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$A, P \in M_n(\mathbb{R})$, A simmetrica e definita positiva.

Dimostrare che $L_P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $X \longmapsto PX$ è simmetrica rispetto a $\langle\, , \,\rangle_A$ $\boxed{\Longleftrightarrow}$ AP è simmetrica.

$\langle X, Y \rangle_A = {}^tX A Y$

$$\boxed{\langle L_P X, Y \rangle_A = \langle X, L_P Y \rangle_A}$$

Dim

$\langle L_P(X), Y \rangle_A = {}^t(PX) A Y = {}^tX\, {}^tP A Y$

$\langle X, L_P(Y) \rangle_A = {}^tX A P Y$

L_P simmetrica $\Longleftrightarrow$ ${}^tX\, {}^tP A Y = {}^tX A P Y$ $\forall X, Y \in \mathbb{R}^n$

$\Longleftrightarrow$ ${}^tP A = AP$

$\quad\; \|$

${}^t(AP)$ $\quad ({}^tA = A)$

$\Longleftrightarrow$ AP simm.

1/10 Diagonalizzazione $A = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$ in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_2$, $\mathbb{Z}_5$

$\boxed{\mathbb{R}}$

$$P_A = \det(A - x I_2) = \begin{vmatrix} 1-x & 3 \\ -1 & 1-x \end{vmatrix} = \underline{(1-x)^2} \underline{+} 3$$

non ha zeri reali $\Rightarrow$ non diag. su $\mathbb{R}$.

$\boxed{\mathbb{C}}$

$$1 - x = \pm i\sqrt{3} \quad \Rightarrow \quad x = 1 \pm i\sqrt{3}$$

$\lambda_1 = 1 + i\sqrt{3}$, $\lambda_2 = 1 - i\sqrt{3}$ autovalori di A

$m_a(\lambda_{1,2}) = \underline{m_g(\lambda_{1,2})} = 1 \Rightarrow A$ diagon. su $\mathbb{C}$

Autovettori (su $\mathbb{C}$)

$\lambda_1 = 1 + i\sqrt{3}$

$$(A - \lambda_1 I_2) X = 0$$

$$\begin{pmatrix} -i\sqrt{3} & 3 \\ -1 & -i\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + i\sqrt{3}y = 0 \qquad v_1 = \begin{pmatrix} -i\sqrt{3} \\ 1 \end{pmatrix}$$

$\lambda_2 = 1 - i\sqrt{3}$

$$\begin{pmatrix} i\sqrt{3} & 3 \\ -1 & i\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x + i\sqrt{3}y = 0$$

$$v_2 = \begin{pmatrix} i\sqrt{3} \\ 1 \end{pmatrix}$$

(v_1, v_2) base di $\mathbb{C}^2$ diagonalizzante per A.

$$S = \begin{pmatrix} -i\sqrt{3} & i\sqrt{3} \\ 1 & 1 \end{pmatrix} \qquad \mathcal{V} \overset{S}{\rightsquigarrow} \mathcal{E}$$

$$S^{-1} A S = \begin{pmatrix} 1+i\sqrt{3} & 0 \\ 0 & 1-i\sqrt{3} \end{pmatrix}$$

$\underline{\mathbb{Z}_2}$ $\quad P_A(x) = (1-x)^2 + 3 = \underbrace{1 + x^2}\,+1 = x^2$

$\lambda = 0$ unico autovalore in $\mathbb{Z}_2$ $\qquad L_A : \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X \longmapsto AX$

$m_a(0) = 2 \qquad \underline{m_g(0) = \begin{cases} 1 \\ 2 \end{cases} ?}$

$$\boxed{\mathbb{Z}_5\ ?}$$

<u>Autovettore</u> $\quad (A - 0 \cdot I_2)\,X = 0$

$$AX = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\operatorname{rg} A = 1$ (in $\mathbb{Z}_2$) $\Rightarrow$

$\dim V_0 = 1 = m_g(0)$

$\Rightarrow$ non diag. in $\mathbb{Z}_2$.

4/10 Siano $A \in M_n(\mathbb{K})$, $S \in GL_n(\mathbb{K})$, $B = SAS^{-1}$.

Dimostrare che $B^k = SA^kS^{-1}$ $\forall\, k \in \mathbb{N} - \{0\}$.

Inoltre se $A \in GL_n(\mathbb{K})$, dimostrare che vale $\forall\, k \in \mathbb{Z}$. $(A^0 := I_n)$

Dim

$k \geq 1$

$$B^k = \underbrace{B \cdot \ldots \cdot B}_{k \text{ volte}} = \underbrace{SA\cancel{S^{-1}} \cdot \cancel{S}A\cancel{S^{-1}} \cdots \cancel{S}AS^{-1}}_{k} = SA^kS^{-1}$$

Dim per induzione base dell'induzione $k = 1$

$B = SAS^{-1}$ banale ✓

Supponiamo che sia vero per $k-1$ e dimostriamolo per $k \geq 2$

$$B^k = B\,B^{k-1} = SA\cancel{S^{-1}}\,\cancel{S}A^{k-1}S^{-1} = SA^kS^{-1}$$ ✓

Sia ora $A \in GL_n(\mathbb{K}) \Rightarrow B = S A S^{-1}$ invertibile

$k \geq 1$

$$\underline{B^{-k}} = (B^{-1})^k = (S A^{-1} S^{-1})^k = S (A^{-1})^k S^{-1} = \underline{S A^{-k} S^{-1}}$$

$$B^{-1} = S A^{-1} S^{-1}$$

$k = 0$ $\quad B^0 = I_n = S \underset{\overset{\shortparallel}{I_n}}{A^0} S^{-1}$ ✓

2/11 Calcolare $A^n \ \forall n \in \mathbb{Z}$, con $A = \begin{pmatrix} 5/2 & -1 \\ 3 & -1 \end{pmatrix}$

$$D = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

Idea diagonalizzare A, $A = S D S^{-1}$

$$S : \mathcal{V} \longrightarrow \mathcal{E}$$

$$\mathcal{E} \xrightarrow{S^{-1}} \mathcal{V}$$

Diagonalizziamo A

1) <u>Autovalori</u>

$$P_A = \det\begin{pmatrix} \frac{5}{2}-x & -1 \\ 3 & -1-x \end{pmatrix} = \left(\frac{5}{2}-x\right)(-1-x) + 3 =$$

$$= x^2 + x - \frac{5}{2}x - \frac{5}{2} + 3 =$$

$$= x^2 - \frac{3}{2}x + \frac{1}{2}$$

$$2x^2 - 3x + 1 = 0$$

$$x = \frac{3 \pm \sqrt{9-8}}{4} = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$$

$\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$ $m_a(1) = m_a\left(\frac{1}{2}\right) \Rightarrow$ diagonalizz.

Antwort:

$\lambda_1 = 1$

$$\begin{pmatrix} \frac{3}{2} & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$3x - 2y = 0 \qquad v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\lambda_2 = \frac{1}{2}$

$$\begin{pmatrix} 2 & -1 \\ 3 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$2x - y = 0 \qquad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$S = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \implies A = S D S^{-1} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$A^n = S \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} S^{-1}$$

$$= \dots \begin{pmatrix} & \\ & \end{pmatrix}$$

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1) Dim. che $O(2)$ è costituito dalle matrici del tipo R_θ, S_θ

2) $SO(2) = \{ R_\theta \mid \theta \in [0, 2\pi] \}$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

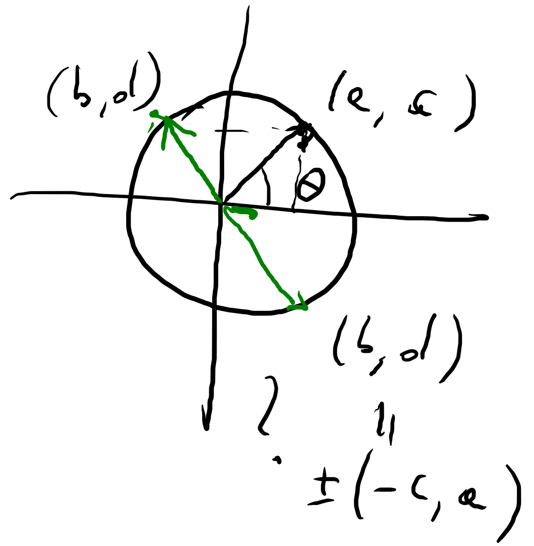
Dim

1) $R_\theta \in SO(2)$, $S_\theta \in O(2) - SO(2)$ $(\det S_\theta = -1)$

Sia $M \in O(2) \iff \{M_{(1)}, M_{(2)}\}$ base ortonormale di $\mathbb{R}^2$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{cases} a^2 + c^2 = 1 & \|M_{(1)}\| = 1 \\ b^2 + d^2 = 1 & \|M_{(2)}\| = 1 \\ ab + cd = 0 & \langle M_{(1)}, M_{(2)} \rangle = 0 \end{cases}$$

$\begin{cases} a = \cos\theta \\ c = \sin\theta \end{cases}$ per un certo $\theta \in [0, 2\pi]$



$$(b, d) = \pm (-c, a) = \pm (-\sin\theta, \cos\theta)$$

$$\Rightarrow \quad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_\theta$$ (rotazioni attorno a O)

oppure $$M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = S_\theta$$ (riflessione rispetto ad una retta per O)

2) $M \in SO(2) \Rightarrow M = R_\theta$ per un certo $\theta \in [0, 2\pi]$

$(\det S_\theta) = -1$.