Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

LINEAR SYSTEMS

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Green's function

Green's function (GF) is a basic solution to a linear differential equation, a building block that can be used to construct many useful solutions.

If one considers a linear differential equation written as:

$$L(x)u(x)=f(x)$$

where L(x) is a linear, self-adjoint differential operator, u(x) is the unknown function, and f(x) is a known non-homogeneous term, the GF is a solution of:

$$L(x)u(x,s)=\delta(x-s)$$

$$G(x,s)$$

Why GF is important?

If such a function G can be found for the operator L, then if we multiply the second equation for the Green's function by f(s), and then perform an integration in the s variable, we obtain:

$$\int L(x)G(x,s)f(s)ds = \int \delta(x-s)f(s)ds = f(x) = Lu(x)$$

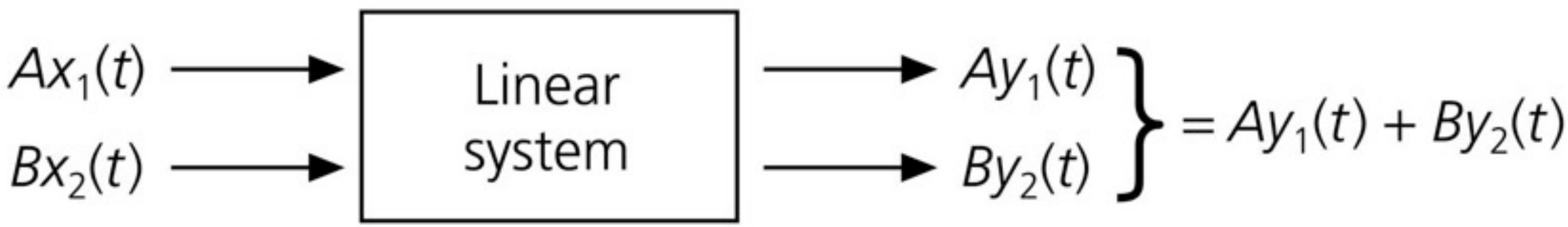
$$L\int G(x,s)f(s)ds = Lu(x)$$

$$u(x) = \int G(x,s)f(s)ds$$

Thus, we can obtain the function u(x) through knowledge of the Green's function, and the source term. This process has resulted from the linearity of the operator L.

Linear Systems

Figure 6.3-1: Definition of a linear system.



$$x(t) = \int x(\tau)\delta(\tau - t)d\tau$$

$$\int x(\tau)h(\tau - t)d\tau$$

$$x(t) * h(t) = y(t)$$

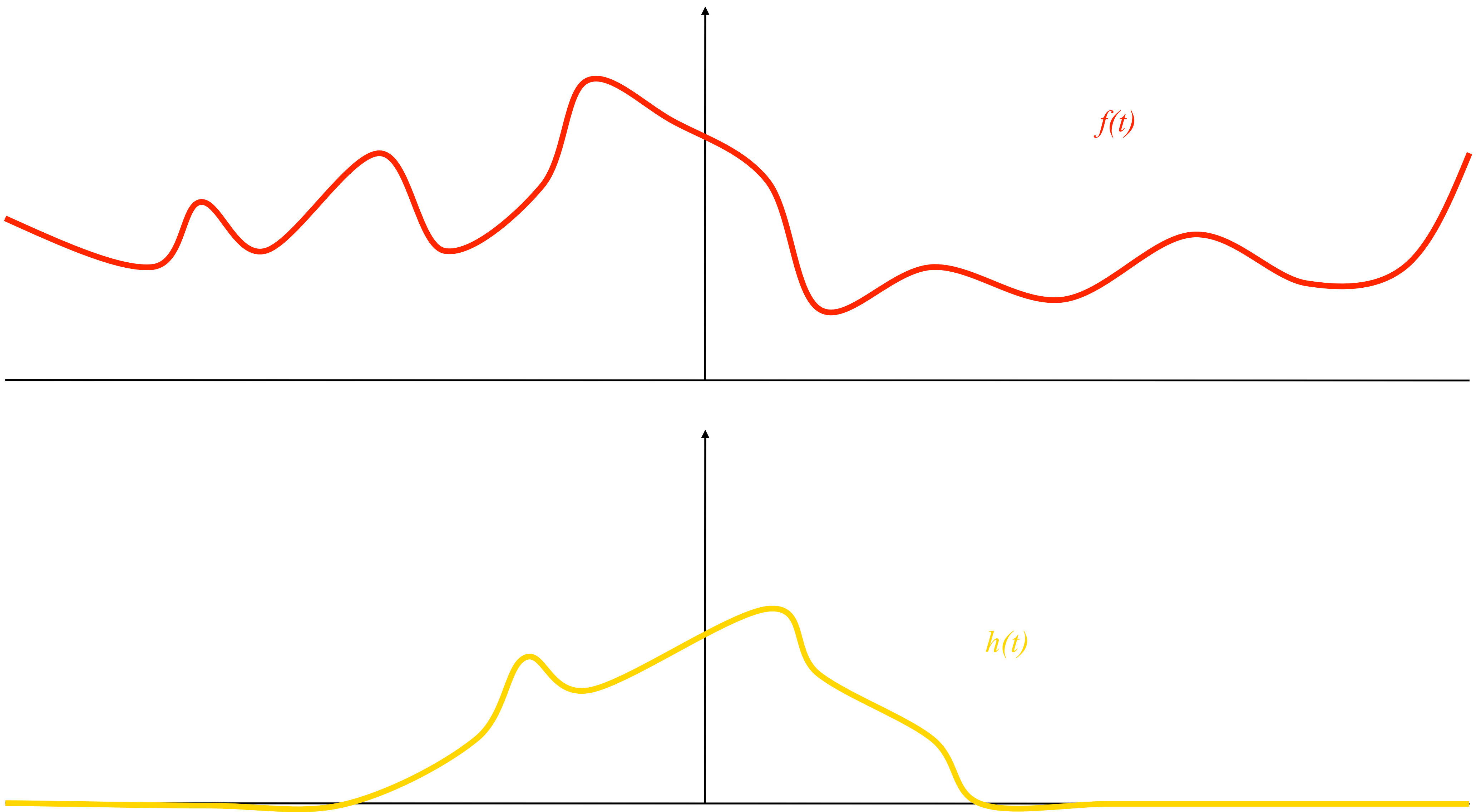
(remember GF definition)

Convolution

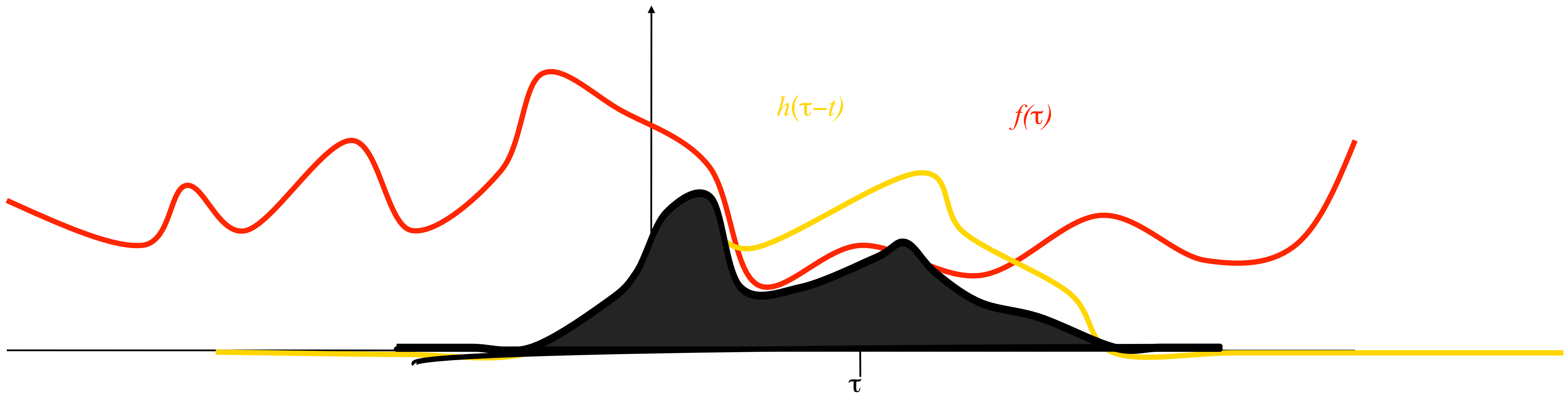
- Definition:

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

- Pictorially

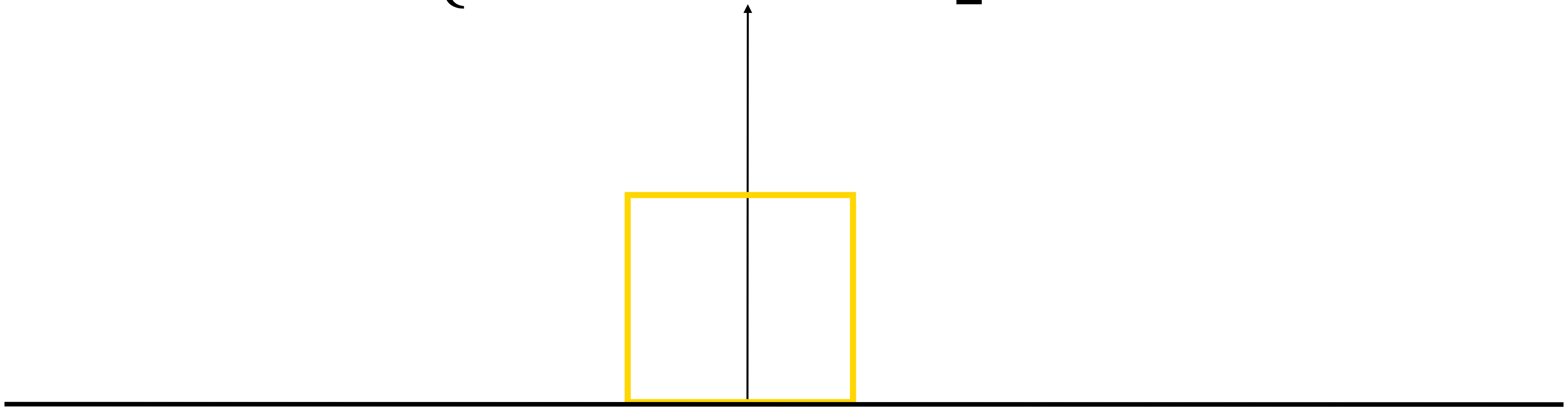


Convolution

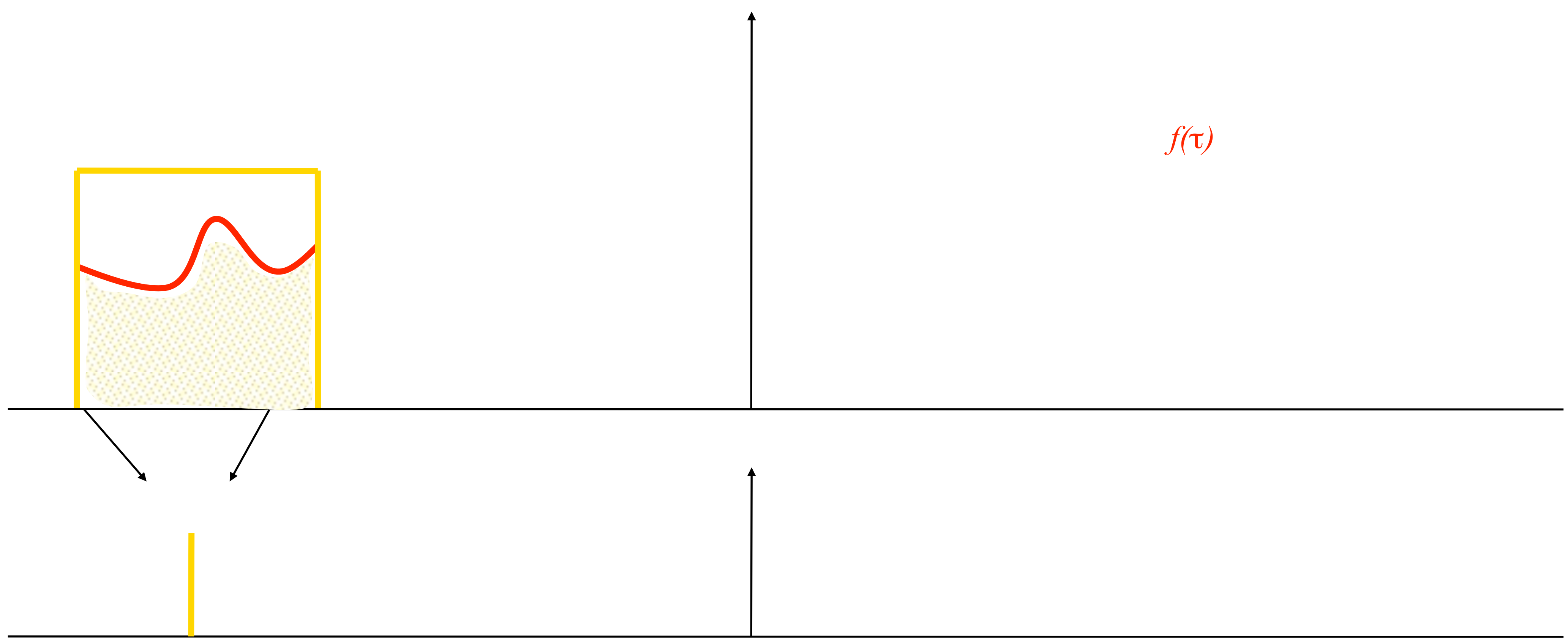


- Consider the function (box filter):

$$h(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}$$



- This function windows our function f(t)



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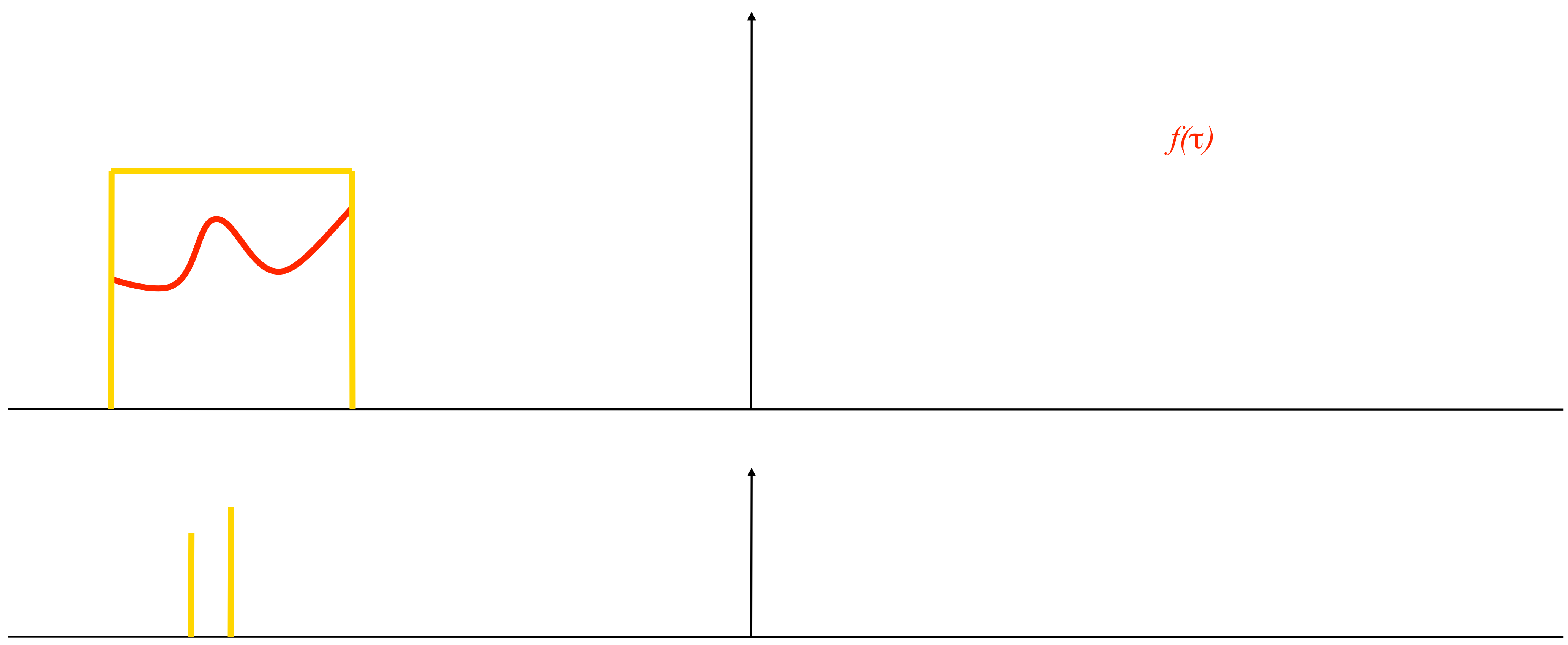

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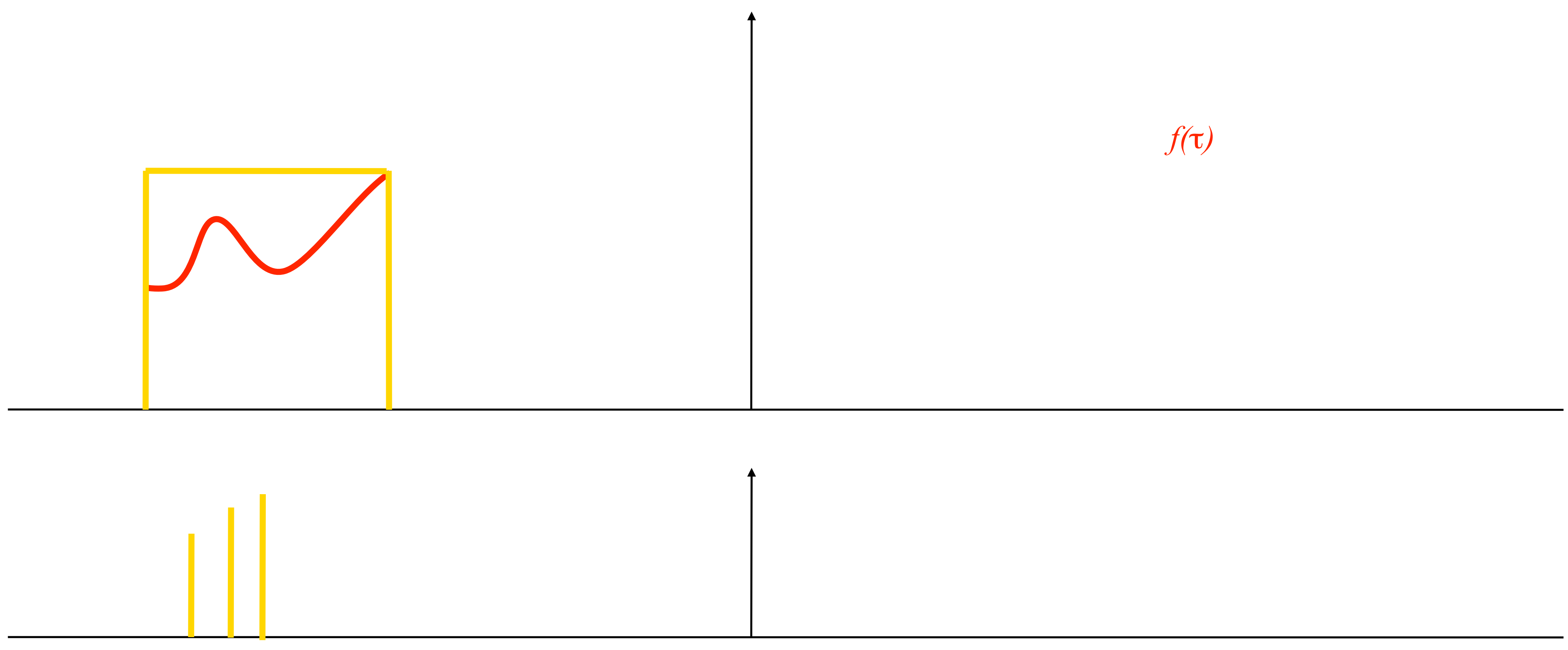

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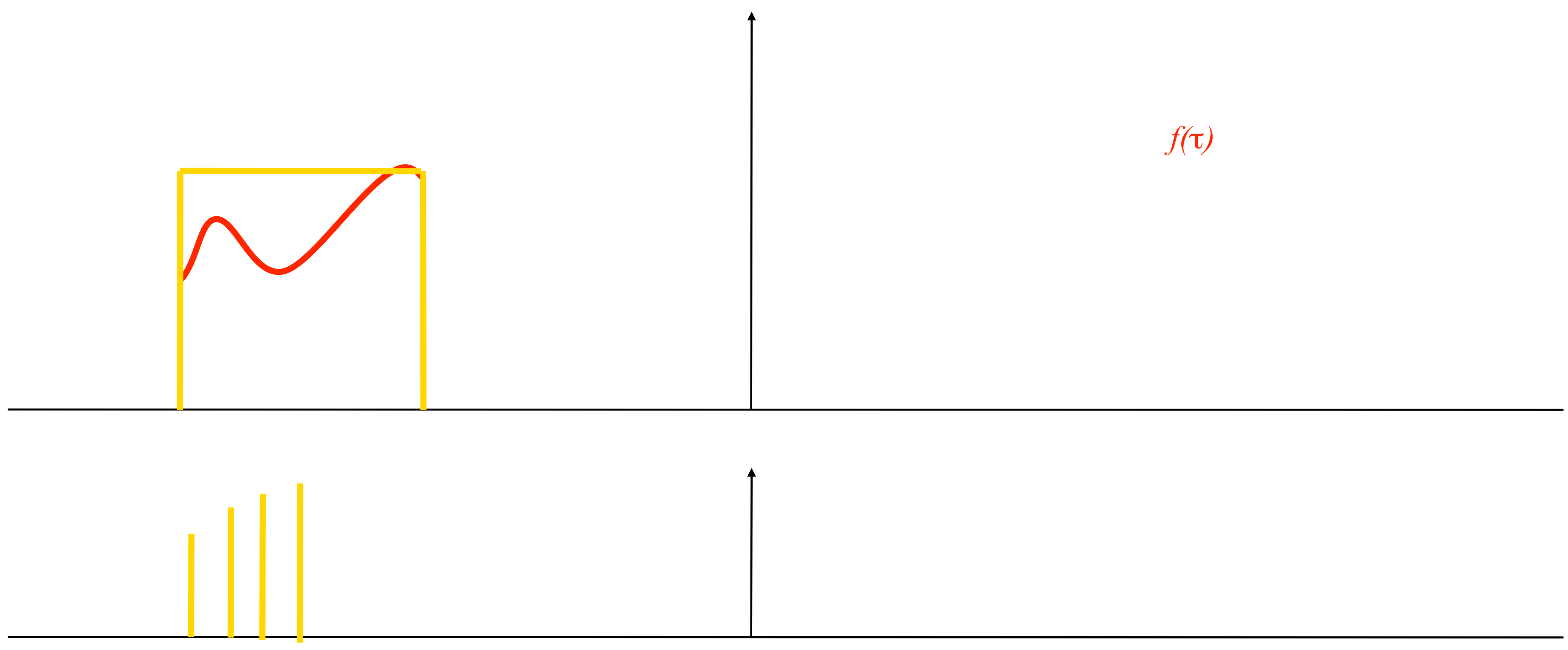

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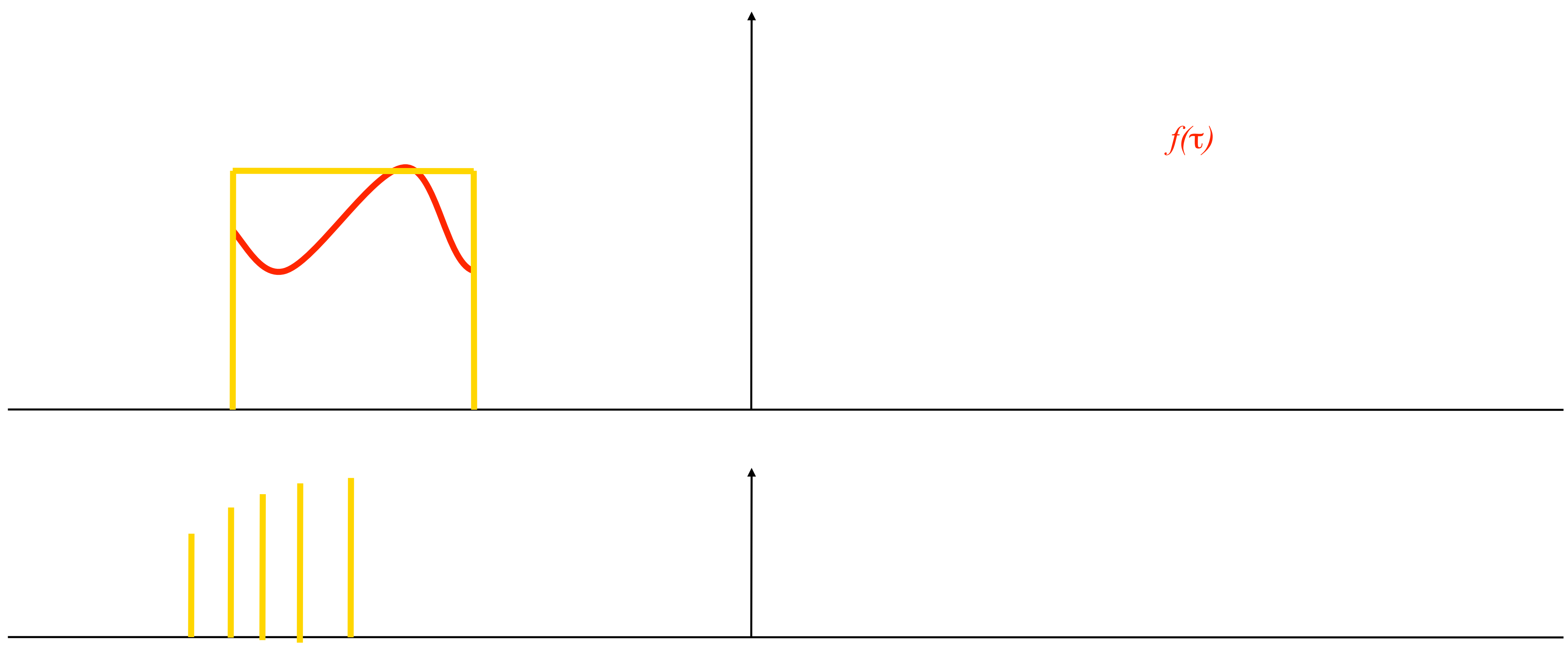

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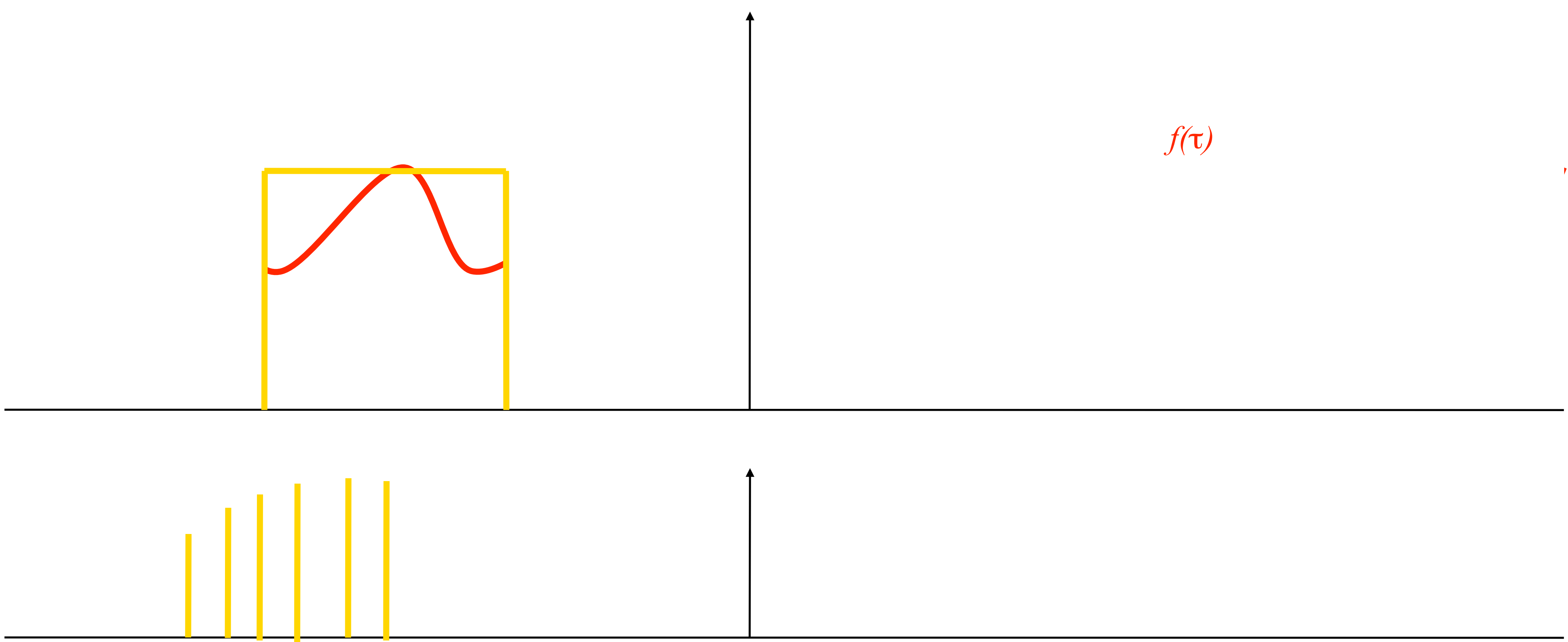

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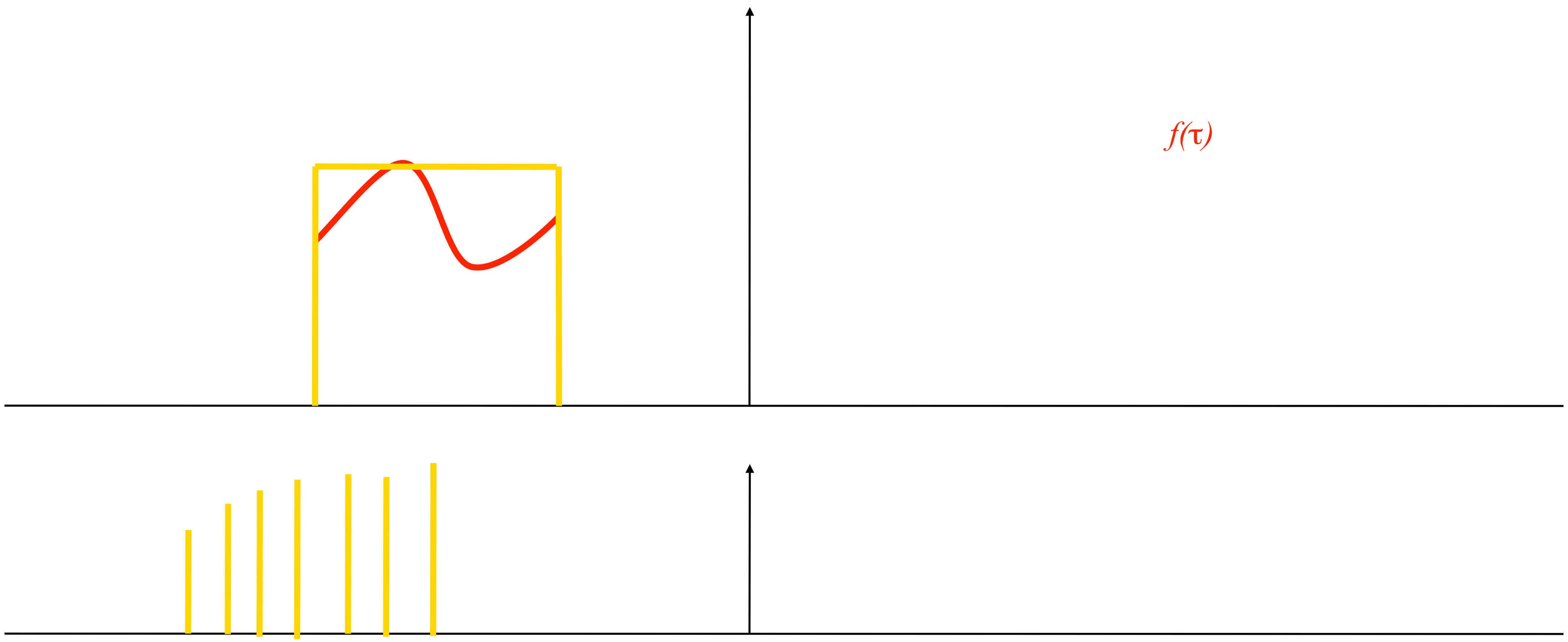

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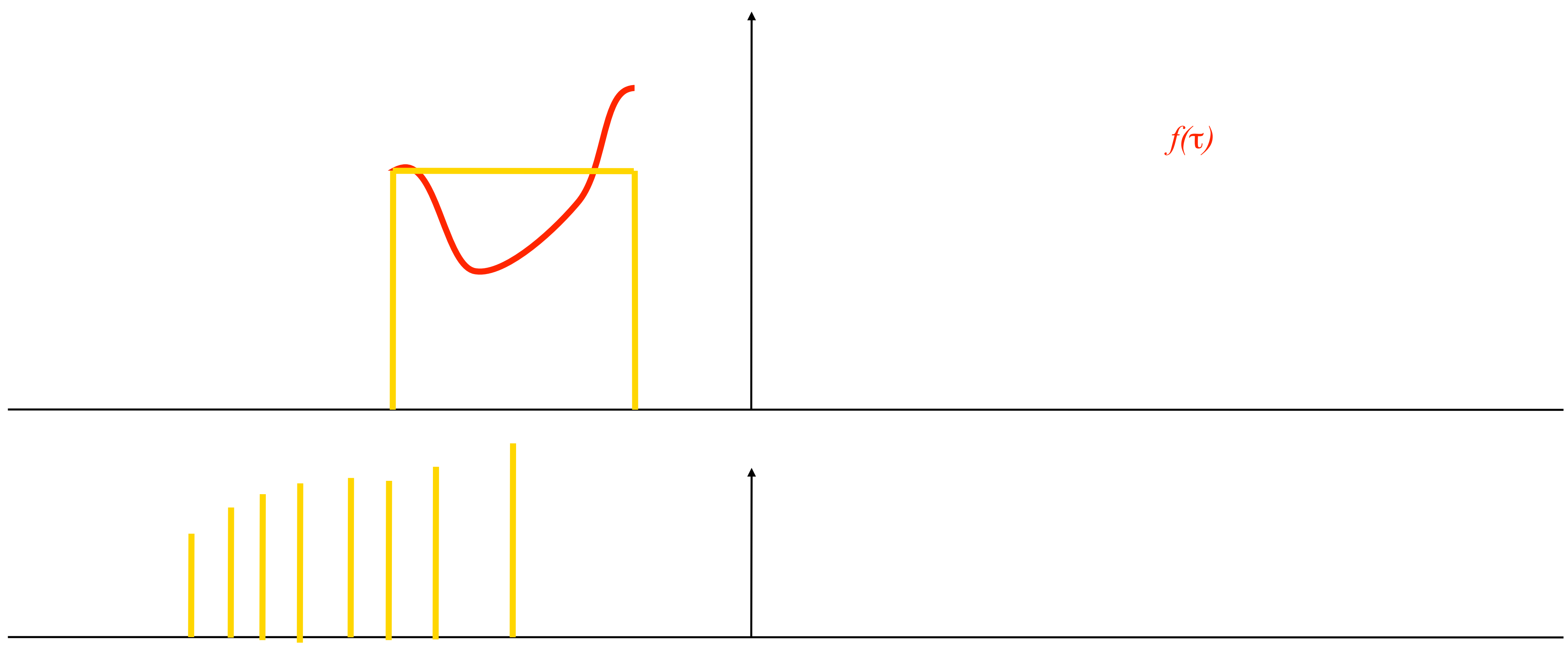

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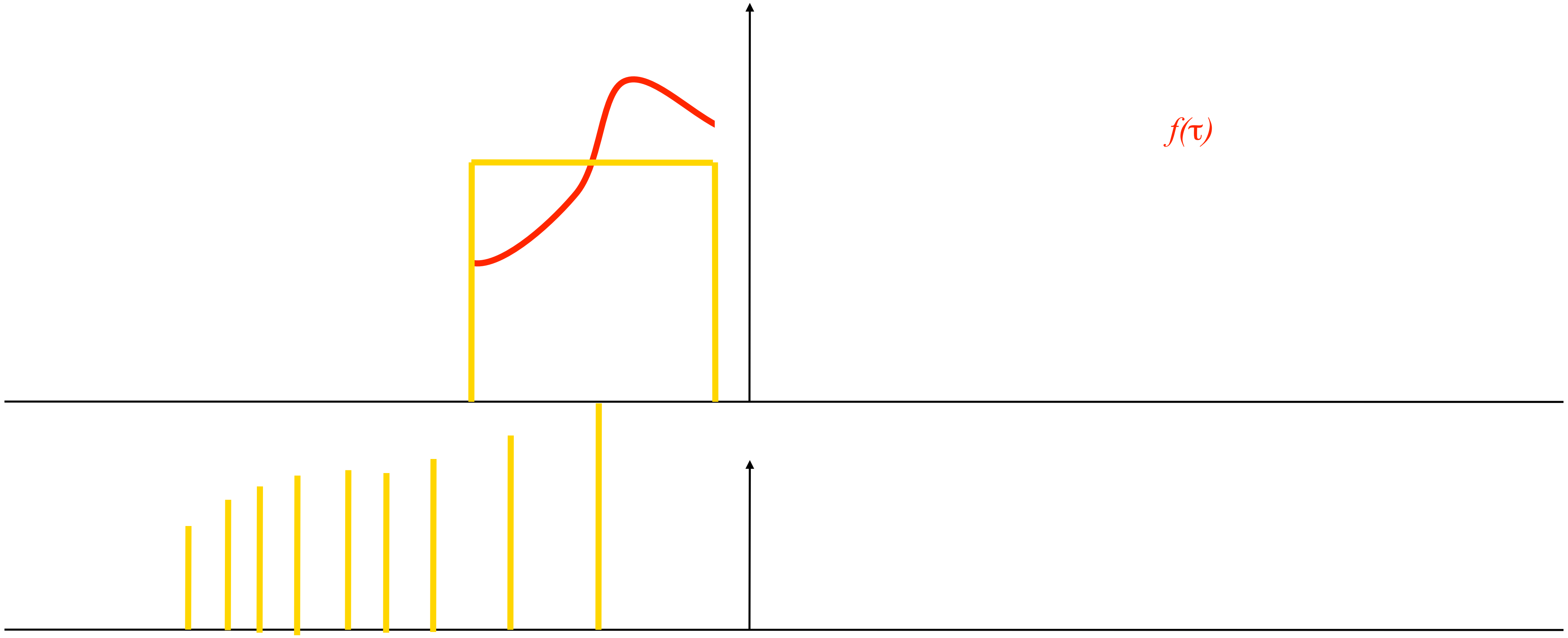

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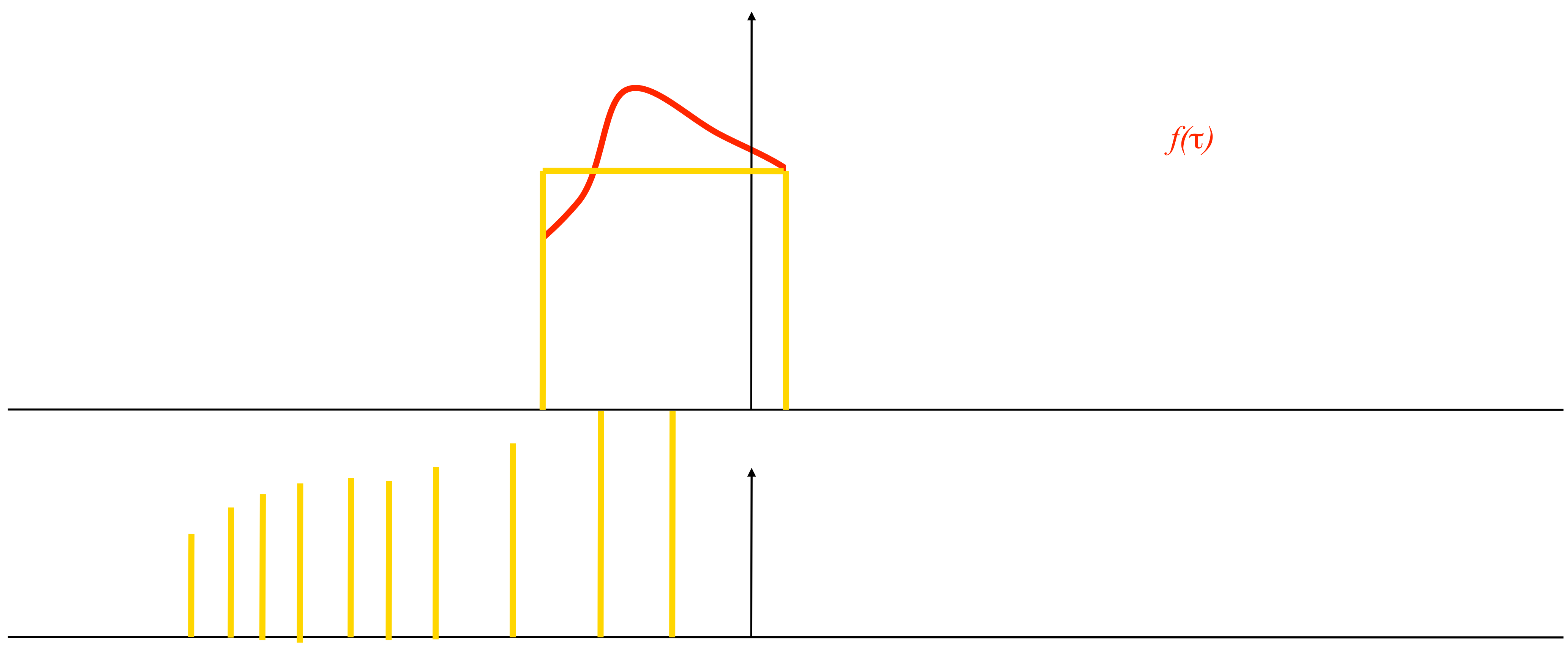

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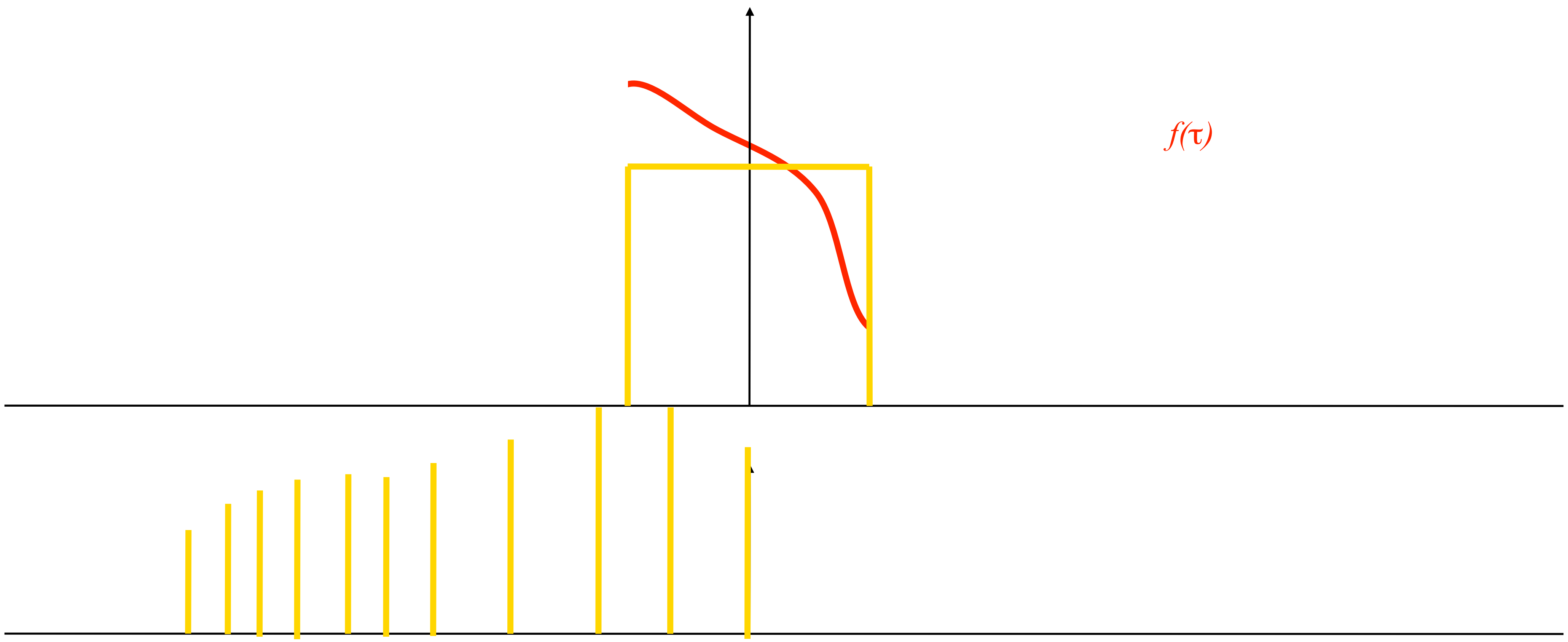

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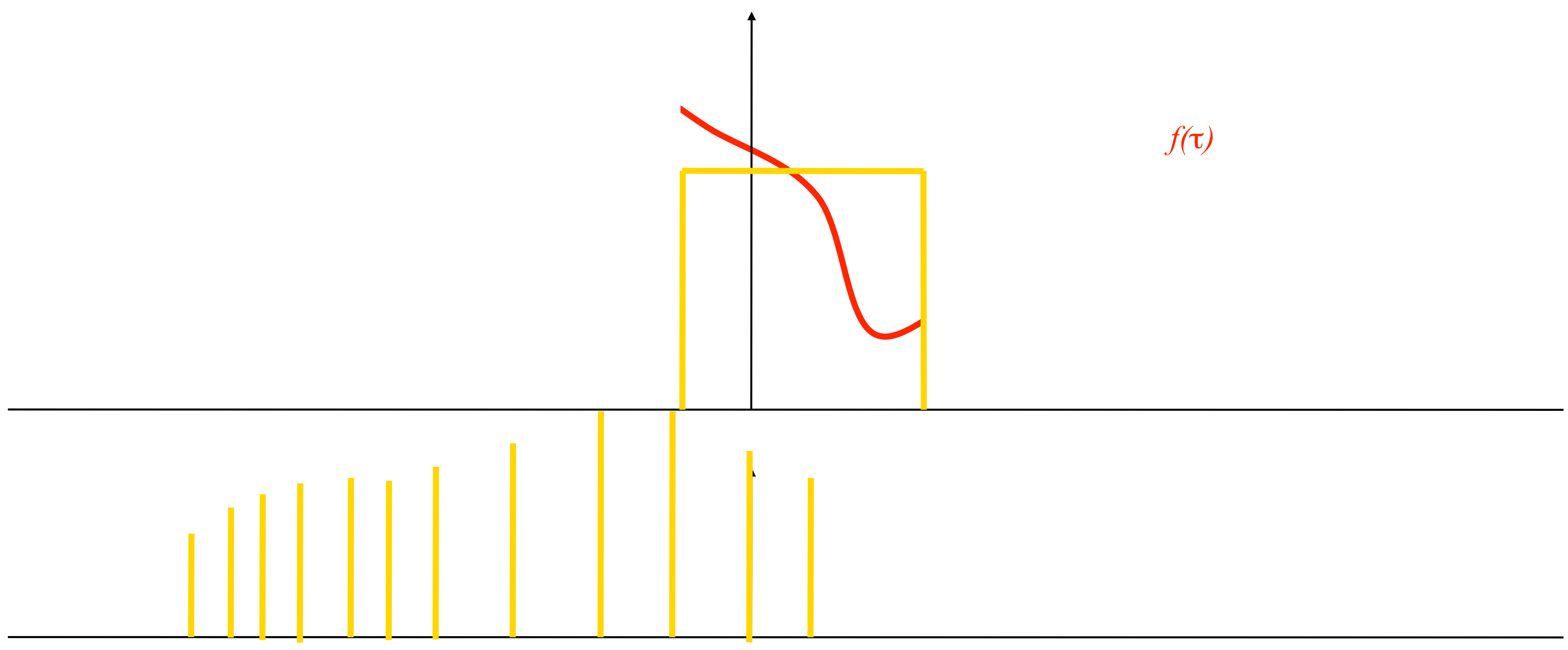

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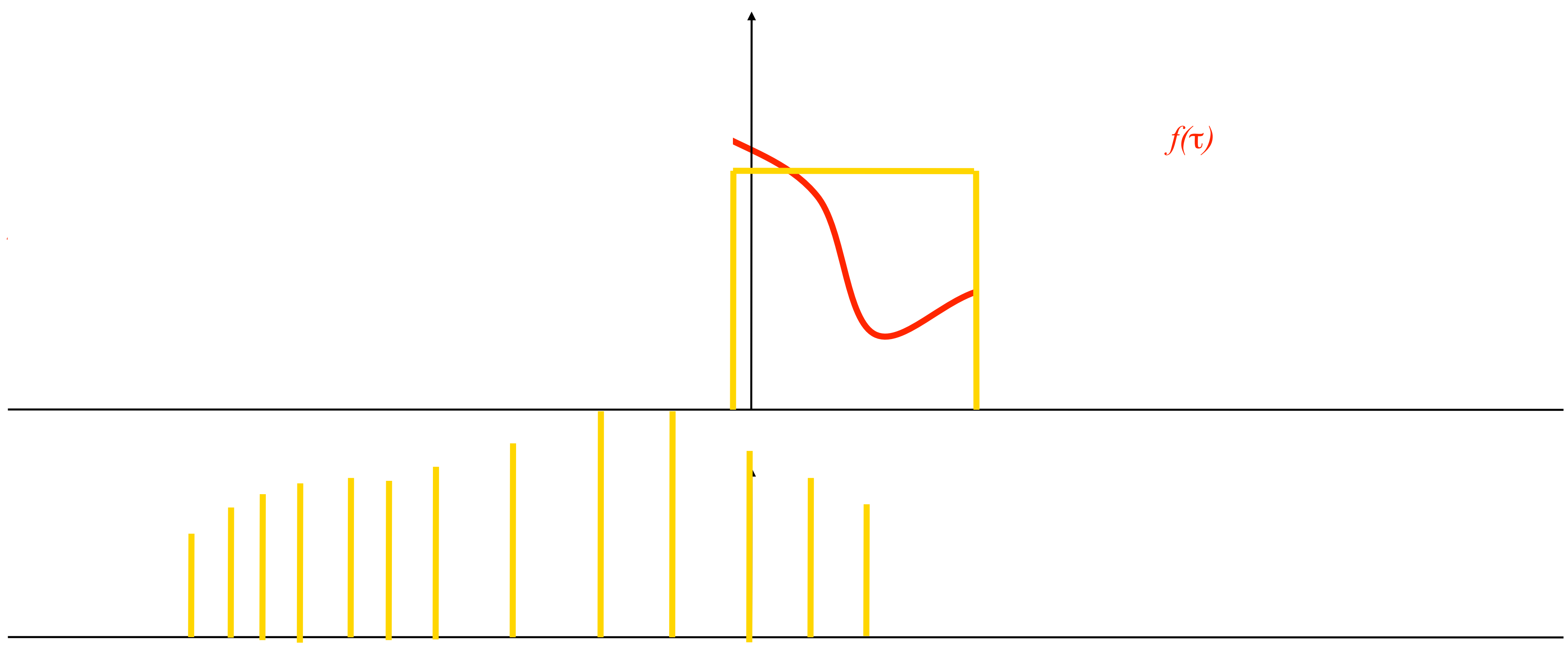

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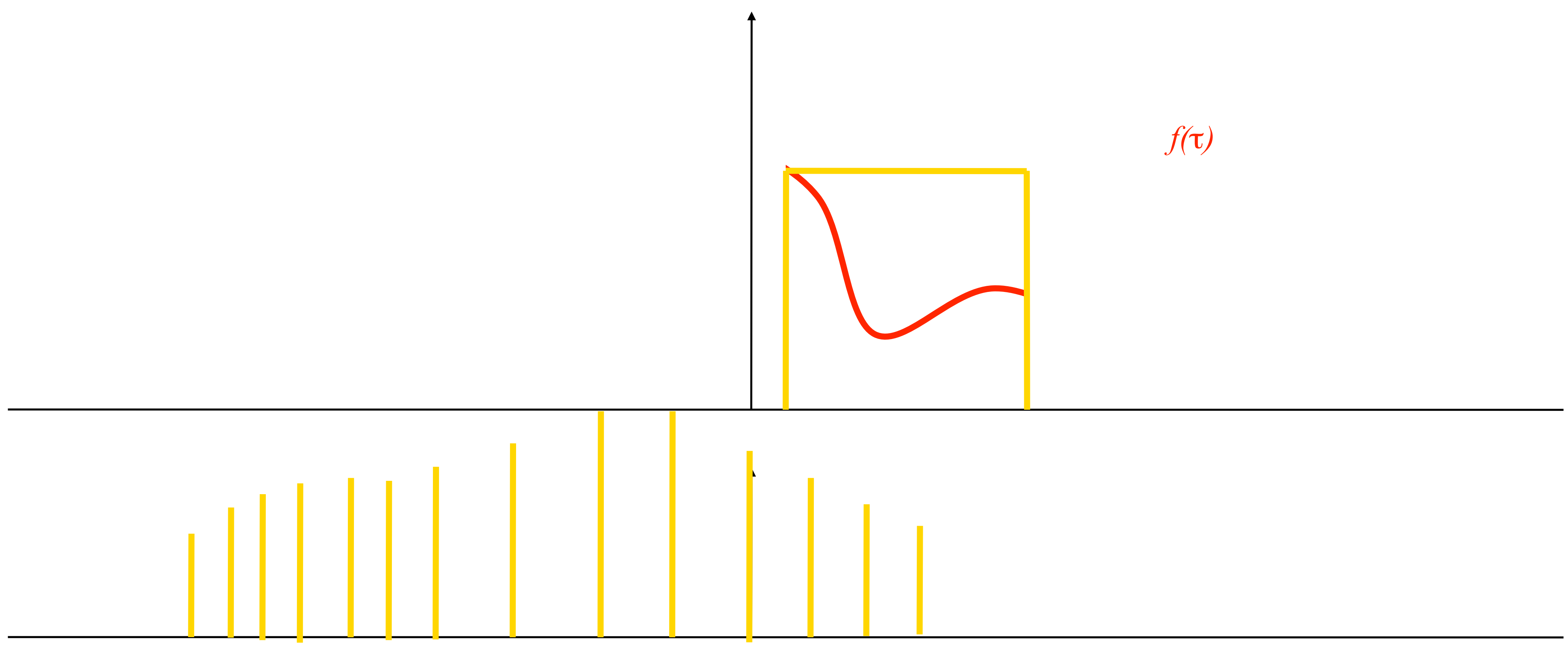

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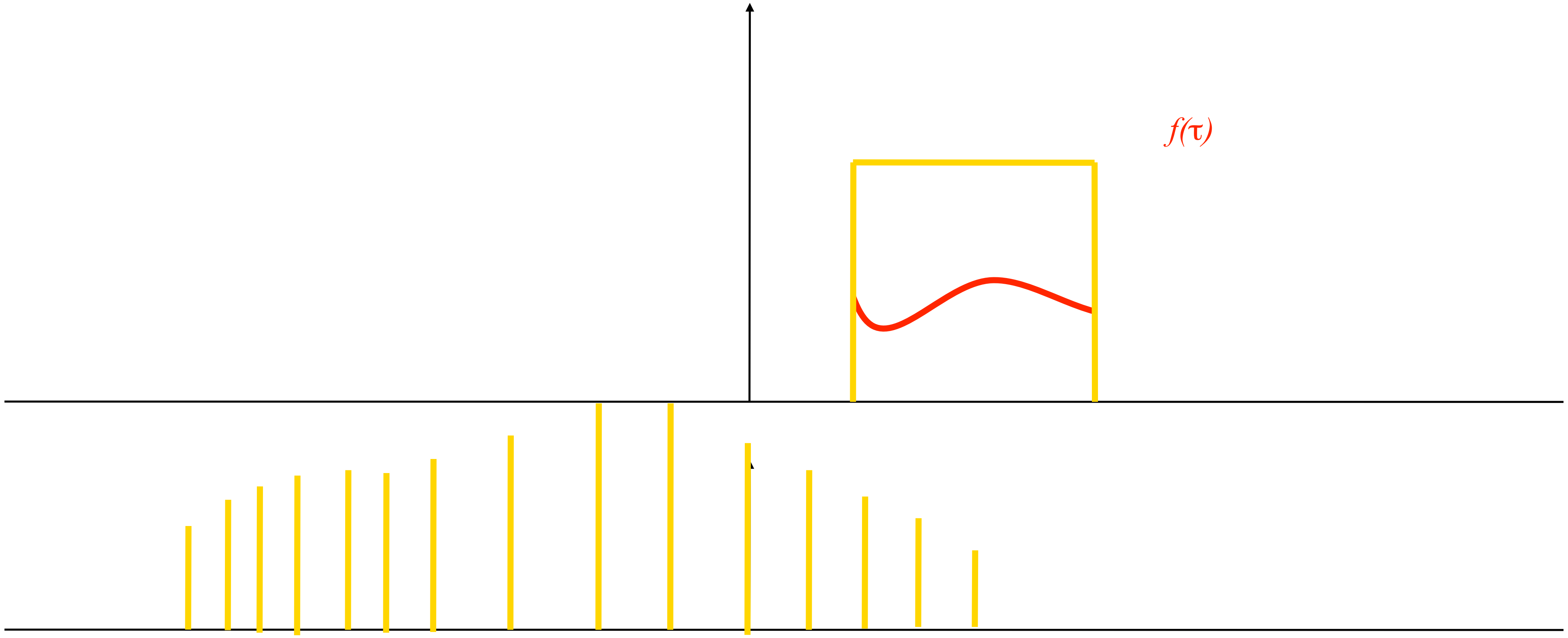

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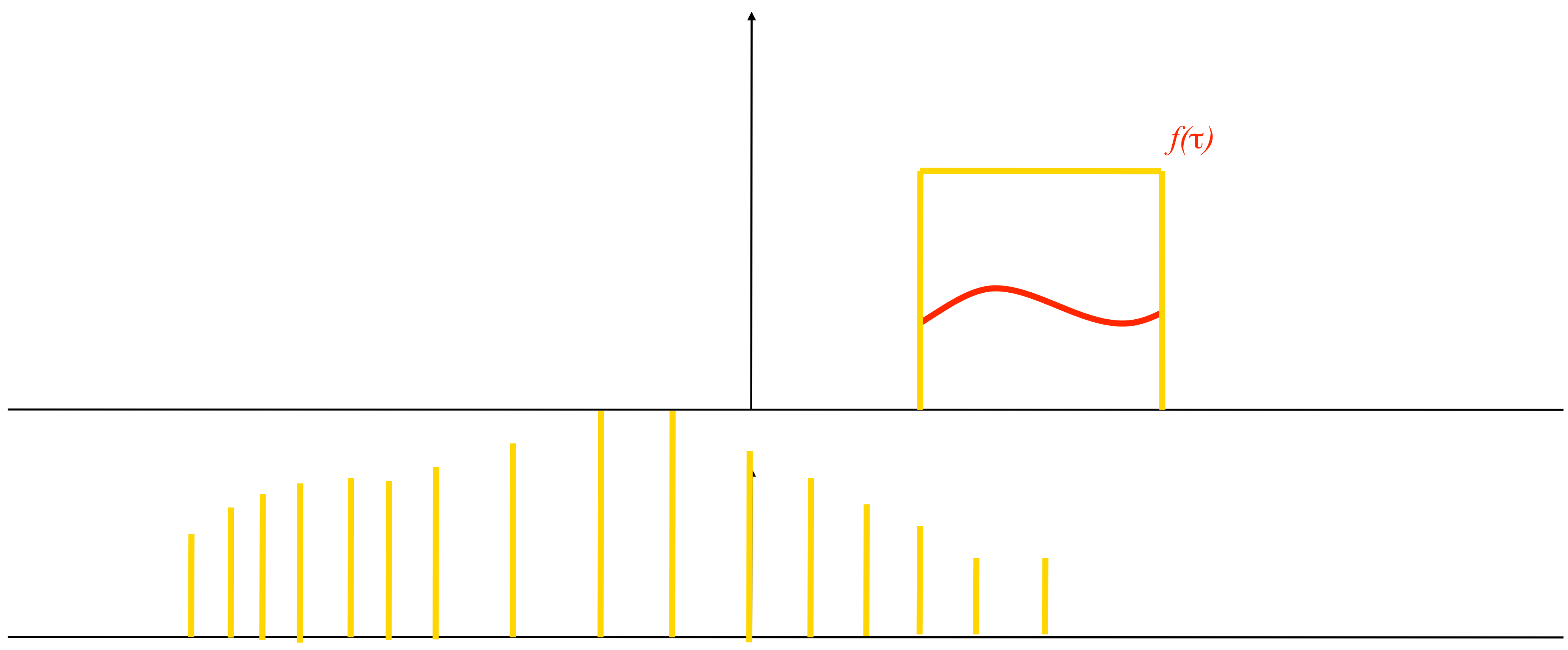

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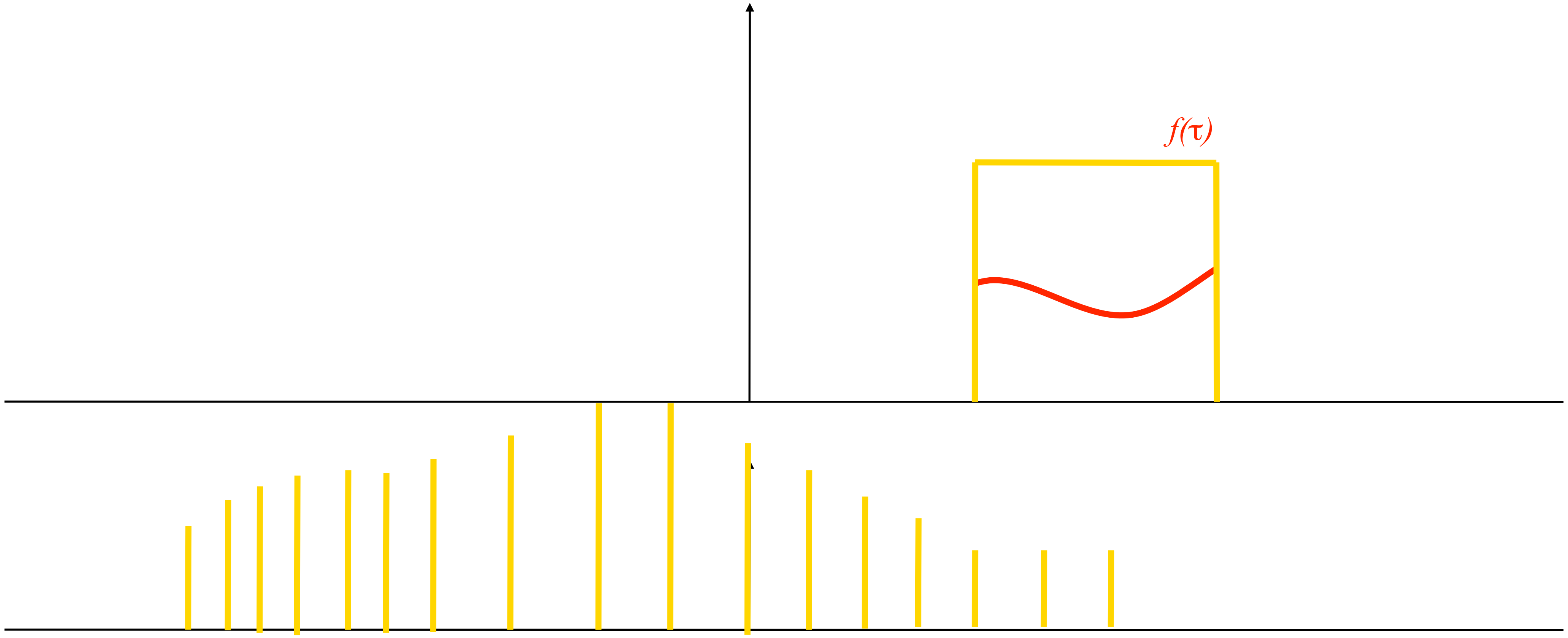

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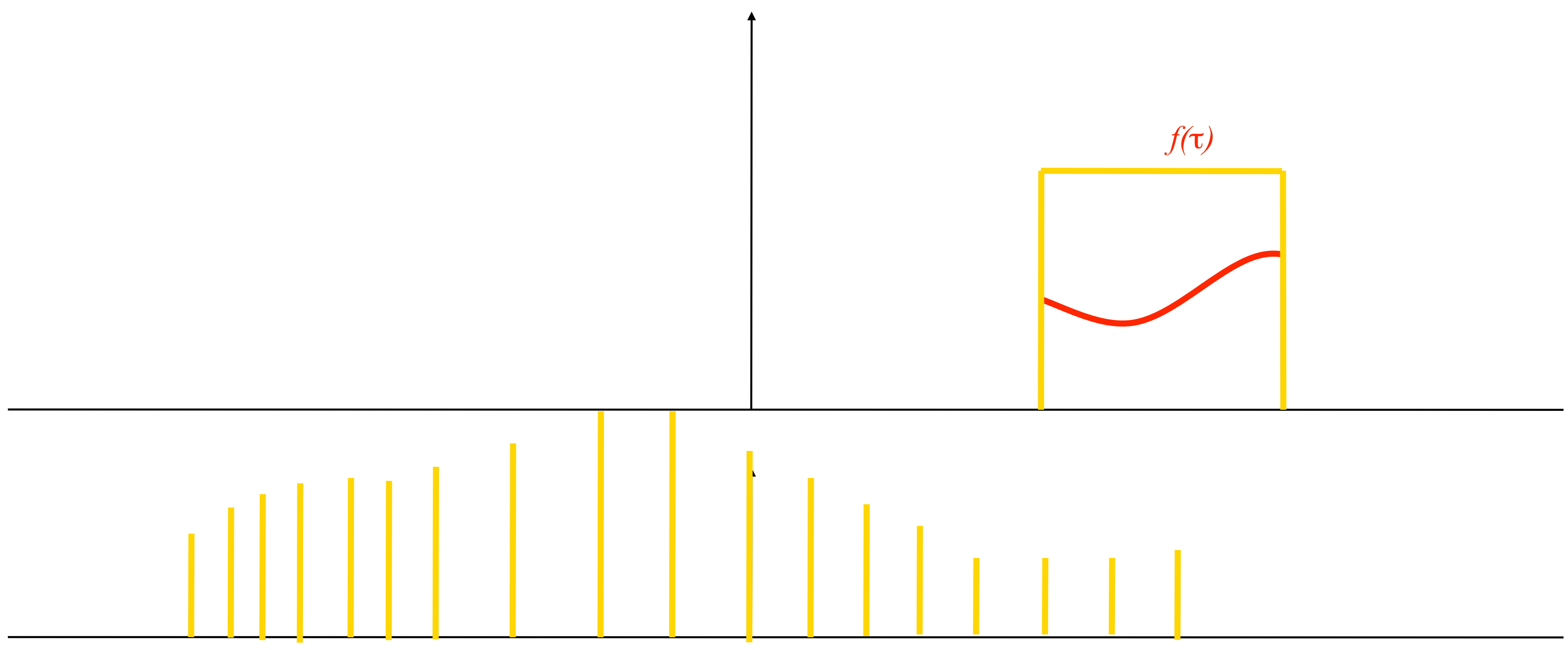

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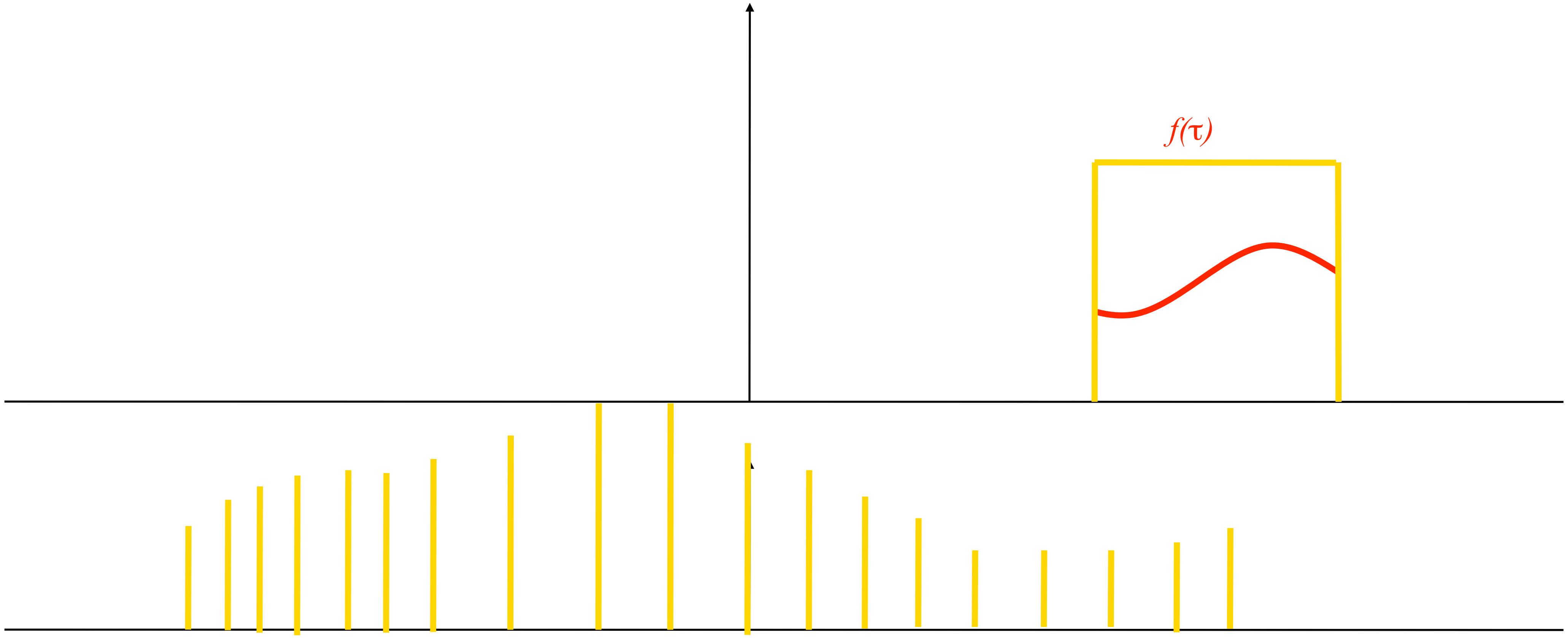
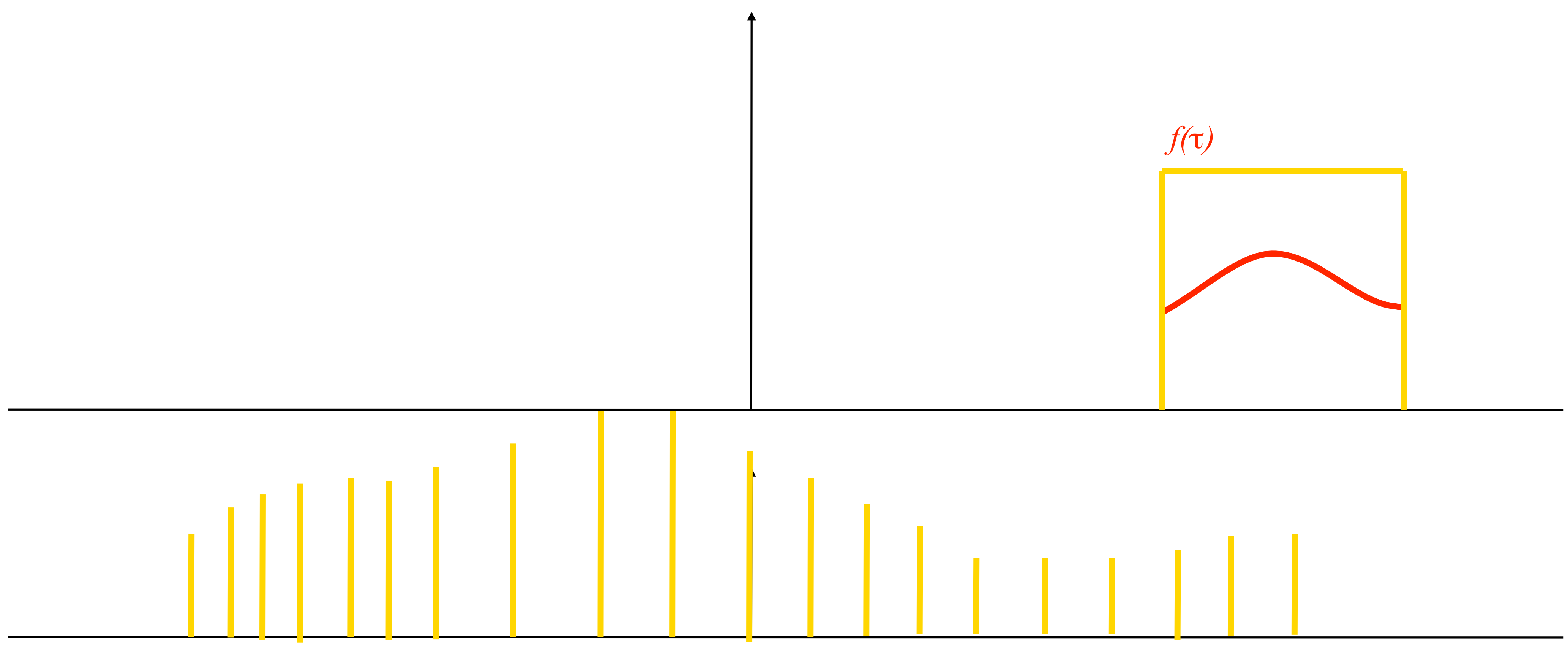


Convolution

- This function windows our function f(t)



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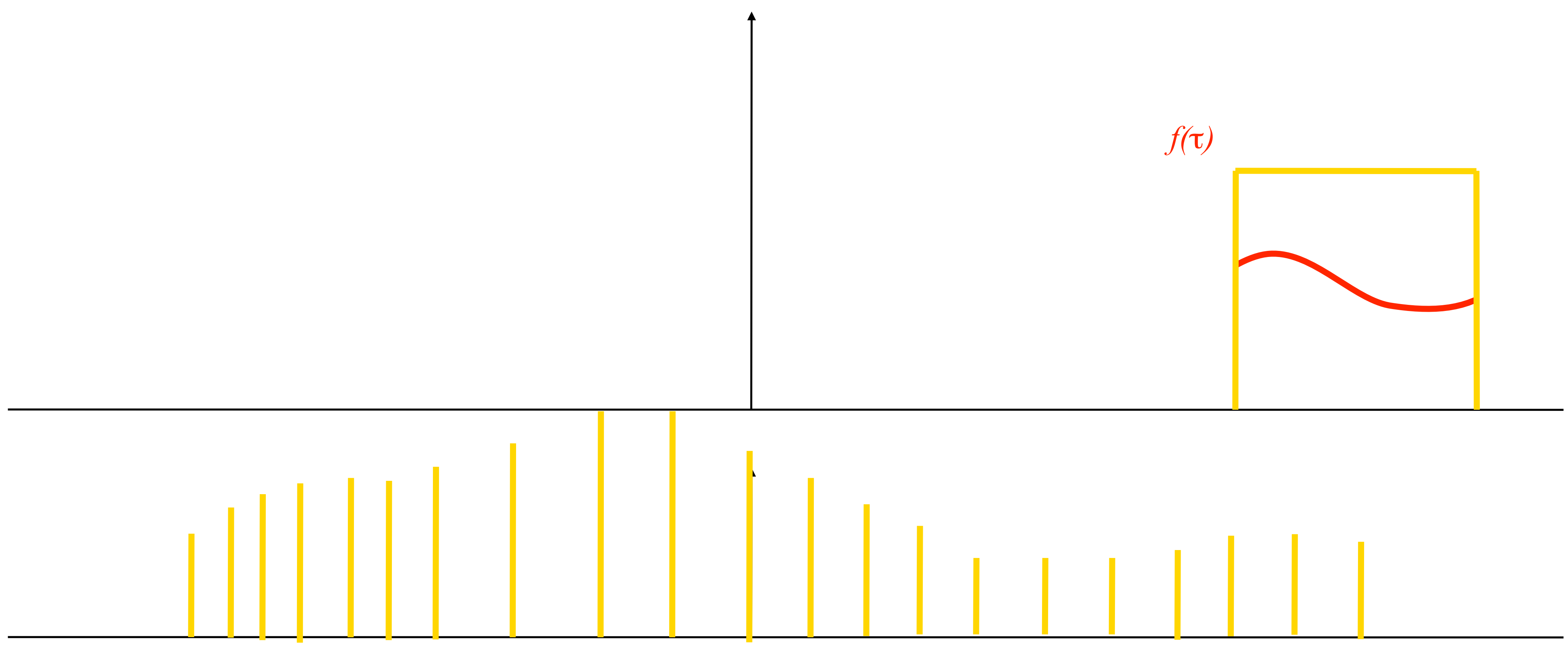

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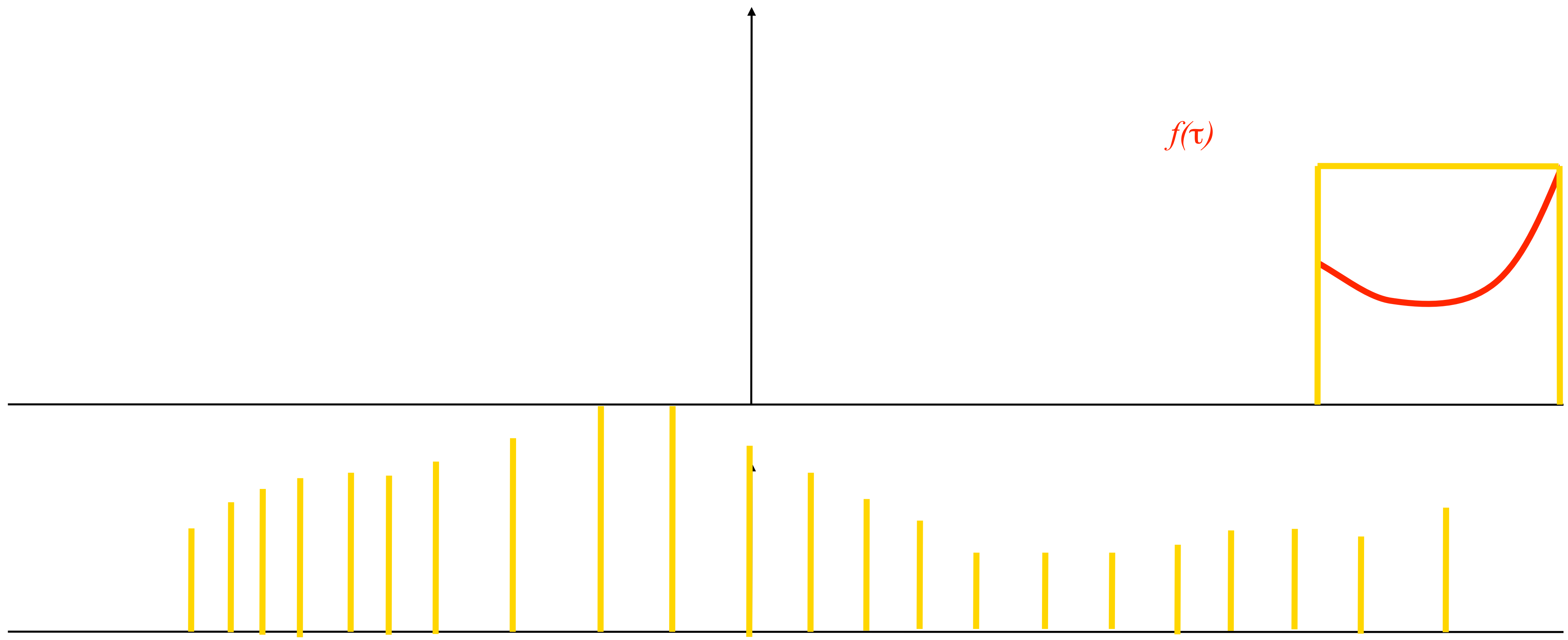
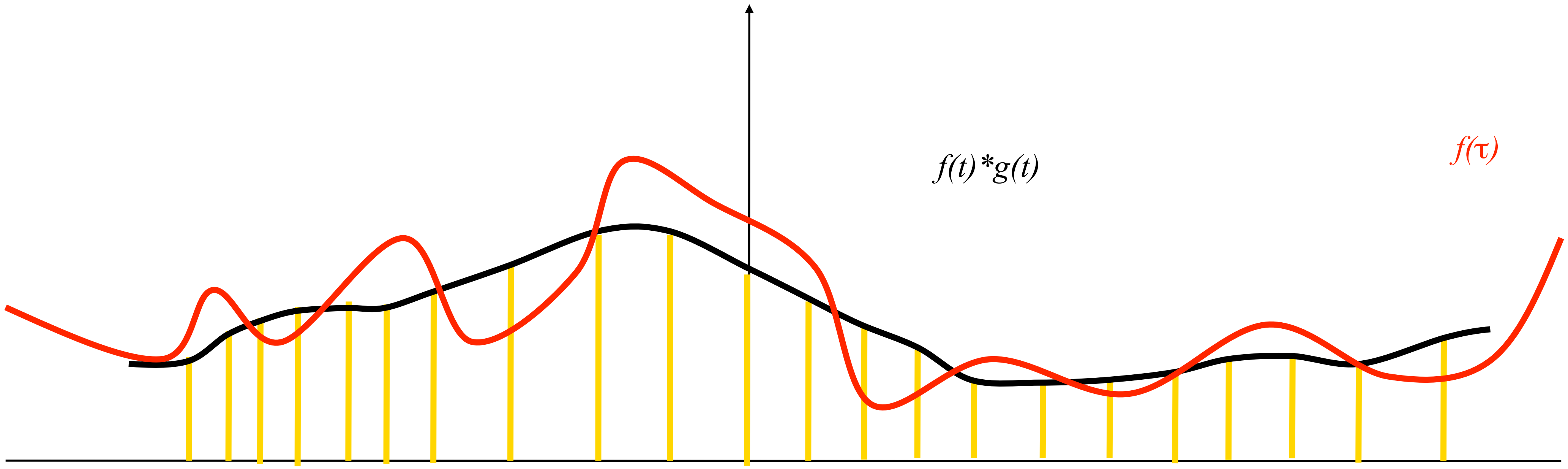
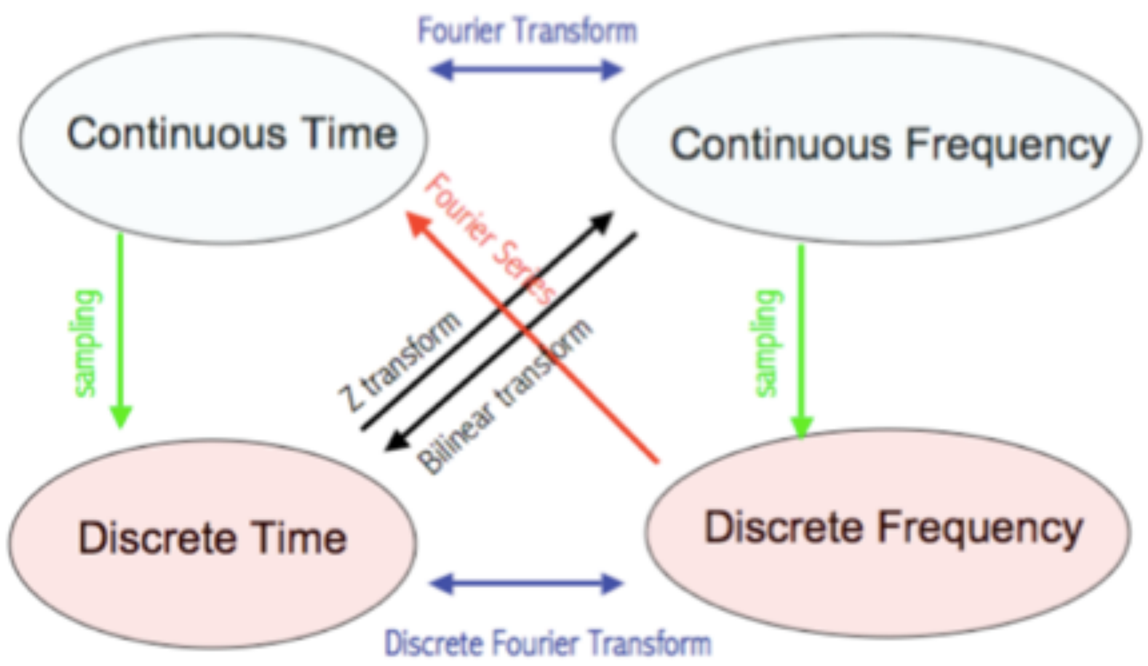


- This particular convolution smooths out some of the high frequencies in f(t).





Signal type	Continuous time	Discrete time	Transform Domain
Finite duration	Laplace	z	Continuous complex frequency (s-plane)
Finite duration	Fourier	Discrete-time Fourier (DTFT)	Continuous real frequency
Periodic	Fourier Series	Discrete Fourier Series (DFS)	Discrete real frequency

LS & FT

Figure 6.3-1: Definition of a linear system.

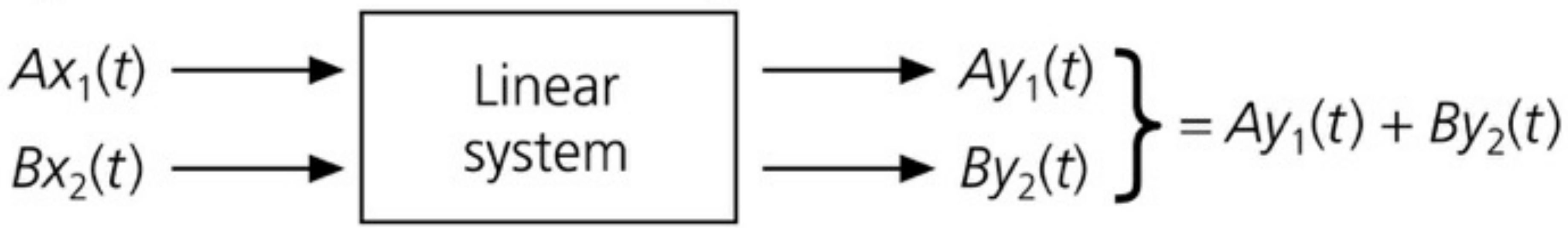
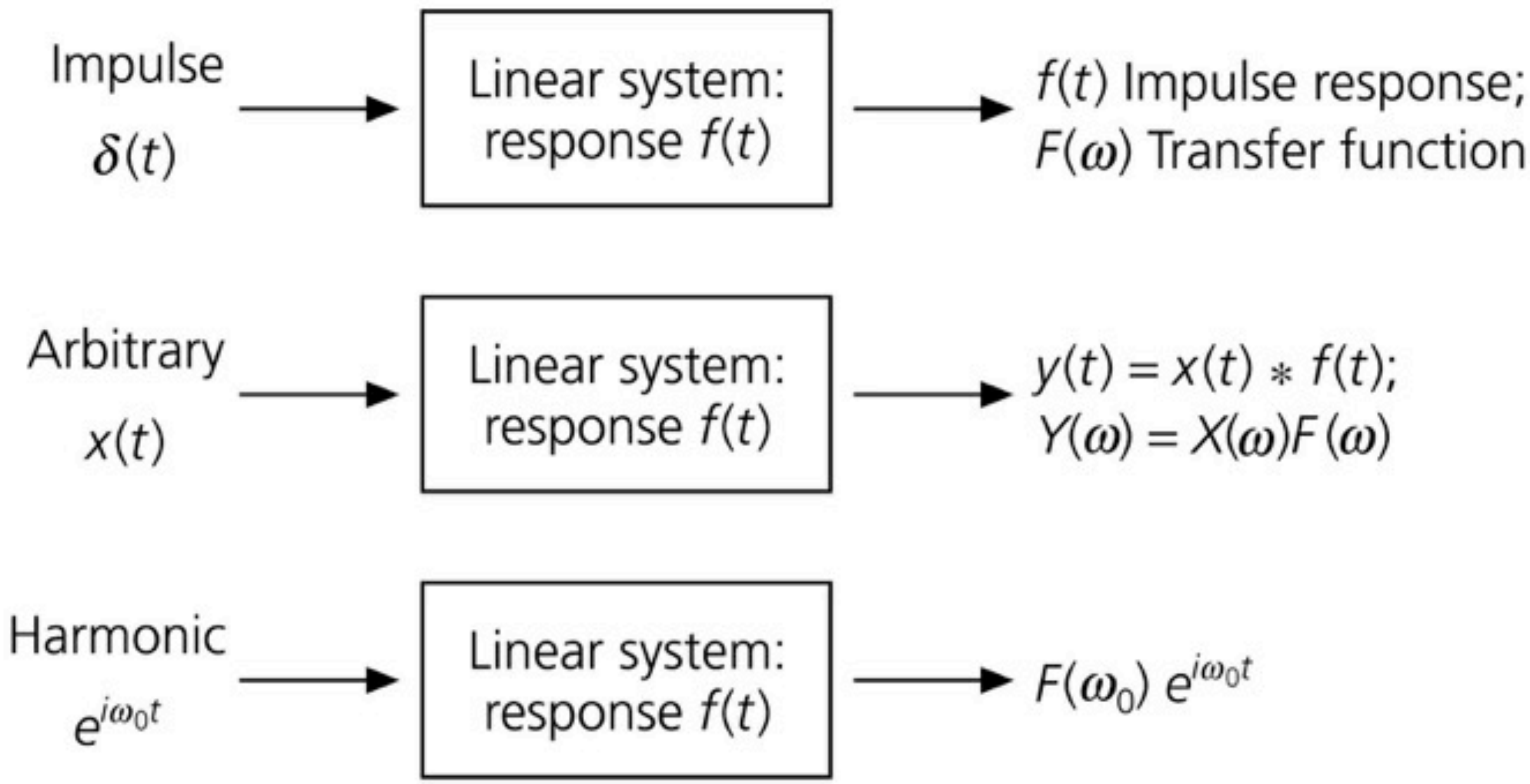


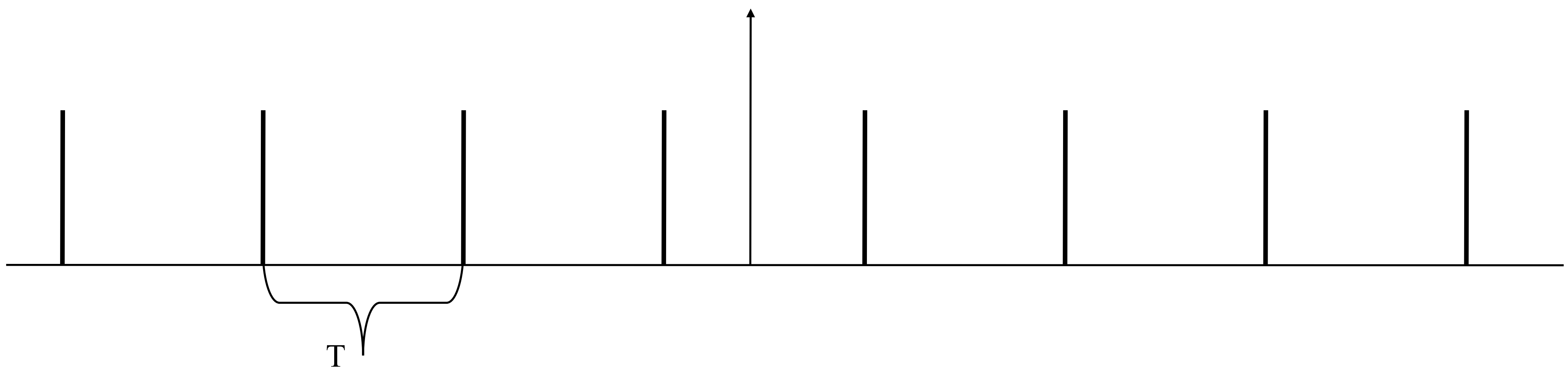
Figure 3.3-29: Seismic section before and after deconvolution.



Sampling Function

- A Sampling Function or Impulse Train is defined by:

$$S_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



Sampling Function

- The Fourier Transform of the Sampling Function is itself a sampling function.
- The sample spacing is the inverse.

$$S_T(t) \Leftrightarrow S_{\frac{1}{T}}(\omega)$$

Convolution Theorem

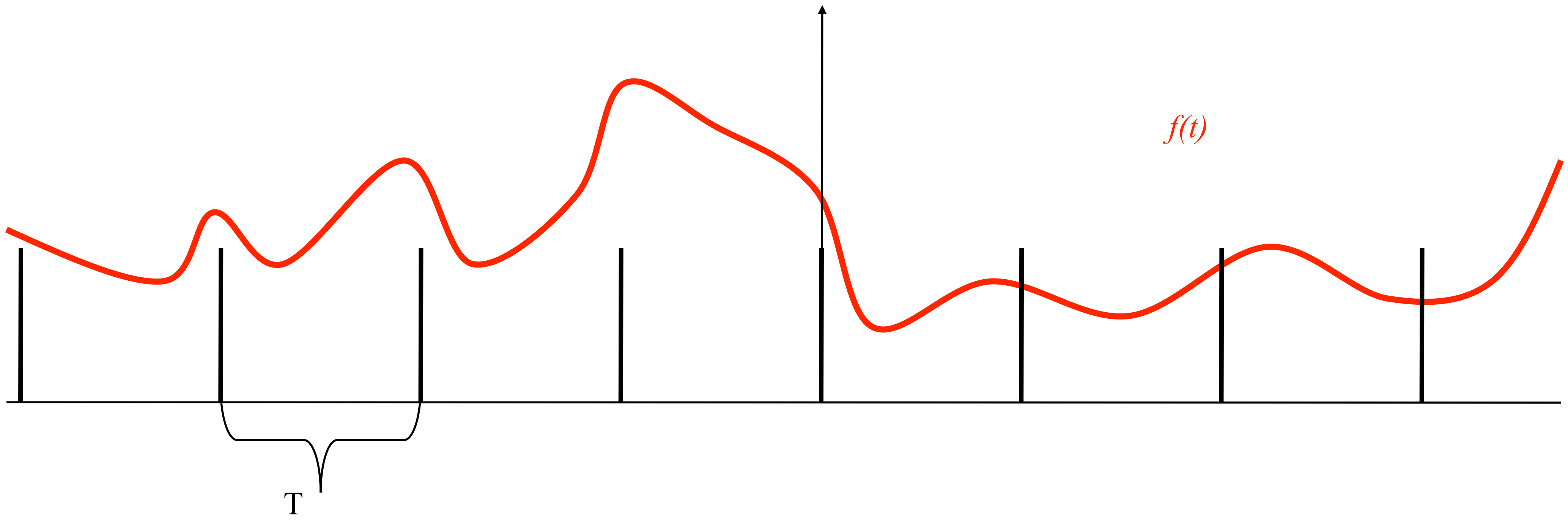
- The convolution theorem states that convolution in the spatial domain is equivalent to multiplication in the frequency domain, and viceversa.

$$f(t) * g(t) \Leftrightarrow F(\omega)G(\omega)$$

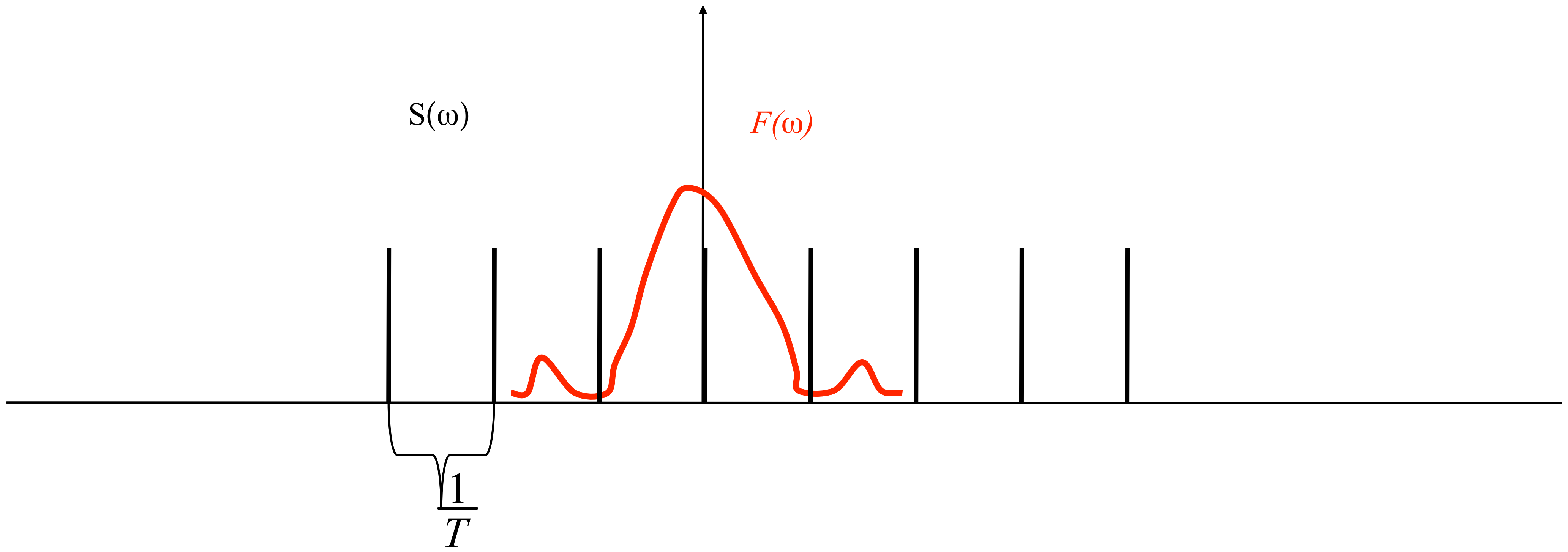
$$f(t)g(t) \Leftrightarrow F(\omega) * G(\omega)$$

Convolution Theorem

- This powerful theorem can illustrate the problems with our point sampling and provide guidance on avoiding aliasing.
- Consider: $f(t)\, S_T(t)$

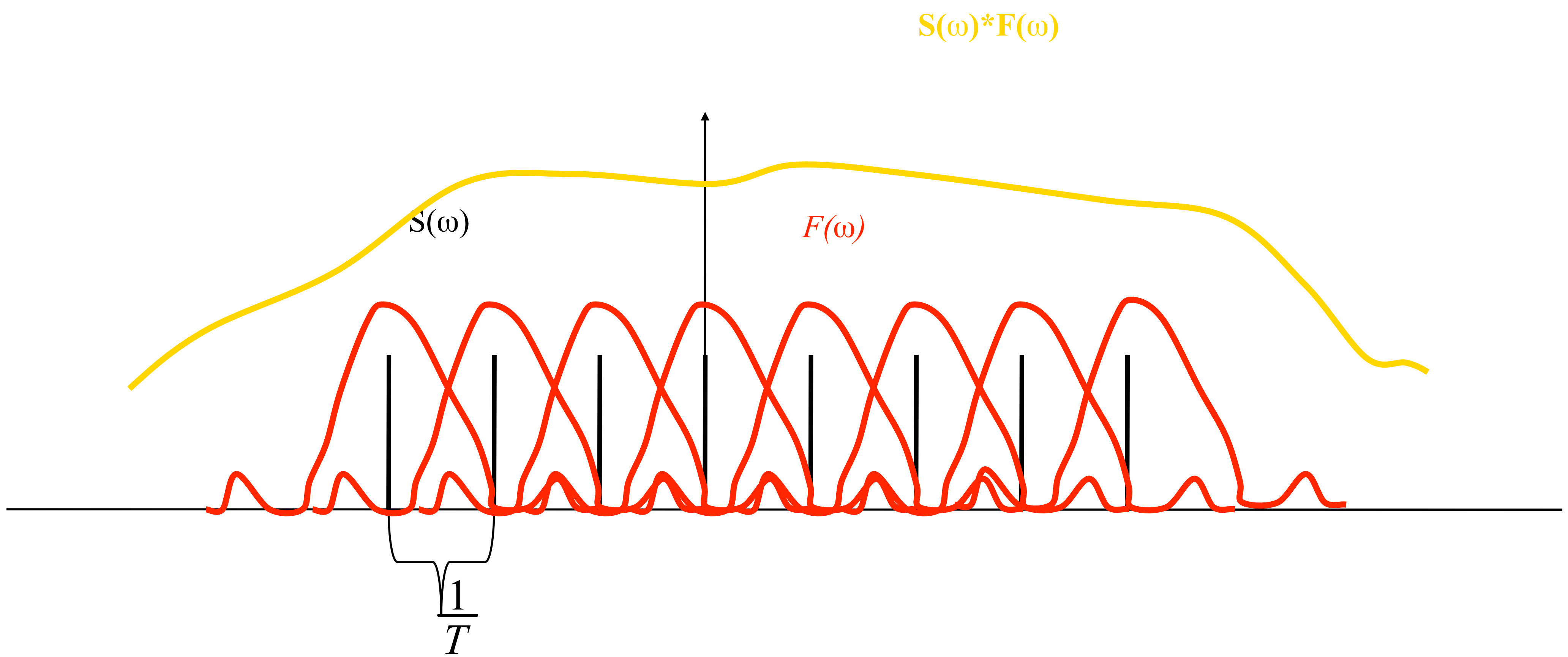


- What does this look like in the Fourier domain?



Convolution Theorem

- In Fourier domain we would convolve



Aliasing

- What this says, is that any frequencies greater than a certain amount will appear intermixed with other frequencies.
- In particular, the higher frequencies for the copy at 1/T intermix with the low frequencies centered at the origin.

Aliasing and Sampling

- Note, that the sampling process introduces frequencies out to infinity.
- We have also lost the function f(t), and now have only the discrete samples.
- This brings us to our next powerful theory.

- The Shannon Sampling Theorem
- A band-limited signal f(t), with a cutoff frequency of λ, that is sampled with a sampling spacing of T may be perfectly reconstructed from the discrete values f[nT] by convolution with the sinc(t) function, provided:

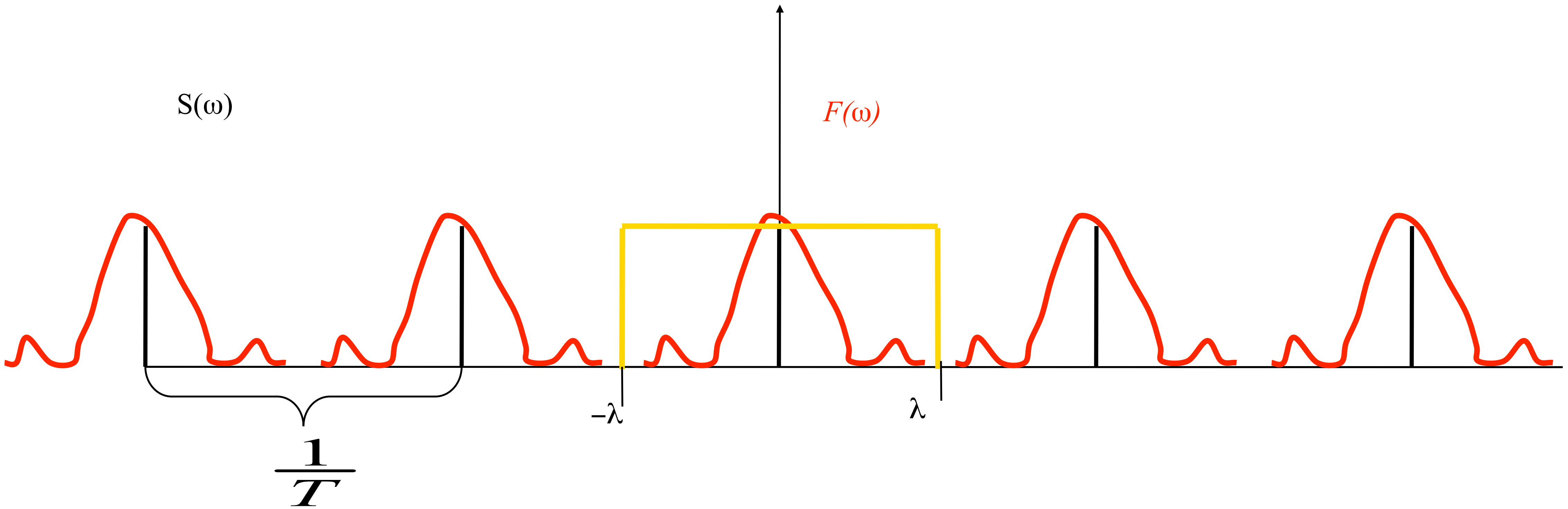
$$\lambda < \frac{1}{2T}$$

Sampling Theory

- Why is this?
- The Nyquist limit will ensure that the copies of $F(\omega)$ do not overlap in the frequency domain.
- I can completely reconstruct or determine $f(t)$ from $F(\omega)$ using the Inverse Fourier Transform.

Sampling Theory

- In order to do this, I need to remove all of the shifted copies of F(ω) first.
- This is done by simply multiplying F(ω) by a box function of width 2λ.



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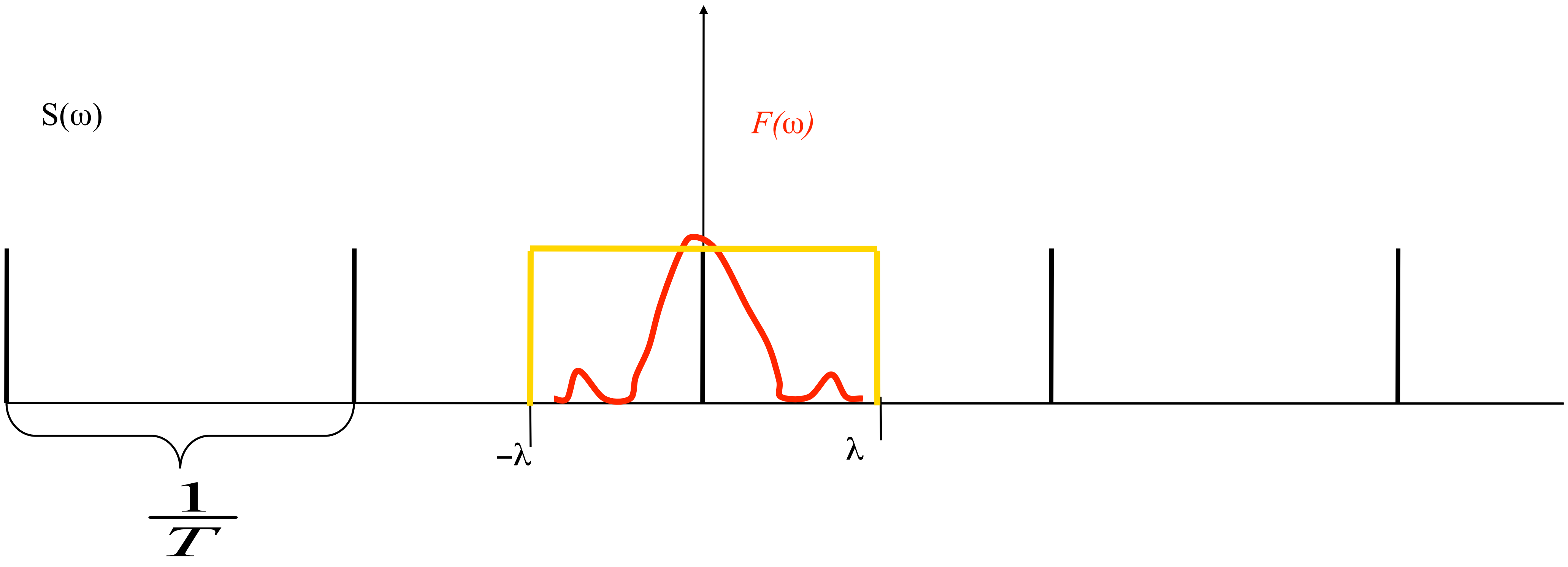
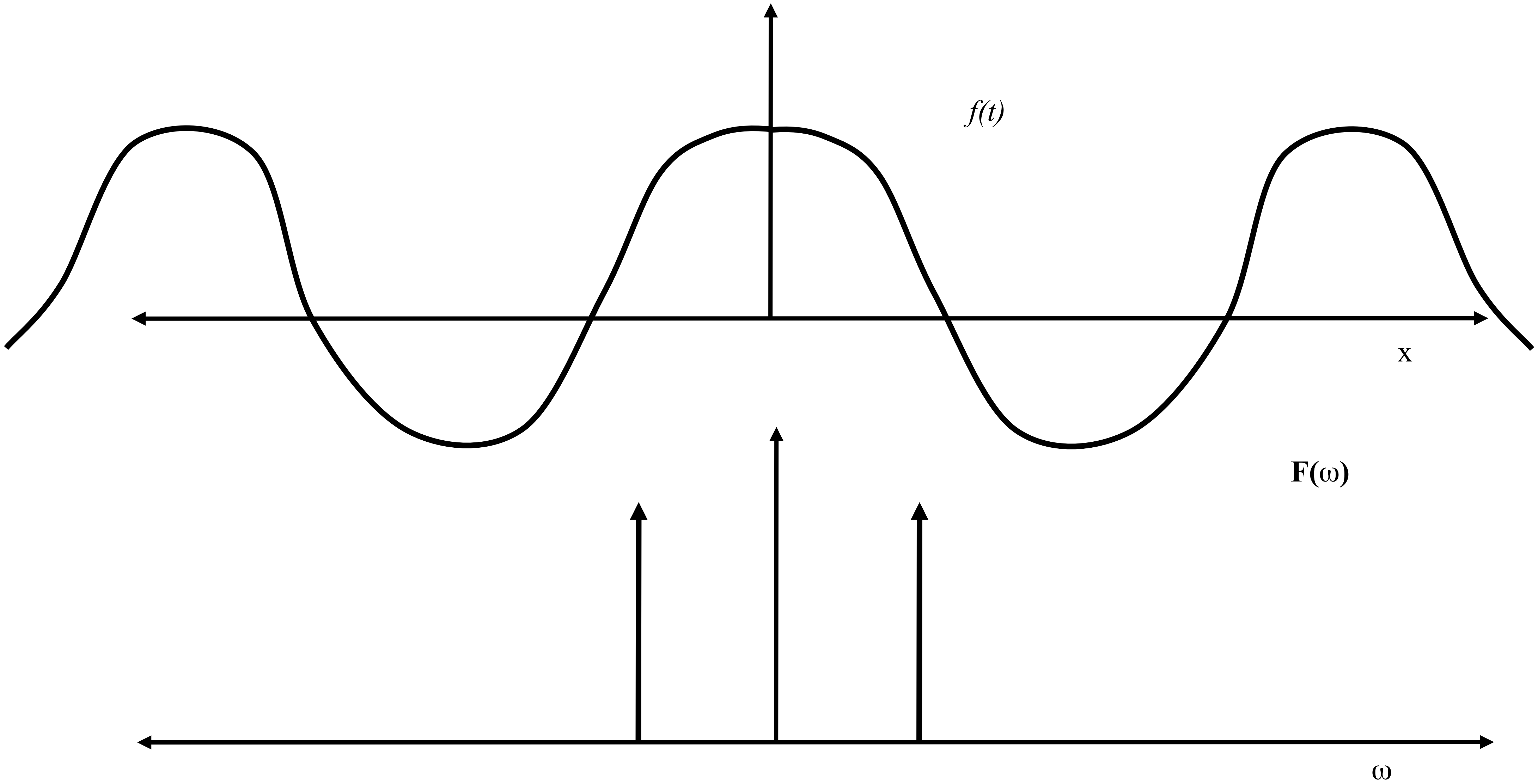


A Cosine Example

- Consider the function $f(t)=\cos(2\pi t)$.



Sampling Theory

- So, given f[nT] and an assumption that f(t) does not have frequencies greater than 1/2T, we can write the formula:

$$f[nT] = f(t)\, S_T(t) \Leftrightarrow F(\omega) * S_T(\omega)$$

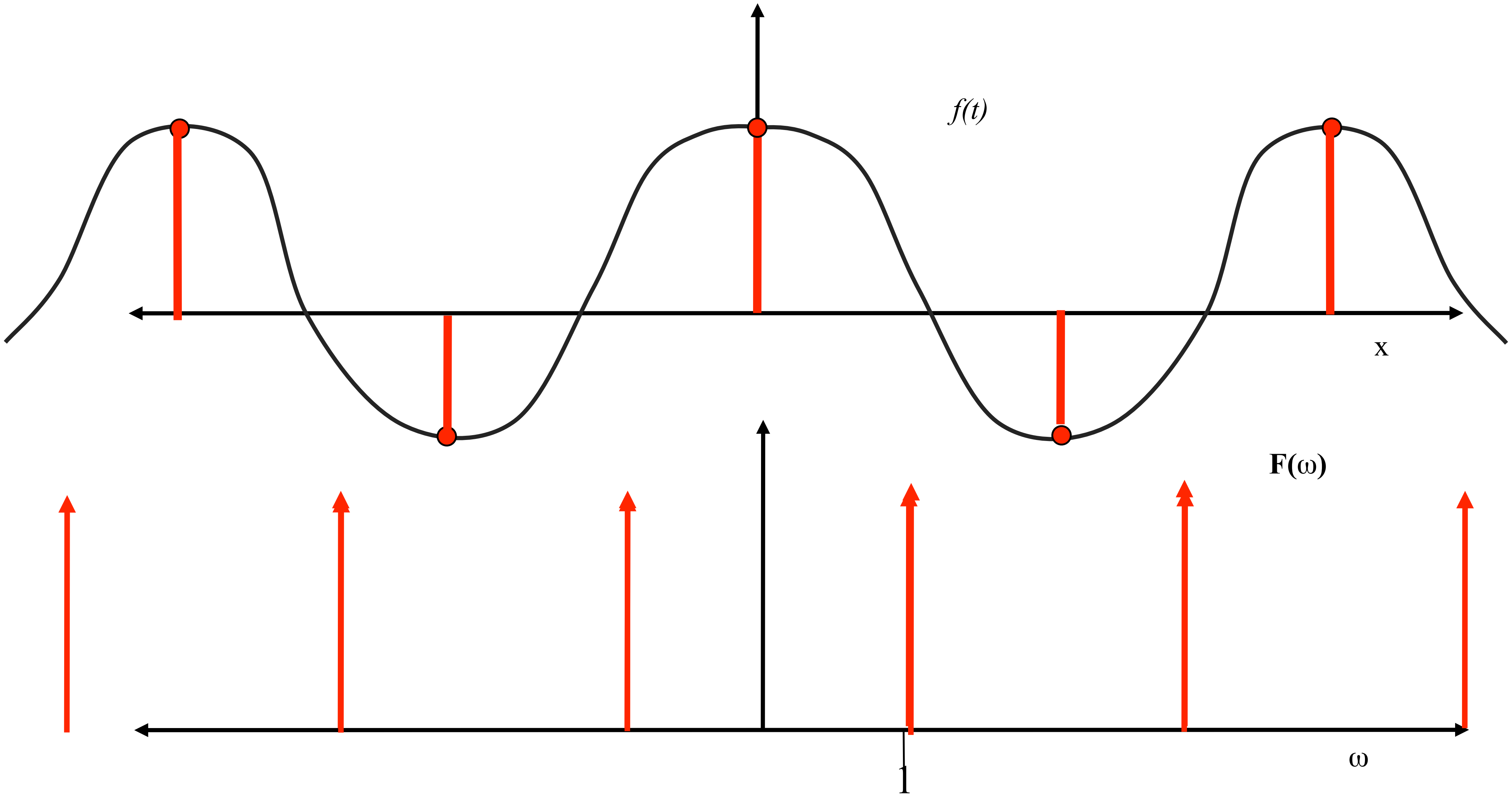
$$F(\omega) = (F(\omega) * S_T(\omega))\, Box_{1/2T}(\omega)$$

therefore,

$$f(t) = f[nT] * \operatorname{sinc}(t)$$

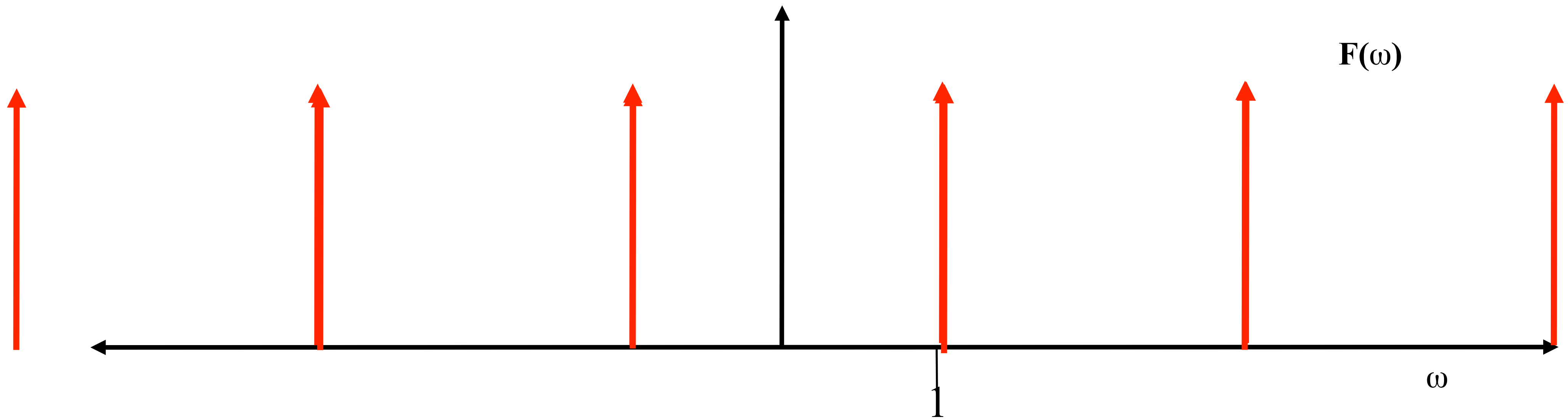
A Cosine Example

- Now sample it at T=1/2



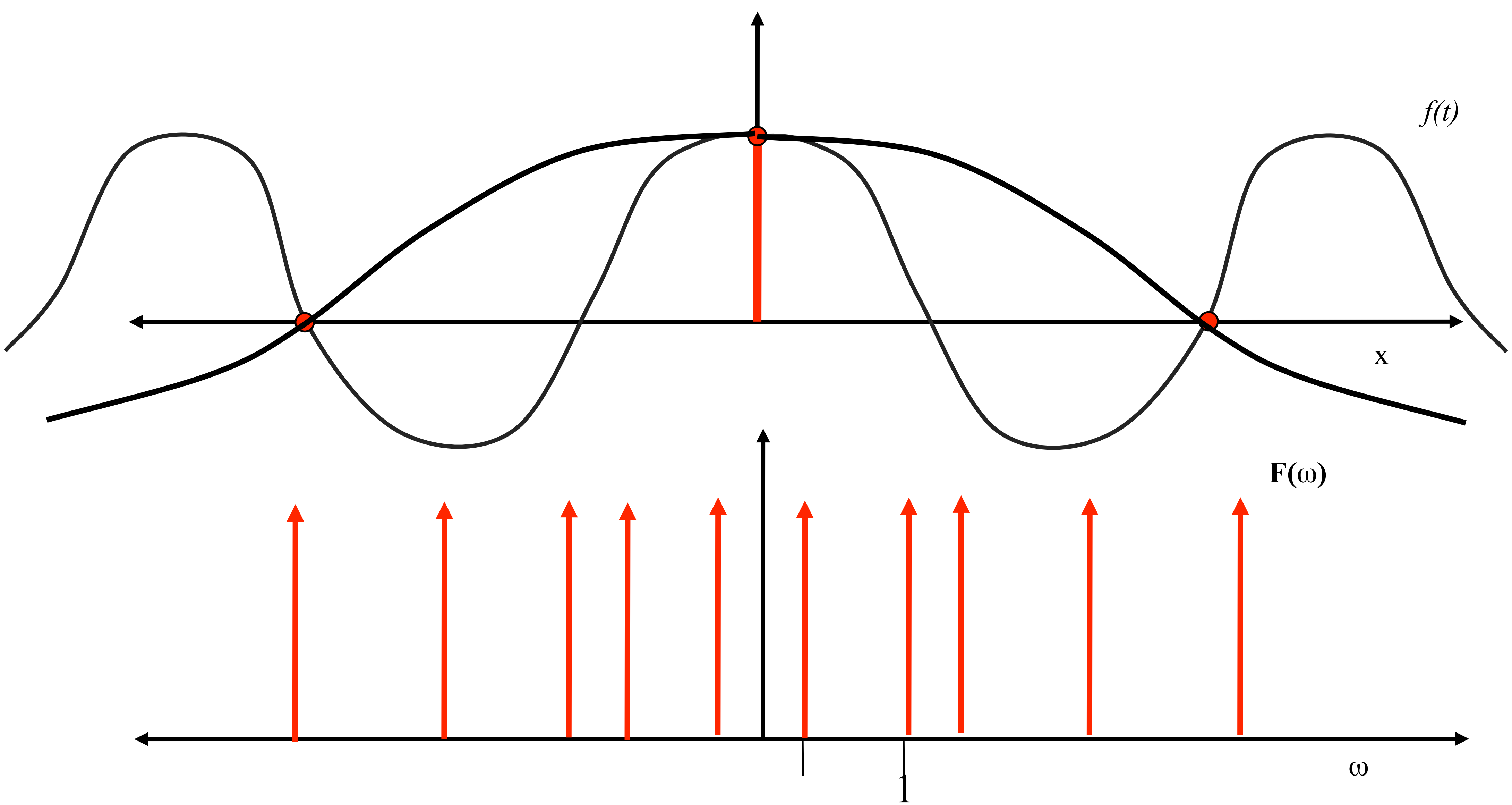
A Cosine Example

- Problem:
- The amplitude is now wrong or undefined.
- Note however, that there is one and only one cosine with a frequency less than or equal to 1 that goes through the sample pts.

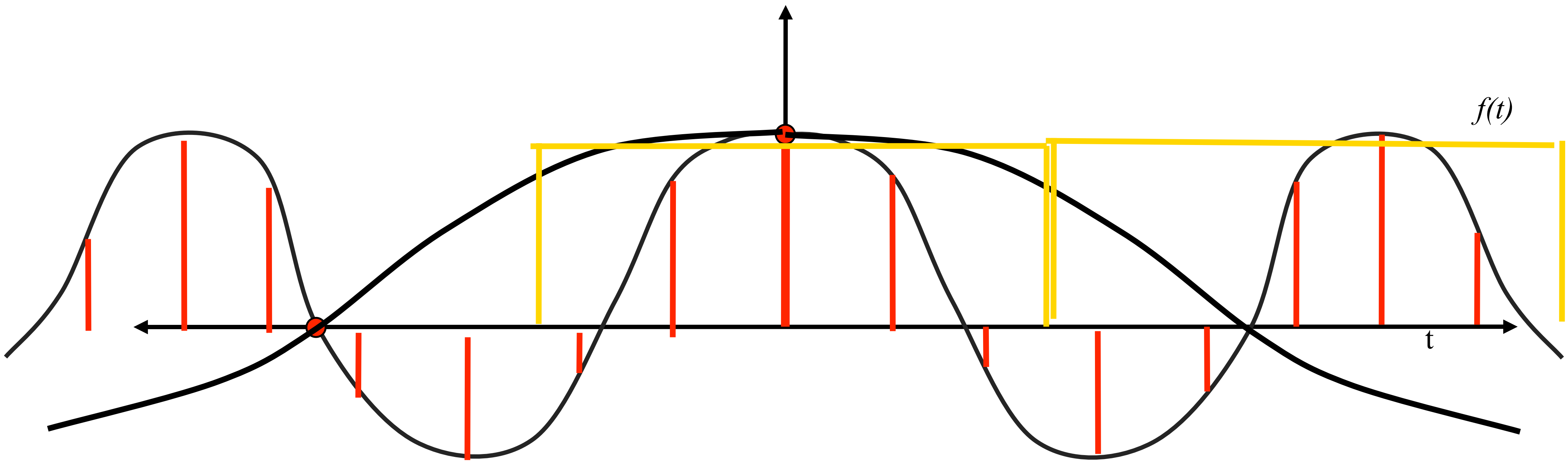


A Cosine Example

- What if we sample at T=2/3?



- Supersampling increases the sampling rate, and then integrates or convolves with a box filter, which is finally followed by the output sampling function.



Sampling and Anti-aliasing

- The problem:
- The signal is not band-limited.
- Uniform sampling can pick-up higher frequency patterns and represent them as low-frequency patterns.