b forma bilin. simm. su $\mathbb{R}^n$, sp. euclideo con prod. scal. canonico

q forma quadr. assoc. a b

$A = M_{\mathcal{C}}(b)$ $\quad$ $\mathcal{C}$ ortonormale

$\Longrightarrow$ A è simmetrica

Conseg. del teor. spettr.: $\exists$ una base B di $\mathbb{R}^n$ ortonormale, di autovettori di A

$$M_B(b) = {}^tSAS = S^{-1}AS = M_B(L(A)) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_1, \dots, \lambda_n$ autovalori di A

$S = M_{\mathcal{C}}^{B}(id_{\mathbb{R}^n})$

$\mathcal{C}, B$ ortonorm. $\Longrightarrow$ S è ortogonale

${}^tS = S^{-1}$

$B = (v_1, \dots, v_n)$ $\qquad$ $\langle v_1 \rangle, \dots, \langle v_n \rangle$ assi principali per b o q

$$q = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$

$$b = \lambda_1 x_1 y_1 + \dots + \lambda_n x_n y_n$$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \underset{\overset{\|}{M}}{M_B(b)} = \left(b(v_i, v_j)\right)_{i,j=1,\dots,n}$$

$$b(x,y) = \sum m_{ij} x_i y_j = {}^t x M y$$

$$b(v_i, v_j) = \begin{cases} 0 & i \neq j \\ \lambda_i & i = j \end{cases}$$

Voglio costruire B' t.c. $M_{B'}(b)$ abbia solo $1, -1, 0$ sulla diag. princ.

Riordino B in modo che

$\lambda_1, \dots, \lambda_r$ siano > 0

$\lambda_{r+1}, \dots, \lambda_{r+s}$ siano < 0

$\lambda_{r+s+1} = \dots = \lambda_n = 0$

$1 \leq i \leq r \qquad \lambda_i > 0, \quad \exists \sqrt{\lambda_i} > 0$

$$v_i' = \frac{v_i}{\sqrt{\lambda_i}} \qquad b(v_i', v_i') = b\left(\frac{1}{\sqrt{\lambda_i}} v_i, \frac{1}{\sqrt{\lambda_i}} v_i\right) =$$

$$= \frac{1}{(\sqrt{\lambda_i})^2} b(v_i, v_i) = \frac{1}{\lambda_i} b(v_i, v_i) = \frac{1}{\lambda_i} \cdot \lambda_i = \underline{1}$$

$r+1 \leq i \leq r+s \qquad \lambda_i < 0, \quad \exists \sqrt{-\lambda_i} > 0$

$$v_i' = \frac{v_i}{\sqrt{-\lambda_i}} \qquad b(v_i', v_i') = \frac{1}{(\sqrt{-\lambda_i})^2} b(v_i, v_i) =$$

$$= \frac{1}{-\lambda_i} \lambda_i = \underline{-1}$$

$r+s+1 \le i \le n \qquad \lambda_i = 0 \qquad v'_i = v_i$

$$b(v'_i, v'_i) = b(v_i, v_i) = 0$$

$b(v'_i, v'_j) = 0$ perché $i \neq j$

$$b(v'_i, v'_j) = b(c_i v_i, c_j v_j) = c_i c_j \cdot b(v_i, v_j) = 0$$

c_i può essere $\frac{1}{\sqrt{\lambda_i}}$, $\frac{1}{\sqrt{-\lambda_i}}$ opp. 1

$$M_{B'}(b) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & 0 & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & 0 & & & & & 0 & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

$\langle v'_i \rangle = \langle v_i \rangle \quad \forall i = 1, \dots, n$

gli assi principali non sono cambiati

$f: V \to V \qquad P_f(x) = (\lambda_1 - x) \cdots (\lambda_n - x)$

$\exists\, B$ base t.c. $M_B(f)$ triangolare ~~inf.~~ sup.

Dim. induz. su n.

$n = 1$ vero

$n > 1$, supp. vero per $n-1$

f ha almeno un autovalore, v autovett. $\neq 0$

$f(v) = \lambda_1 v$,

$\exists$ una base $\mathcal{B}'$ di V: $v_1, v_2, \ldots, v_n$ con $v_1 = v$

$W = \langle v_2, \ldots, v_n \rangle$

$V = \langle v \rangle \oplus W$

$$A = M_{\mathcal{B}'}(f) = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$f(v_1)$ $f(v_2)$

$\to A'$ matrice di un endom. g di W risp. a $(v_2, \ldots, v_n)$

$g(v_2) = a_{21} v_2 + \cdots + a_{n1} v_n$

$\vdots$

$g(v_n) = a_{2n} v_2 + \cdots + a_{nn} v_n$

Vogliamo usare l'ip. induttiva su $g: W \to W$

Ci serve dire: che $p_g(x)$ si fattorizza in fattori lin.

$p_f(x) = (\lambda_1 - x) \cdots (\lambda_n - x)$
$\parallel$

$$|A - xE_n| = \begin{vmatrix} \lambda_1 - x & * \cdots * \\ 0 & \\ \vdots & A' - xE_{n-1} \\ 0 & \end{vmatrix} = (\lambda - x)\,|A' - xE_{n-1}| =$$

$= (\lambda_1 - x)\, p_g(x)$ siamo in $\mathbb{K}[x]$

$\lambda_1 - x \neq 0$

$\Rightarrow \quad p_g(x) = (\lambda_2 - x) \cdots (\lambda_n - x)$

Possiamo usare l'ip. induttiva:

$\exists$ base di W $w_2, \dots, w_n$ t.c.

$$M_{(w_2 \dots w_n)}(g) = \begin{pmatrix} b_{22} & \cdots & b_{2n} \\ & \ddots & \vdots \\ 0 & & b_{nn} \end{pmatrix}$$

$B = (v_1, w_2, \dots, w_n)$ è una base di V

Tesi: $M_B(f)$ è triangolare sup. e precisam. è del tipo
$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & b_{nn} \end{pmatrix}$$

$f(w_2)$
$\vdots$
$f(w_n)$ — vogliamo le loro coord. risp. a B

$f(w_2)$

$$f(v_2) = \cancel{* v_1} + \underbrace{a_{22} v_2 + \cdots + a_{n2} v_n}_{\in W} = * v_1 + \overset{\varphi}{g(v_2)}$$
$\vdots$
$$f(v_n) = * v_1 + a_{2n} v_2 + \cdots + a_{nn} v_n = * v_1 + \underset{W}{g(v_n)}$$

$V = \langle v_1 \rangle \oplus W$

Ogni $v \in V$ ha 1! espress. del tipo

$$v = \underbrace{\boxed{* v_1} + w}$$

$$\pi : V \longrightarrow W \quad \text{è lin.}$$
$$v \longrightarrow w \quad \text{proiezione}$$

$$g(v_2) = \pi(f(v_2))$$
$$\vdots$$
$$g(v_n) = \pi(f(v_n))$$

$$\Longrightarrow \boxed{g|_W = \pi \circ f}$$

per il teor. di det. di un'appl. lin.

$$f(w_2) = * v_1 + \pi f(w_2) = * v_1 + g(w_2) = * v_1 + b_{22} w_2$$
$$\vdots$$
$$f(w_n) = * v_1 + \pi(f(w_n)) = * v_1 + g(w_n)$$

coord. di $f(w_2)$ risp a $\{v_1, \underline{w_2, \dots, w_n}\}$

$$f(w_2) = * v_1 +$$

$$\boxed{M_{(w_2 \dots w_n)}(g)} = \begin{pmatrix} b_{22} & \dots & \dots \\ & \ddots & \dots \\ 0 & & b_{nn} \end{pmatrix}$$

$$M_{\mathcal{B}}(f) = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & & & \\ \vdots & & M_{(w_2 \dots w_n)}(g) & \\ 0 & & & \end{pmatrix}$$

i)

$$A_t = \begin{pmatrix} t & 1 & 2 \\ 1 & t & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_{A_t}(x) = \begin{vmatrix} t-x & 1 & 2 \\ 1 & t-x & t \\ 0 & 0 & 1-x \end{vmatrix} = (1-x)\left[(t-x)^2-1\right] =$$

$$= (1-x)(t-x-1)(t-x+1) =$$

$$= (1-x)(t-1-x)(t+1-x)$$

$\underset{\lambda_1}{\overset{\parallel}{1}}, \underset{\lambda_2}{\overset{\parallel}{t-1}}, \underset{\lambda_3}{\overset{\parallel}{t+1}}$ sono gli autovalori di A_t

1) $\lambda_1, \lambda_2, \lambda_3$ distinti se $t-1 \neq 1$ e $t+1 \neq 1$

cioè $t \neq 2, t \neq 0$

$\Rightarrow A_t$ diag.

2) $t = 2$ o $t = 0$

$t = 2$ $\quad A_2 = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \lambda_1 = \lambda_2 = 1$, $\lambda_3 = 3$

dim Aut$(3) = 1$

A_2 diag. $\iff mg(1) = 2 \iff \underset{1}{\overset{}{rg(A - E_3)}}$ $= 1$

$$A - E_3 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow A_2 \text{ diag.}$$

$$t=0 \quad A_0 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_0 \text{ diag.} \Leftrightarrow rg(A_0 - E_3) = 1$$

$$A_0 - E_3 = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ ha rg } 2$$

A_0 non è diag.

$$\Rightarrow A_t \text{ è diag.} \Leftrightarrow t \neq 0$$

Matrice diag. simile ad A_t

$$\underline{\begin{pmatrix} 1 & & 0 \\ & t-1 & \\ 0 & & t+1 \end{pmatrix}} \qquad t=2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix}$$

$Aut(t+1)$ si può calcolare $\forall t \neq 0$
senza dist $\qquad t=2$

$$A_t - (t+1)E_3 = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & t \\ 0 & 0 & -t \end{pmatrix} \boxed{\text{ha rg } 2}$$

1) $t \neq 0$, le righe 2 e 3 sono lin. indip.

$$\begin{cases} x_1 - x_2 + t x_3 = 0 \\ -t x_3 = 0 \end{cases} \qquad \begin{array}{l} x_1 = x_2 \\ x_3 = 0 \quad (1,1,0) \end{array}$$

$$Aut(t+1) = \langle (1,1,0) \rangle$$

2) $$\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & t \\ 0 & 0 & -t \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & t+2 \\ 0 & 0 & -t \end{pmatrix} \to$$

$$\to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & -t \\ 0 & 0 & t+2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & -t \\ 0 & 0 & 0 \end{pmatrix}$$

$$\neq 0$$

$$Aut(1) = \ker(A_t - E_3)$$

$$A_t - E_3 = \begin{pmatrix} t-1 & 1 & 2 \\ 1 & t-1 & t \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} t-1 & 1 & 2 \\ 1 & t-1 & t \end{pmatrix} \xrightarrow{IV} \begin{pmatrix} 1 & t-1 & t \\ t-1 & 1 & 2 \end{pmatrix} \xrightarrow{III}$$

$$\to \begin{pmatrix} 1 & t-1 & t \\ 0 & 1-(t-1)^2 & 2-t(t-1) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & t-1 & t \\ 0 & -t^2+2t & -t^2+t+2 \end{pmatrix} = \begin{pmatrix} 1 & t-1 & t \\ 0 & -t(t-2) & -(t-2)(t+1) \end{pmatrix}$$

$$1-(t^2-2t+1) = -t^2+2t = -t(t-2)$$

$2-t^2+t$ ha 2 come radice

$$-t^2+t+2 = -(t^2-t-2) = -(t-2)(t+1)$$

$t \neq 2$ $\begin{pmatrix} 1 & t-1 & t \\ 0 & +t & t+1 \end{pmatrix}$

$$\begin{cases} x_1 + (t-1)x_2 + t\,x_3 = 0 \\ t\,x_2 + (t+1)x_3 = 0 \end{cases} \qquad x_2 = -\frac{t+1}{t}x_3$$

$$x_1 = \frac{(t-1)(t+1)}{t}x_3 + t\,x_3 = \frac{\cancel{t^2}-1+\cancel{t^2}}{t}x_3$$

$$\left(-\frac{1}{t}, -\frac{t+1}{t}, 1\right) \qquad (1, t+1, -t)$$

base per Aut(1)
se $t \neq 2$

$t=2$ $\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}$ —

$t \neq 2$ —— -

2) $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\langle\, , \rangle : M(2\times 2, \mathbb{R}) \times M(2\times 2, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$(A, B) \longrightarrow \langle A, B\rangle = tr({}^tBPA)$$

$V = M(2\times 2, \mathbb{R})$ $\quad \mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$

$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$M_{\mathcal{B}}(\langle\, , \rangle)$

$A = (a_{ij})$ $\quad B = (b_{ij})$

$$tr({}^tBPA) = tr\left({}^tB \begin{pmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{11}+2a_{21} & a_{12}+2a_{22} \end{pmatrix}\right) =$$

$$= tr\left(\begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{11}+2a_{21} & a_{12}+2a_{22} \end{pmatrix}\right) =$$

$$= b_{11}(a_{11}+a_{21}) + b_{21}(a_{11}+2a_{21}) +$$
$$+ b_{12}(a_{12}+a_{22}) + b_{22}(a_{12}+2a_{22}) =$$

$= b_{11}a_{11} + 0 \cdot b_{11}a_{12} + 1\, b_{11}a_{21} + 0\, b_{11}a_{22} +$

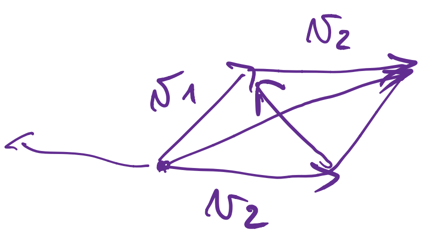
$+ 0 \cdot b_{12}a_{11} + 1\, b_{12}a_{12} + 0\, b_{12}a_{21} + 1\, b_{12}a_{22} +$

$+ 1\, b_{21}a_{11} + 0\, b_{21}a_{12} + 2\, b_{21}a_{21} + 0\, b_{21}a_{22} +$

$+ 0\, b_{22}a_{11} + 1\, b_{22}a_{12} + 0\, b_{22}a_{21} + 2\, b_{22}a_{22}$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$ rappresenta $\langle \, , \rangle$ rip. a $\mathcal{B}$

$\Longrightarrow$ forma bilin. simm.



$\| v_1 \| = \| v_2 \|$

$\langle v_1, v_1 \rangle = \langle v_2, v_2 \rangle$

$\langle v_1 + v_2 , v_2 - v_1 \rangle$

3) $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ $\quad$ $\dim W = 2$

$\cup\!|$

W

$\boxed{\operatorname{Ker} f = W}$

$\mathcal{B} = (\underbrace{v_1, v_2}_{\in W}, e_3, e_4, e_5)$

Per def. f basta fissare $f(v_1)=0$

$$\underset{\substack{\parallel \\ 0}}{f(v_2)},\ \underbrace{f(e_3),\ f(e_4),\ f(e_5)}_{\text{devono essere scelti in modo che } \underline{\operatorname{Ker} f = W}}$$

$$5 = \underbrace{\dim W}_{2} + \underbrace{\dim \operatorname{Im} f}_{3}$$

$\operatorname{Im} f$ è gen. da $\underline{f(e_3), f(e_4), f(e_5)}$

bisogna sceglierli

lin. indip.

Altrim. $W \subsetneq \operatorname{Ker} f$