_n \varphi_n^+ \gamma_5 \varphi_n \right\| = \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \text{tr} \left[ \gamma_5 f \left( - \frac{(\not{D}_\mu + ik_\mu)(\not{D}_\mu + ik_\mu)}{\Lambda^2} + i \frac{\gamma^\mu \gamma^\nu F_{\mu\nu}}{2\Lambda^2} \right) \right]$$

$$= \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \text{tr} \left[ \gamma_5 f \left( \frac{k^2}{\Lambda^2} - \frac{2ik_\mu \not{D}_\mu + D^2}{\Lambda^2} + i \frac{\gamma^\mu \gamma^\nu F_{\mu\nu}}{2\Lambda^2} \right) \right]$$

- $\text{tr } \gamma_5 = 0$ ,  $\text{tr } \gamma_5 \gamma^{\mu_1} \dots \gamma^{\mu_l} = 0$  se  $l \neq d$ ,  
 $\text{tr } \gamma_5 \gamma^{\mu_1} \dots \gamma^{\mu_d} = i^{\frac{3d}{2}} 2^{d/2} \epsilon^{\mu_1 \dots \mu_d}$  ( $\epsilon^{12 \dots d} = 1$ )

- espandiamo  $f$  in Taylor attorno a  $k^2/\Lambda^2$

Simplify  
 solve in  
 terms  
 at leading order  
 in  $1/n^2$

$$= \lim_{n \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \frac{i^{\frac{3d}{2}} 2^{d/2}}{(\frac{d}{2})! (2n^2)^{d/2}} f^{(d/2)}(k^2/n^2) \epsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_{d/2} \nu_{d/2}} \text{tr } i F_{\mu_1 \nu_1} \dots i F_{\mu_{d/2} \nu_{d/2}}$$

$k \rightarrow Mk$

$$= \lim_{n \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \frac{i^{\frac{3d}{2}} i^{d/2}}{(\frac{d}{2})!} f^{(d/2)}(k^2) \epsilon^{\mu_1 \dots \nu_{d/2}} \text{tr } F_{\mu_1 \nu_1} \dots F_{\mu_{d/2} \nu_{d/2}} + \mathcal{O}(1/n)$$

$$= \epsilon^{\mu_1 \dots \nu_{d/2}} \text{tr } F \dots F \frac{1}{(\frac{d}{2})!} \int \frac{d^d k}{(2\pi)^d} f^{(d/2)}(k^2)$$

coord. polar

$r = k^2$   
 $dr = 2k dk$

$$\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{k^{d-1} dk}{(2\pi)^d} f^{(d/2)}(k^2)$$

$$\frac{\pi^{d/2}}{(\frac{d}{2}-1)!} \int_0^\infty \frac{r^{d/2-1} dr}{(2\pi)^d} f^{(d/2)}(r) = \frac{(-1)^{d/2}}{2^d \pi^{d/2}}$$

$$\int_0^\infty r^{n-1} f^{(n)}(r) dr = \int_0^\infty \left[ \frac{d}{dr} (r^{n-1} f^{(n-1)}) - (n-1) r^{n-2} f^{(n-1)}(r) \right] dr$$

$$= - (n-1) \int_0^\infty r^{n-2} f^{(n-1)} dr = \dots = (-1)^{n-1} (n-1)! \int_0^\infty f'(r) dr =$$

$$= (-1)^{n-1} (n-1)! f|_0^\infty = (-1)^n (n-1)!$$

" $\sum_n \mathcal{L}_n^+ \mathcal{P}_n$ " =

$$\epsilon^{\mu_1 \nu_1 \dots \mu_{d/2} \nu_{d/2}} \text{tr } F_{\mu_1 \nu_1} \dots F_{\mu_{d/2} \nu_{d/2}} \frac{(-1)^{d/2}}{2^d (\frac{d}{2})! \pi^{d/2}}$$