Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

# FLUID DYNAMICS (Intro)

FABIO ROMANELLI

Department of Mathematics & Geosciences

University of Trieste

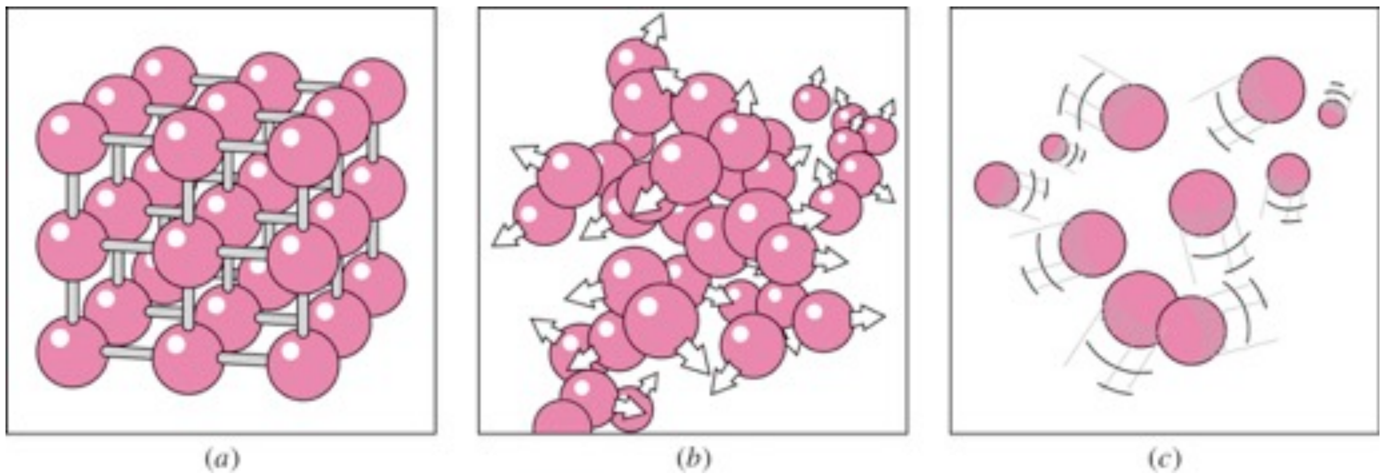
romanel@units.it

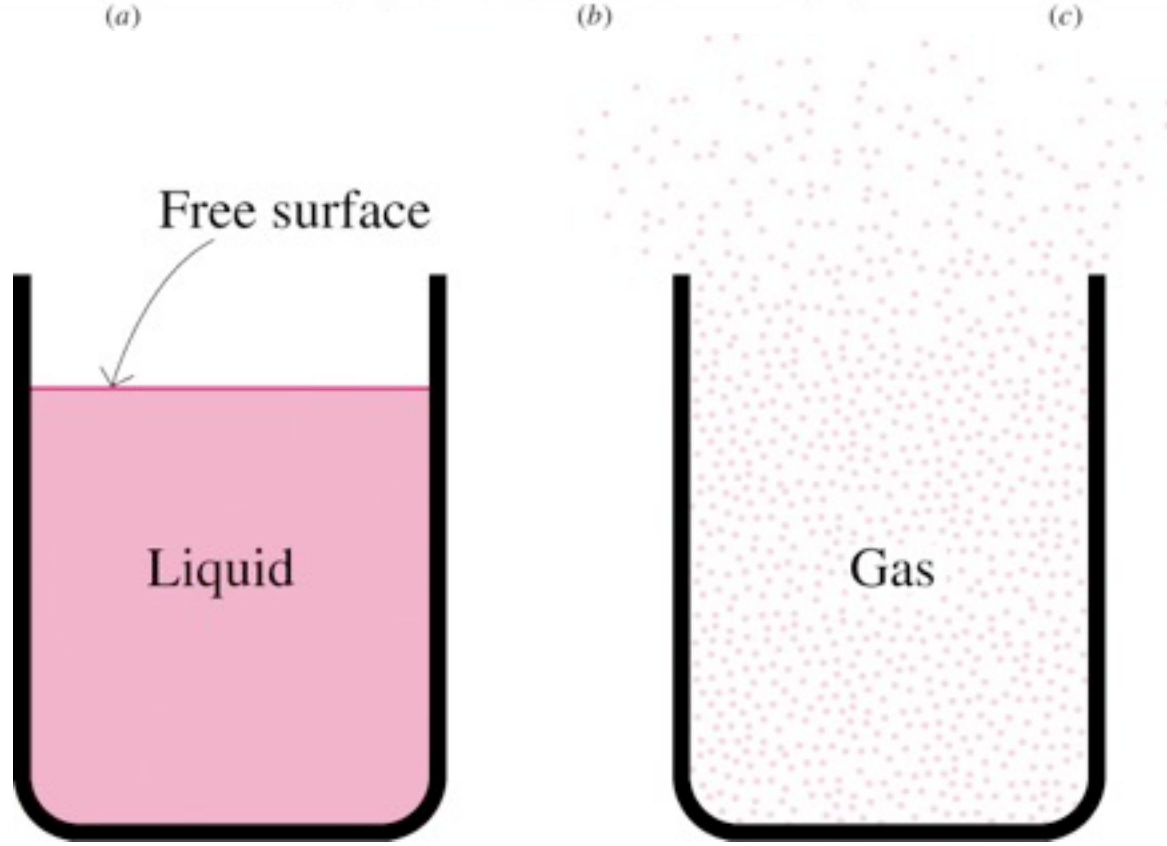
http://moodle2.units.it/course/view.php?id=887

The space occupied by the material will be called the **domain**.

- **Solids** are materials that have a more or less intrinsic configuration or shape and do not conform to their domain under nominal conditions.
- **Fluids** do not have an intrinsic shape; **gases** are fluids that will completely fill their domain (or container) and **liquids** are fluids that form a free surface in the presence of gravity.

# What is a fluid?




- A liquid takes the shape of the container it is in and forms a free surface in the presence of gravity
- A gas expands until it encounters the walls of the container and fills the entire available space. Gases cannot form a free surface
- Gas and vapor are often used as synonymous words

# What is a fluid?

- The word "**fluid**" traditionally refers to one of the states of matter, either liquid or gaseous, in contrast to the "**solid**" state.
- A material that exhibits **flow** if shear forces are applied
- Basically any material that appears as elastic or non-deformable, with a crystalline structure (i.e., belonging to the solid state) or with a disordered structure (e.g., a glass, which from a thermodynamic point of view belongs to the liquid state) can be irreversibly deformed (flow) when subjected to stresses for a long enough time.

https://www.youtube.com/watch?v=sMKJvYSYiOs

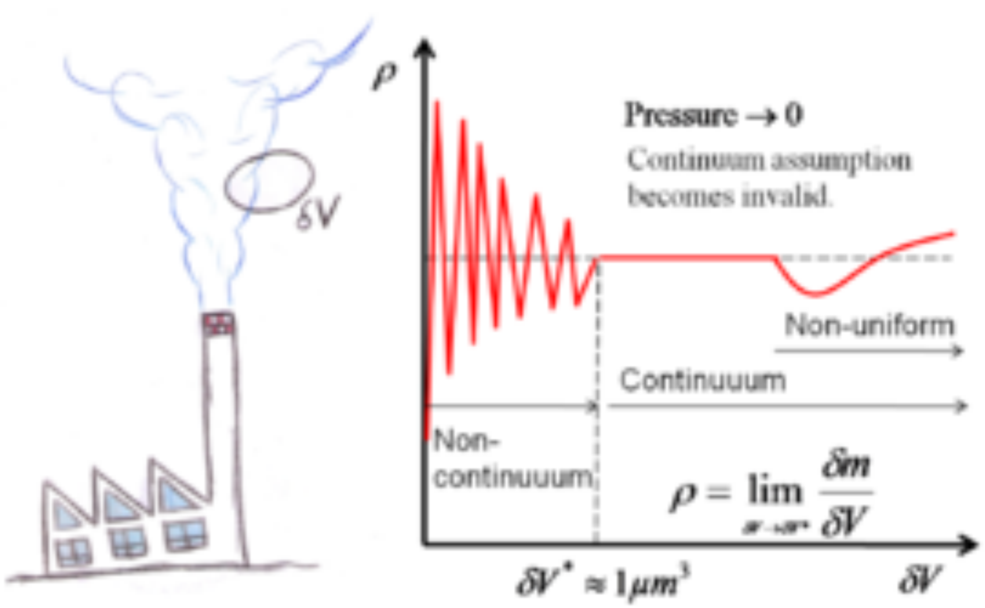
# Fluid mechanics assumptions

- Classical fluid mechanics, like classical thermodynamics, is concerned with **macroscopic phenomena** (bulk properties) rather than microscopic (molecular-scale) phenomena.

- The molecular makeup of a fluid will be ignored in all that follows, and the crucially important **physical properties** of a fluid, e.g., its mass density, $\rho$, and specific heat, $C_p$, among others, must be provided from outside of this theory. It is assumed that these physical properties, along with **flow properties**, e.g., the pressure, $P$, velocity, $\mathbf{v}$, temperature, $T$, etc., are in principle definable at every point in space, as if the fluid was a smoothly varying continuum, rather than a swarm of very fine, discrete particles (molecules).

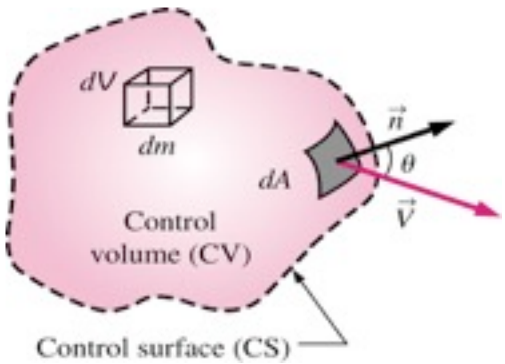
# Continuum

- Matter is made up of atoms that are spaced, but it is very convenient to disregard the atomic nature of a substance and view it as a continuous, homogeneous matter with no holes, that is, a continuum.
- The continuum idealization allows us to treat properties as point functions and to assume the properties vary continually in space with no jump discontinuities.
- This idealization is valid as long as the size of the system we deal with is large relative to the space between the molecules in the fluid.
- For density, the mass (m) per unit volume (V) in a substance, measured at a given point, will tend toward a constant value in the limit as the measuring volume shrinks down to zero.



## Conservation of matter

The total mass of fluid flowing, in unit of time, through a surface S, has to be equal to the decrease, in unit time, in the mass of fluid in the volume V:




$$\int \frac{\partial \rho}{\partial t} dV + \oint \rho(\mathbf{v} \cdot \mathbf{n}) dS = 0$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \rho\,\mathrm{div}(\mathbf{v}) + \mathbf{v} \cdot \mathrm{grad}(\rho) = 0$$

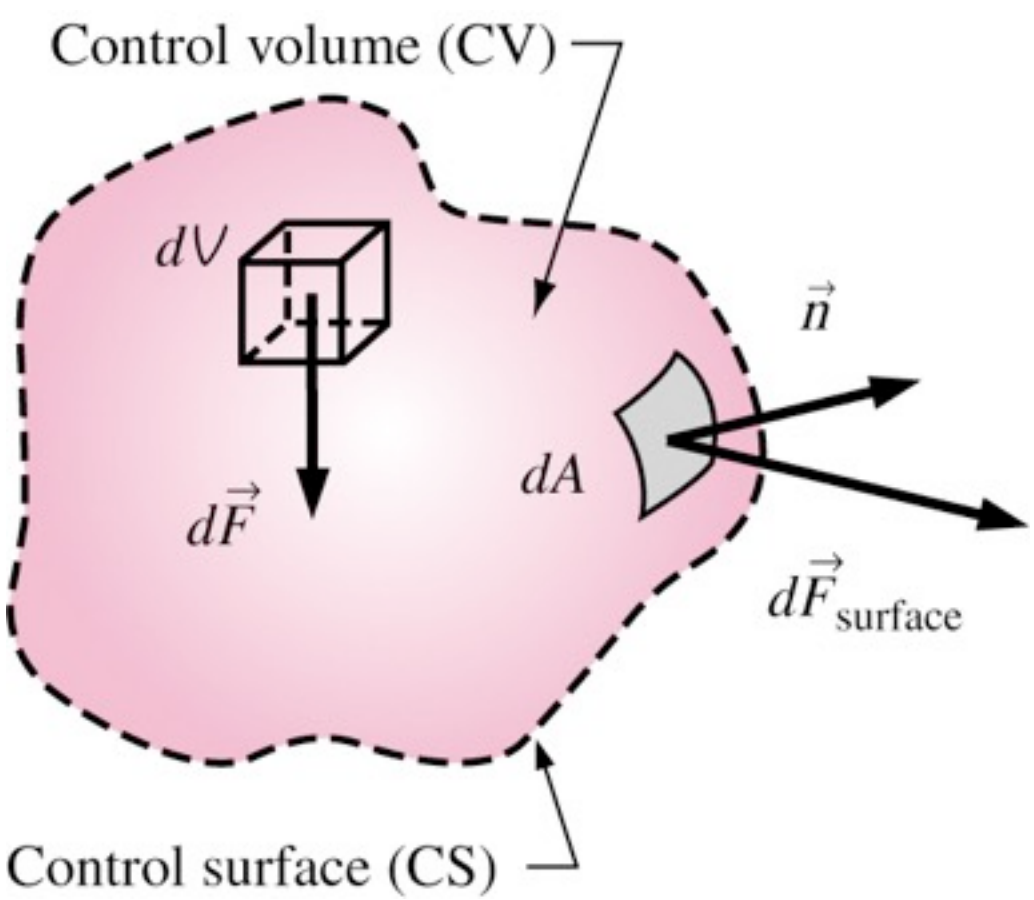
that can be compared with what we obtained considering 1D sound waves:

$$\Delta \rho = -\rho_0 \frac{\partial s}{\partial x}$$

The gas moves and causes density variations

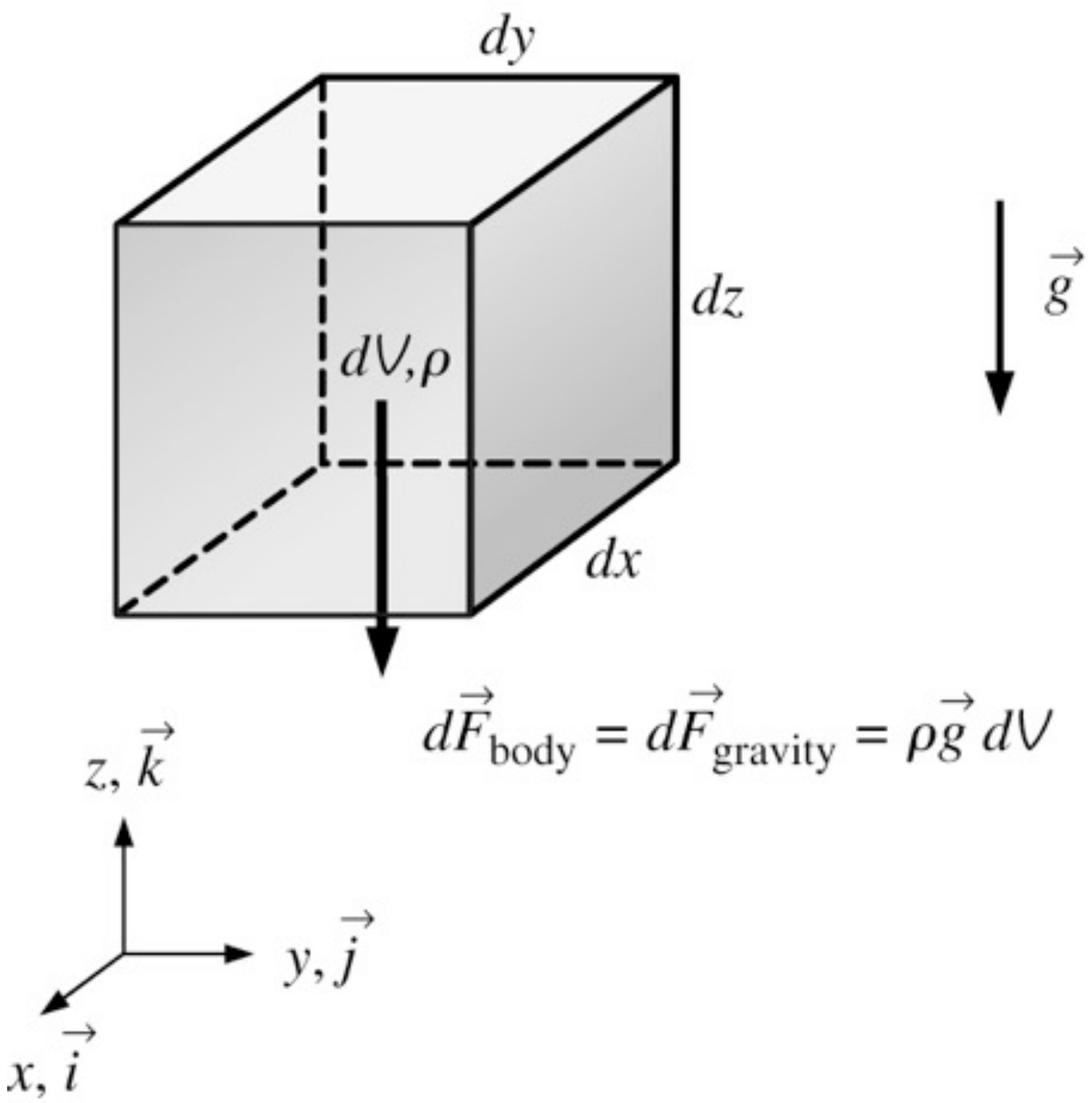
# Forces Acting on a CV

- Forces acting on a Control Volume consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).




- Body forces act on each volumetric portion dV of the CV.
- Surface forces act on each portion dA of the CS.

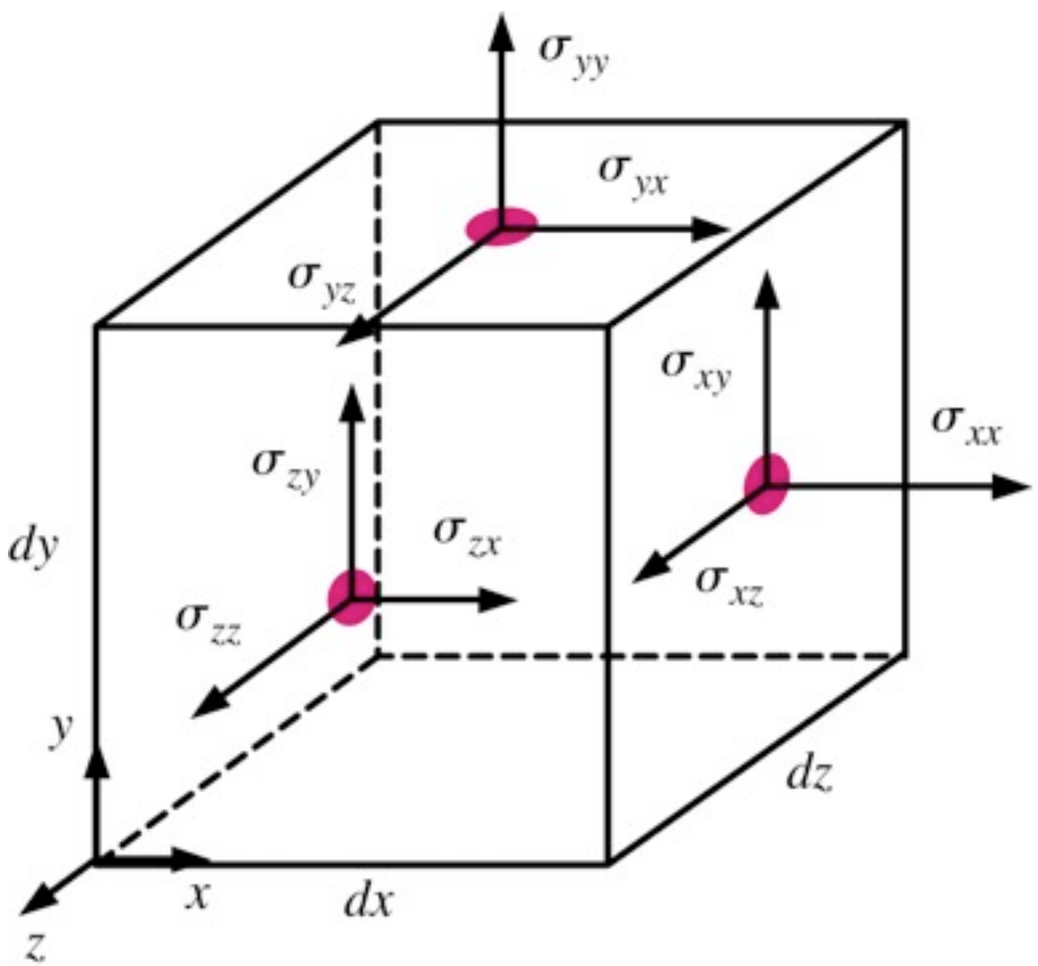
# Body Forces




$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}\, dV$$

- The most common body force is **gravity**, which exerts a downward force on every differential element of the CV

# Surface Forces




- Surface forces are not as simple to analyze since they include both normal and tangential components
- Diagonal components $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$ are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components $\sigma_{xy}$, $\sigma_{xz}$, etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

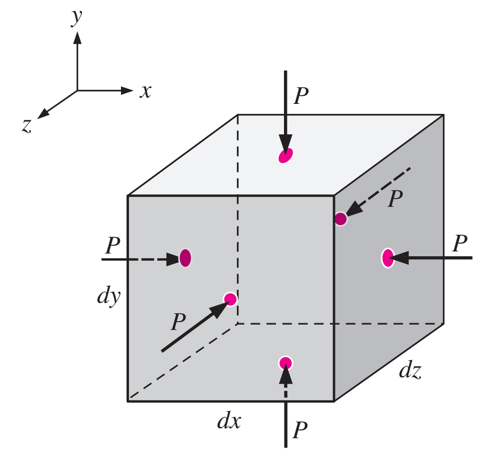
$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\mathbf{F}_S = \oint \sigma_{ij} \cdot \mathbf{n} dS$$

# Stress Tensor

For a fluid at rest, according to Pascal's law, regardless of the orientation the stress reduces to:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$




Hydrostatic pressure is the same as the thermodynamic pressure from study of thermodynamics. P is related to temperature and density through some type of equation of state (e.g., the ideal gas law).

# Stress Tensor

- Separate $\sigma_{ij}$ into pressure and viscous stresses

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \underbrace{\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}}_{\text{Viscous (Deviatoric) Stress Tensor}}$$

$$\sigma_{ij} = -P\delta_{ij} + 2\eta\varepsilon_{ij}$$

**Newton's law**

The fluid in the volume is accelerated by the total force acting on it:

$$\rho \frac{d\mathbf{v}}{dt} = -\mathrm{grad}(P) - \rho\,\mathrm{grad}(\phi) + \mathbf{f}_{visc}$$

Fluid moves from high-pressure areas to low-pressure areas. Moving implies that fluid moves in direction of largest change in pressure

External forces that act at a distance; we can suppose that they are conservative (like **gravity** and electricity)

Internal force due to the fact that in a flowing fluid there can also be a shearing stress, and it is called the **viscous** force

**Newton's law**

The fluid in the volume is accelerated by the total force acting on it:

$$\rho \frac{dv}{dt} = -grad(P) - \rho grad(\phi) + \mathbf{f}_{visc}$$

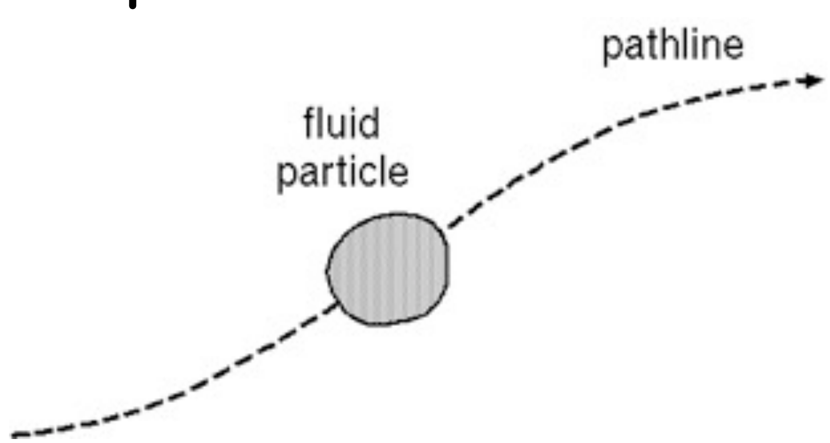
that can be compared with what we obtained considering for 1D sound waves:

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x}$$

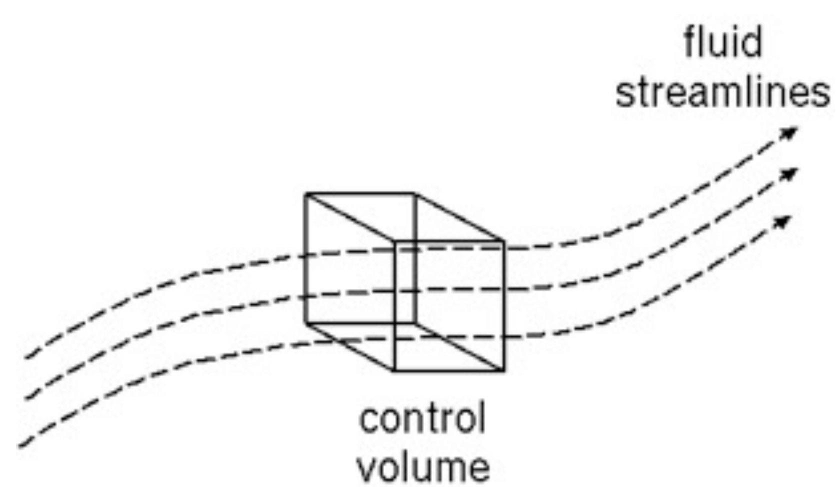
Pressure variations generate gas motion

# Lagrangian vs. Eulerian description

A fluid flow field can be thought of as being comprised of a large number of finite sized **fluid particles** which have mass, momentum, internal energy, and other properties. Mathematical laws can then be written for each fluid particle. This is the Lagrangian description of fluid motion.




Another view of fluid motion is the Eulerian description. In the Eulerian description of fluid motion, we consider how flow properties change at a **fluid element** that is fixed in space and time (x,y,z,t), rather than following individual fluid particles.




Governing equations can be derived using each method and converted to the other form.

# If we move a parcel in time $\Delta t$

Using Taylor series expansion

$$\Delta f = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \boxed{\begin{array}{c}\text{Higher}\\ \text{Order}\\ \text{Terms}\end{array}}$$

Assume increments over $\Delta t$ are small, and ignore Higher Order Terms

# If we move a parcel in time $\Delta t$

Dividing by $\Delta t$ and taking the small limit:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Introduction of convention of d( )/dt ≡ D( )/Dt

$$\frac{Dx}{Dt} = v_x, \frac{Dy}{Dt} = v_y, \frac{Dz}{Dt} = v_z$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z}$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \mathrm{grad}(f)$$

# Advection

In mathematics and continuum mechanics, including fluid dynamics, the **substantive derivative** (sometimes the **Lagrangian derivative**, **material derivative** or **advective derivative**), written D/Dt, is the **rate of change** of some property of a small parcel of fluid.

Note that if the fluid is moving, the substantive derivative is the rate of change of fluid within the small parcel, hence the other names **advective derivative** and **fluid following derivative**. **Advection** is transport of a some conserved scalar quantity in a vector field.

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} = \mathbf{v} \cdot \mathrm{grad}(f)$$

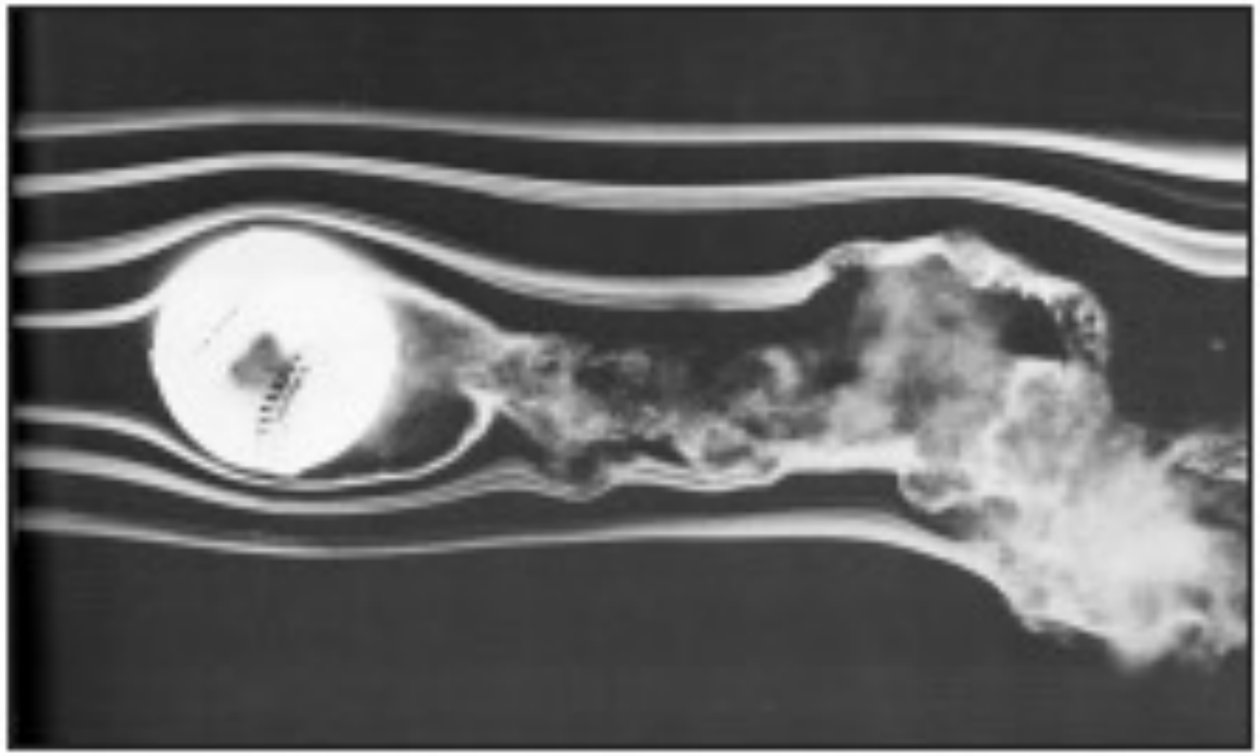
# Euler equations

Newton's law + Conservation of matter - Viscosity

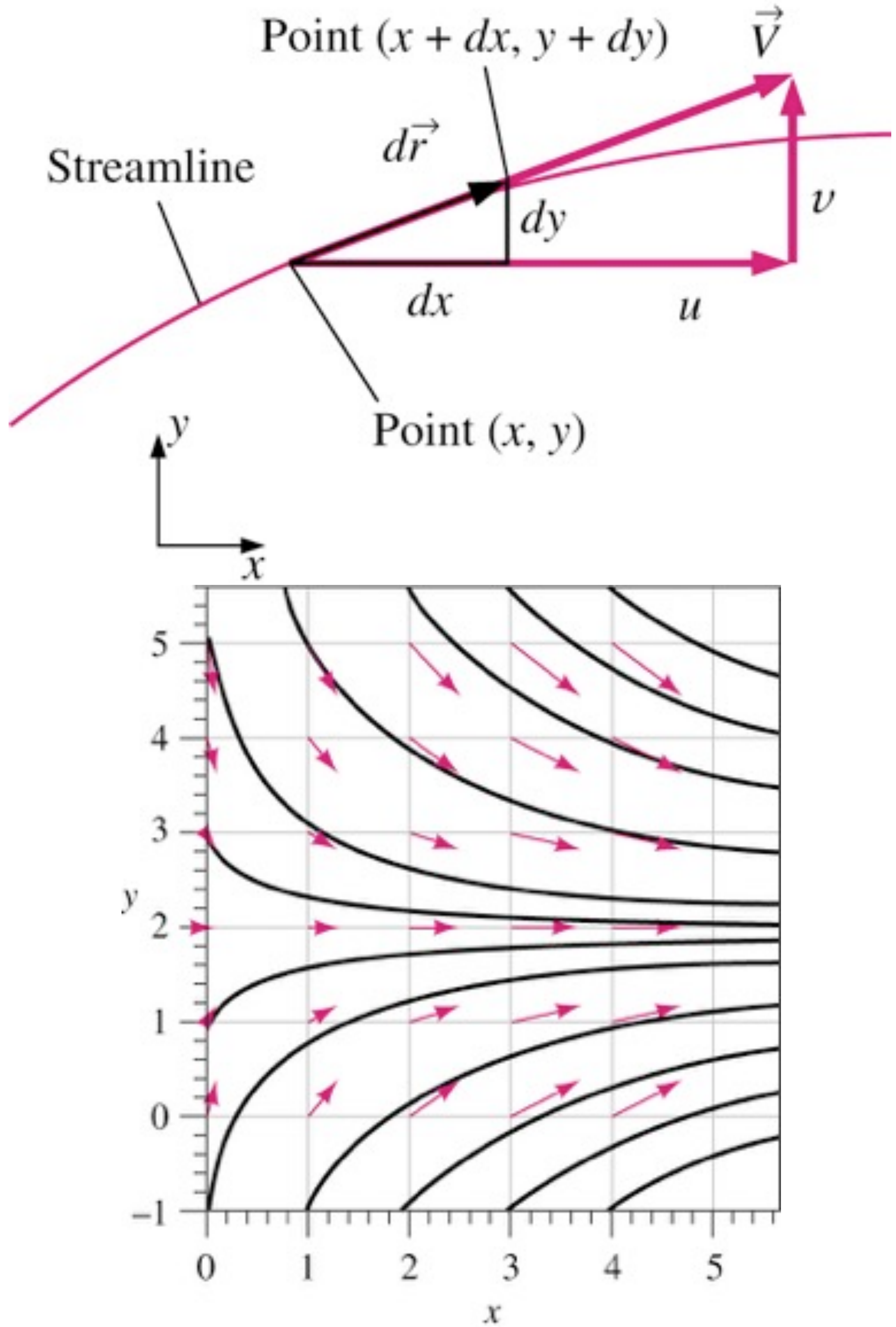
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathrm{grad})\mathbf{v} = -\frac{\mathrm{grad}(P)}{\rho} - \mathrm{grad}(\phi)$$

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines

While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from **flow visualization**

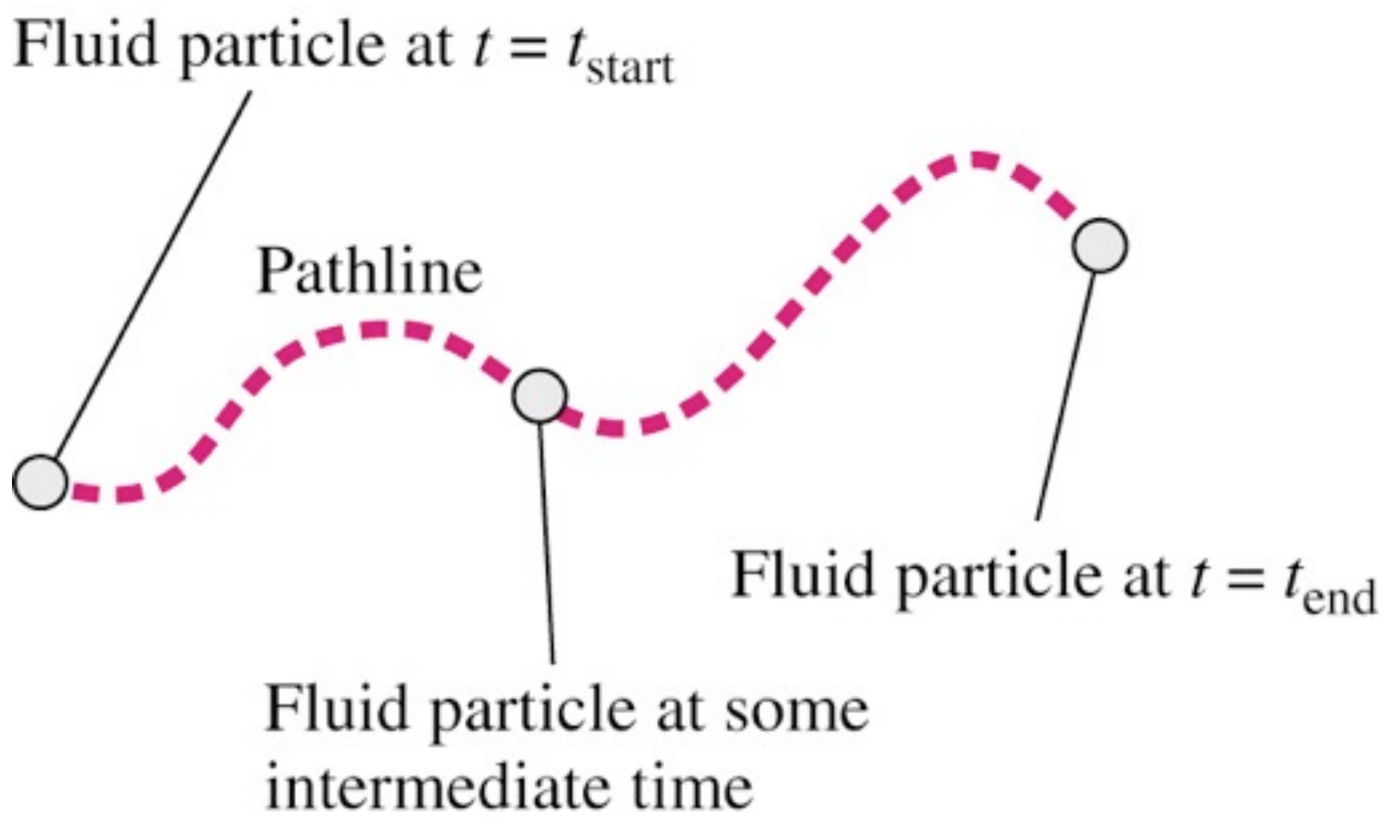


# Streamlines



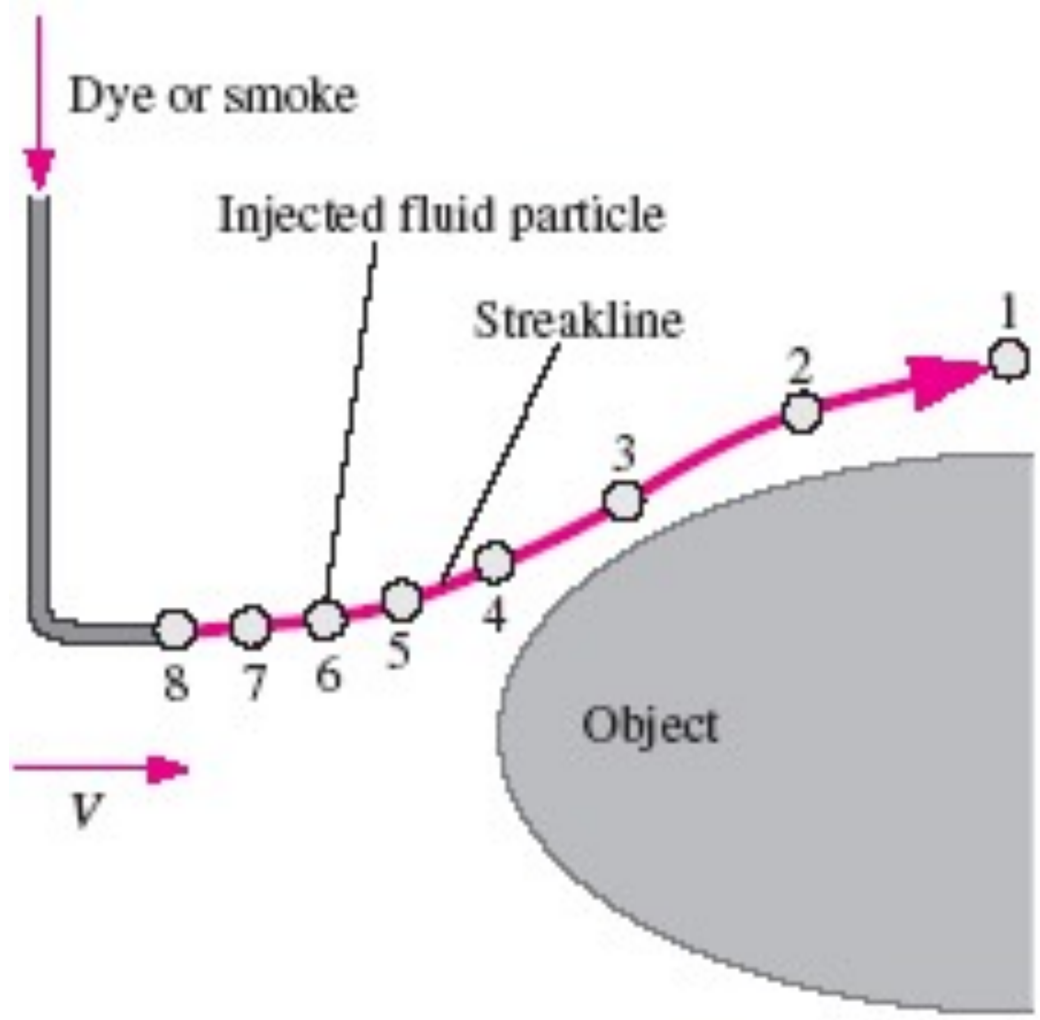

- A Streamline is a curve that is everywhere tangent to the instantaneous local velocity vector.

# Pathlines




- A Pathline is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

# Streaklines

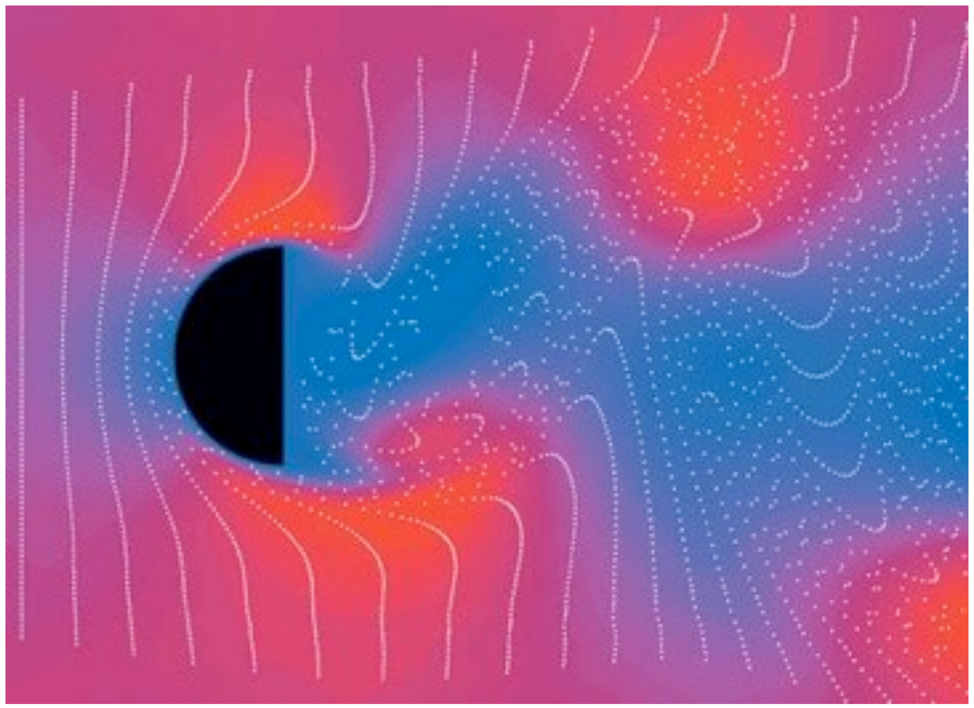


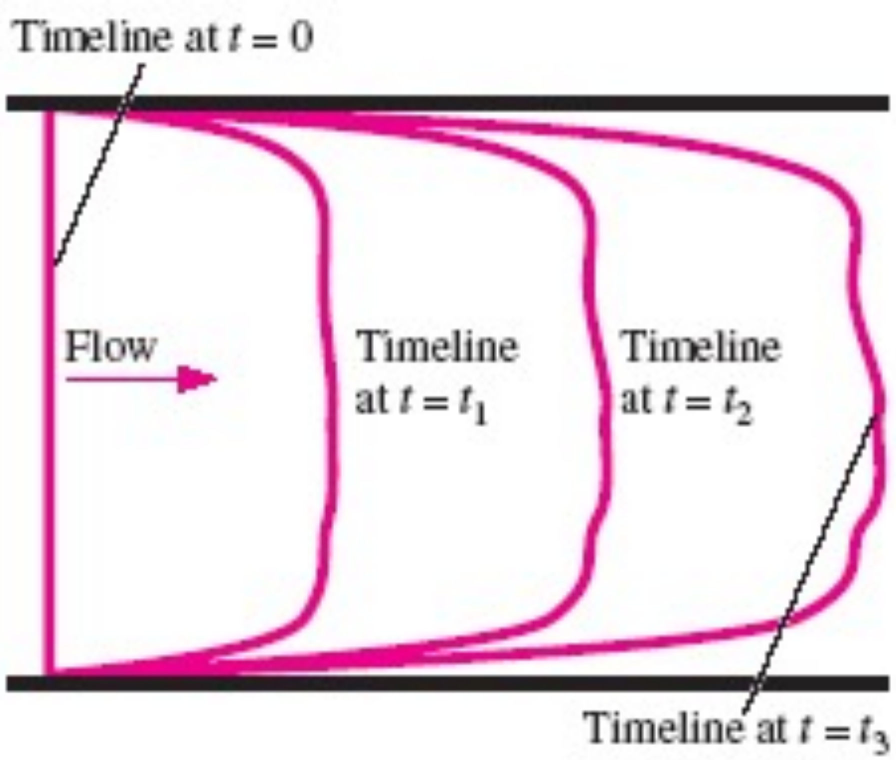

- A Streakline is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: dye in a water flow, or smoke in an airflow.

# Comparisons

- For steady flow, streamlines, pathlines, and streaklines are identical.
- For unsteady flow, they can be very different.
  - Streamlines are an instantaneous picture of the flow field
  - Pathlines and Streaklines are flow patterns that have a time history associated with them.
  - Streakline: instantaneous snapshot of a time-integrated flow pattern.
  - Pathline: time-exposed flow path of an individual particle.

# Timelines



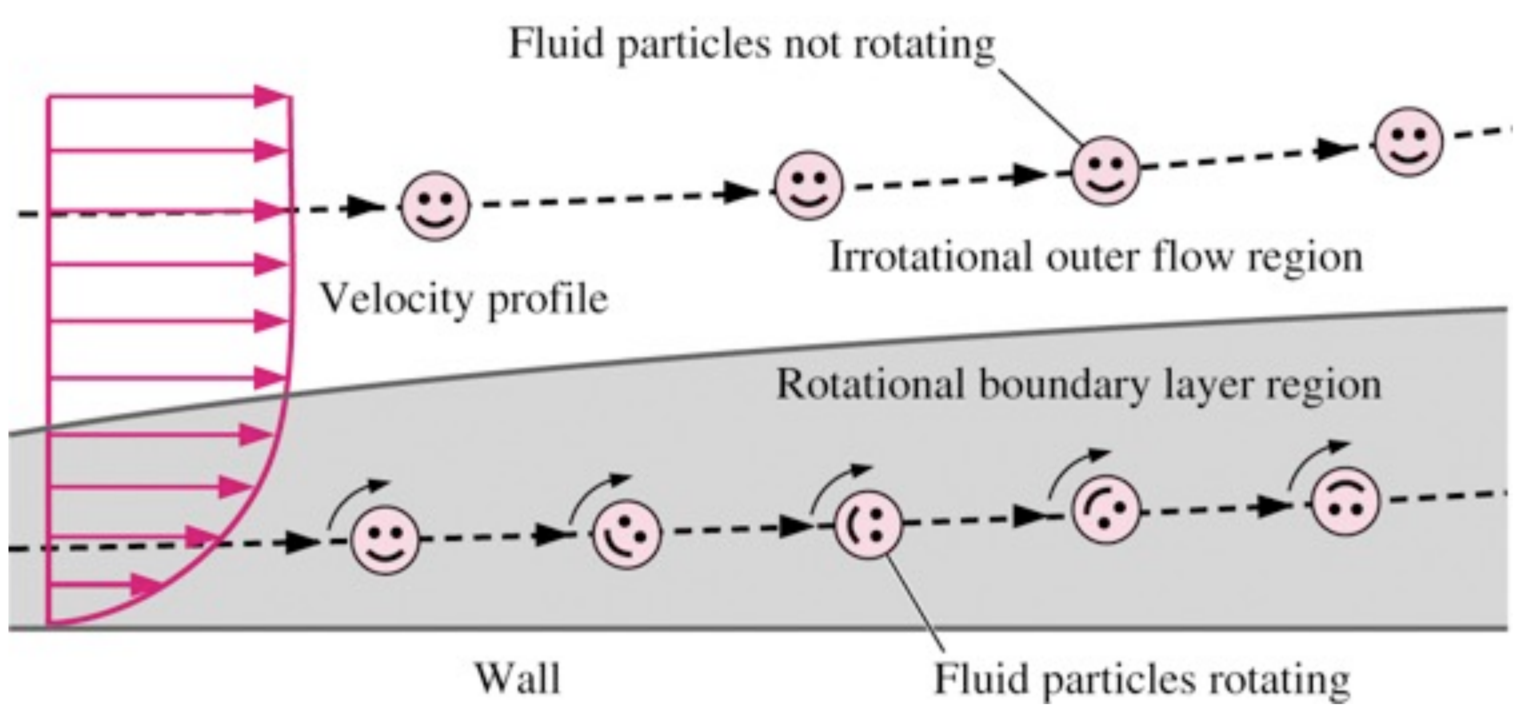



- A Timeline is a set of adjacent fluid particles that were marked at the same (earlier) instant in time.
- Timelines can be generated using a hydrogen bubble wire.

# Vorticity and Rotationality

- The vorticity vector is defined as the curl of the velocity vector $\text{rot}\,\mathbf{v} = \boldsymbol{\Omega}$, is a measure of rotation of a fluid particle.

- Vorticity is equal to twice the angular velocity of a fluid particle
  In regions where $\Omega$= 0, the flow is called irrotational.
  Elsewhere, the flow is called rotational



**Newton's law** + **Conservation of matter** - **Viscosity**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathrm{grad})\mathbf{v} = -\frac{\mathrm{grad}(P)}{\rho} - \mathrm{grad}(\phi) \qquad 1.$$

Using the identity $(\mathbf{v} \cdot \mathrm{grad})\mathbf{v} = (\mathrm{rot}\mathbf{v}) \times \mathbf{v} + \frac{1}{2}\mathrm{grad}(v^2)$

and defining the **vorticity** as $\mathrm{rot}\mathbf{v} = \mathbf{\Omega}$

$$\frac{\partial \mathbf{\Omega}}{\partial t} + \mathrm{rot}(\mathbf{\Omega} \times \mathbf{v}) = 0 \qquad 2.$$

# Euler equations - Steady flow

Euler eq. + $\frac{\partial \mathbf{v}}{\partial t} = 0$ + $(\mathbf{v} \cdot )$

$$\mathbf{v} \cdot grad\left(\frac{P}{\rho} + \phi + \frac{v^2}{2}\right) = 0$$

that means that along a **streamline** one has:

$$\frac{P}{\rho} + \phi + \frac{v^2}{2} = constant$$

and if the motion is **irrotational (potential flow)**, it is valid everywhere, i.e.

**Bernoulli's theorem**

# Equation of state

The functional relationship between **density, pressure and temperature**:

$$P = P(\rho, T) \text{ or equivalently, } \rho = \rho(P, T)$$

with T the absolute temperature in Kelvin.

The archetype of an equation of state is that of an ideal gas, $P = \rho RT/M$
where R=8.31 (Joule moles$^{-1}$ K$^{-1}$) is the universal gas constant and M is the molecular weight (kg/mole).

If the composition of the material changes, then the appropriate equation of state will involve more than three variables, for example the concentration of salt if sea water, or water vapor if air.

An important class of phenomenon may be described by a reduced equation of state having state variables density and pressure alone,

$$P = P(\rho) \text{ or equivalently, } \rho = \rho(P)$$

and the fluid is said to be **barotropic**.

# Equation of state - barotropic

The temperature of the fluid will change as pressure work is done on or by the fluid, and yet temperature need not appear as a separate, independent state variable provided conditions approximate one of two limiting cases:

1) If the fluid is a fixed mass of ideal gas, say, that can readily exchange heat with a heat reservoir having a constant temperature, then the gas may remain **isothermal** under pressure changes;

2) the other limit, which is more likely to be relevant, is that heat exchange with the surroundings is negligible because the time scale for significant conduction is very long compared to the time scale (lifetime or period) of the phenomenon. In that event the system is said to be **adiabatic** and in the case of an ideal gas the density and pressure are related by the well-known adiabatic law.

$$\Delta\rho = \frac{\partial\rho}{\partial P}\Delta P$$

that can be compared with what we obtained considering sound waves:

$$\Delta P = \kappa\Delta\rho = c^2\Delta\rho$$

**Density variations cause pressure variations**

# Incompressible fluids

In many cases of the flow of fluids their density may be supposed invariable, i.e. constant throughout the volume and its motion and we speak of **incompressible flow**

$\rho$ = constant

**Conservation of matter** div (**v**)=0

**Euler equation**

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\Omega} \times \mathbf{v} = -\mathrm{grad}\left(\frac{1}{2}v^2 + \frac{P}{\rho} + \phi\right)$$

The conditions under which the fluid can be considered incompressible are:

$$\frac{\partial \rho}{\partial t} << \rho \,\mathrm{div}(\mathbf{v}) \Rightarrow \frac{\Delta\rho}{\tau} << \frac{\rho v}{\lambda}$$

$$\Delta\rho = \frac{\Delta P}{c^2} \approx \frac{1}{c^2}\left(\rho \frac{\partial v}{\partial t}\lambda\right) \approx \frac{1}{c^2}\left(\rho \frac{v}{\tau}\lambda\right)$$

i.e. $\tau >> \dfrac{\lambda}{c}$

i.e. **v<<c**

i.e. the time taken by a sound signal to traverse distances must be small compared with that during which flow changes appreciably

# Incompressible & Irrotational flow

From Euler equations we have that only viscosity can generate vorticity if none exists initially. And if the flow is irrotational rot(**v**)=0, and thus **v**=grad(θ) and the flow is called **potential**.

**Euler equation** rot (**v**)=0

and if it is also incompressible:

**Conservation of matter** div (**v**)=0

the potential has to satisfy Laplace equation:

$$\nabla^2(\theta)=0$$

and we can separate the variables...

# Separation of variables + BC at bottom

Let us consider a velocity potential propagating along the x-axis and uniform in the y- direction: all quantities are independent of y.
We shall seek a solution which is a simple periodic function of time and of the coordinate x, i.e. we put

$$\theta = F(z)\cos(kx - \omega t)$$

$$\frac{d^2F}{dz^2} - k^2F = 0 \qquad F(z) = \left[Ae^{kz} + Be^{-kz}\right]$$

and if the liquid container has depth h, there the vertical flow has to be 0:

$$v_z = \left.\frac{dF}{dz}\right|_{z=-h} = 0 \quad \Rightarrow \quad B = e^{-2kh}A$$

# BC at bottom

and this leads to:

$$F(z) = 2Ae^{-kh}\cosh[k(z+h)]$$

Thus, at the bottom (z=-h) the cosh(0)=1, while at top it is cosh(kh), thus F grows as z goes from bottom to top values.

If the container is infinitely deep (h goes to infinity) we have that B has to be 0 and the potential as well is going to 0:

$$F(z) = Ae^{kz}$$

# Gravity waves

- The free surface of a liquid in equilibrium in a gravitational field is a plane.
- If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid.
- This motion will be propagated over the whole surface in the form of waves, which are called **gravity waves**, since they are due to the action of the gravitational field.
- We shall here consider gravity waves in which the velocity of the moving fluid particles is so small that we may neglect the term (v•grad)v in comparison with ∂/∂t in Euler's equation.

# Gravity waves

The physical significance of this is easily seen. During a time interval of the order of the period, $\tau$, of the oscillations of the fluid particles in the wave, these particles travel a distance of the order of the amplitude, a, of the wave. Their velocity v is therefore of the order of $a/\tau$. It varies noticeably over time intervals of the order of $\tau$ and distances of the order of $\lambda$ in the direction of propagation (where $\lambda$ is the wavelength). Hence the time derivative of the velocity is of the order of $v/\tau$, and the space derivatives are of the order of $v/\lambda$.

Thus the condition

$$(\mathbf{v}\cdot\mathrm{grad})\mathbf{v} << \frac{\partial \mathbf{v}}{\partial t}$$

is equivalent to

$$\frac{1}{\lambda}\left(\frac{a}{\tau}\right)^2 << \frac{a}{\tau}\frac{1}{\tau} \quad \text{or} \quad a << \lambda$$

i.e. **the amplitude of the oscillations in the wave must be small compared with the wavelength.**

# Small amplitude gravity waves

For waves whose amplitude of motion is smaller than the wavelength, all significant terms in the fluid equation are gradients, and the Euler equation can be expressed as:

$$\mathrm{grad}\left(\frac{\partial\theta}{\partial t}+\frac{P}{\rho}+\phi\right)=0$$

thus, in space:

$$\frac{\partial\theta}{\partial t}+\frac{P}{\rho}+\phi=\mathrm{constant}$$

and assuming a gravitational potential $gz$, we obtain:

$$P=-\rho g z-\rho\frac{\partial\theta}{\partial t}$$

# Gravity waves: BC at the top

Let us denote by f the z coordinate of a point on the surface; f is a function of x, y and t.

In equilibrium f=0, so that f gives the vertical displacement of the surface in its oscillations.

Let a constant pressure $p_0$ act on the surface. Then we have at the surface:

$$p_0 = -\rho g f - \rho \frac{\partial \theta}{\partial t}$$

The constant $p_0$ can be eliminated by redefining the potential, adding to it a quantity independent of the coordinates. We then obtain the condition at the surface as

$$g f + \left. \frac{\partial \theta}{\partial t} \right|_{z=f} = 0$$

# Gravity waves: BC at top

Since the amplitude of the wave oscillations is small, the displacement f is small. Hence we can suppose, to the same degree of approximation, that the vertical component of the velocity of points on the surface is simply the time derivative of f:

$$v_z = \left.\frac{\partial \theta}{\partial z}\right|_{z=f} = \frac{\partial f}{\partial t} = -\left(\frac{1}{g}\frac{\partial^2 \theta}{\partial t^2}\right)$$

Since the oscillations are small, we can take the value of the derivatives at z=0 instead of z=f. Thus we have finally the following system of equations to determine the motion in a gravitational field:

$$\Delta \theta = 0$$

incompressibility

$$\left.\left(\frac{\partial \theta}{\partial z} + \frac{1}{g}\frac{\partial^2 \theta}{\partial t^2}\right)\right|_{z=0} = 0$$

B.C.

# Gravity waves: dispersion

$$F(z) = 2Ae^{-kh}\cosh[k(z+h)]$$

and the boundary at the top gives the **dispersion relation** for incompressible, irrotational, small amplitude "gravity" waves:

$$\omega^2 = kg[\tanh(kh)]$$

**deep water** (kh goes to infinity)

$$\omega^2 = kg$$

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

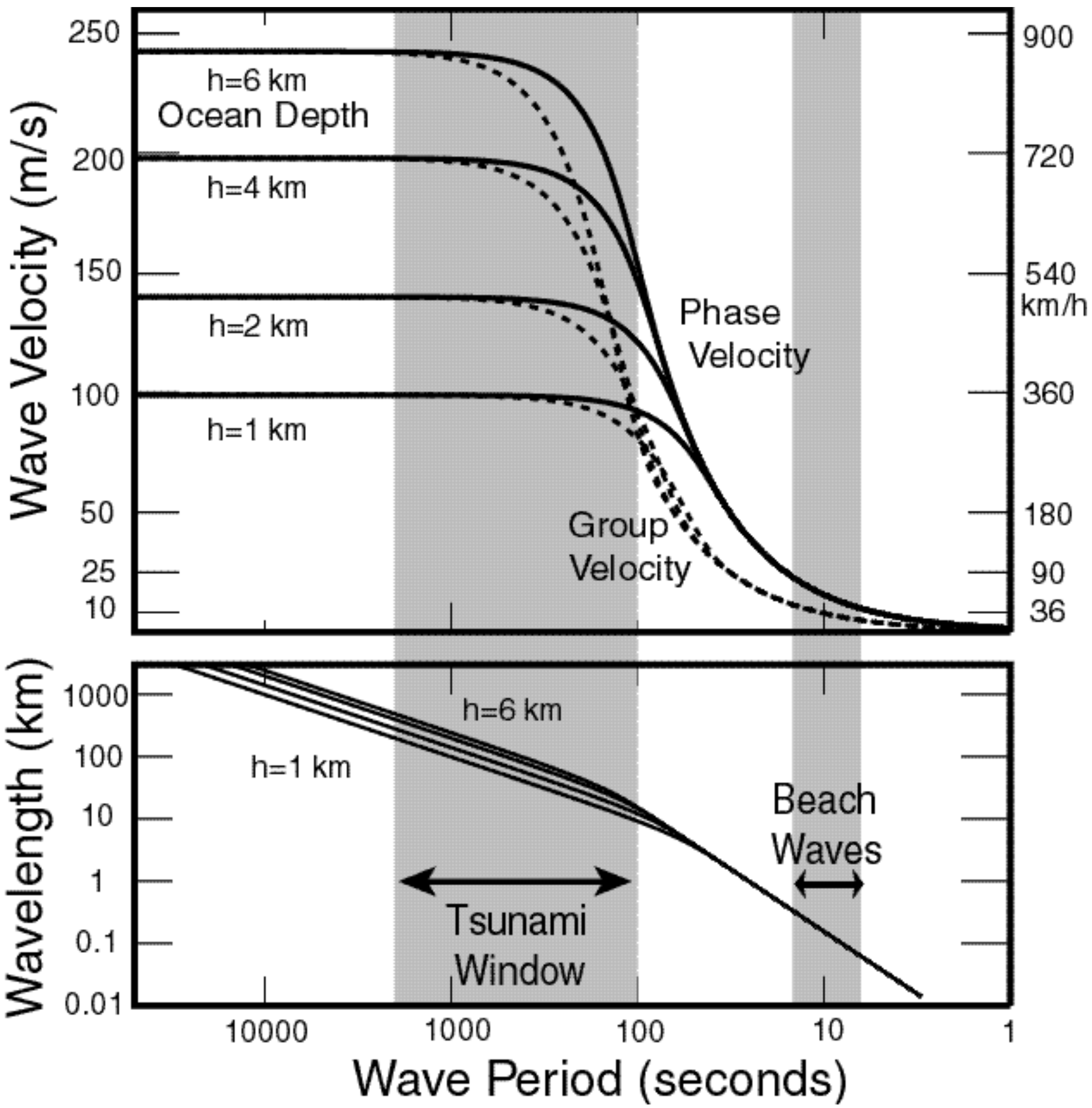
$$u = \frac{\partial\omega}{\partial k} = \frac{1}{2}c = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$

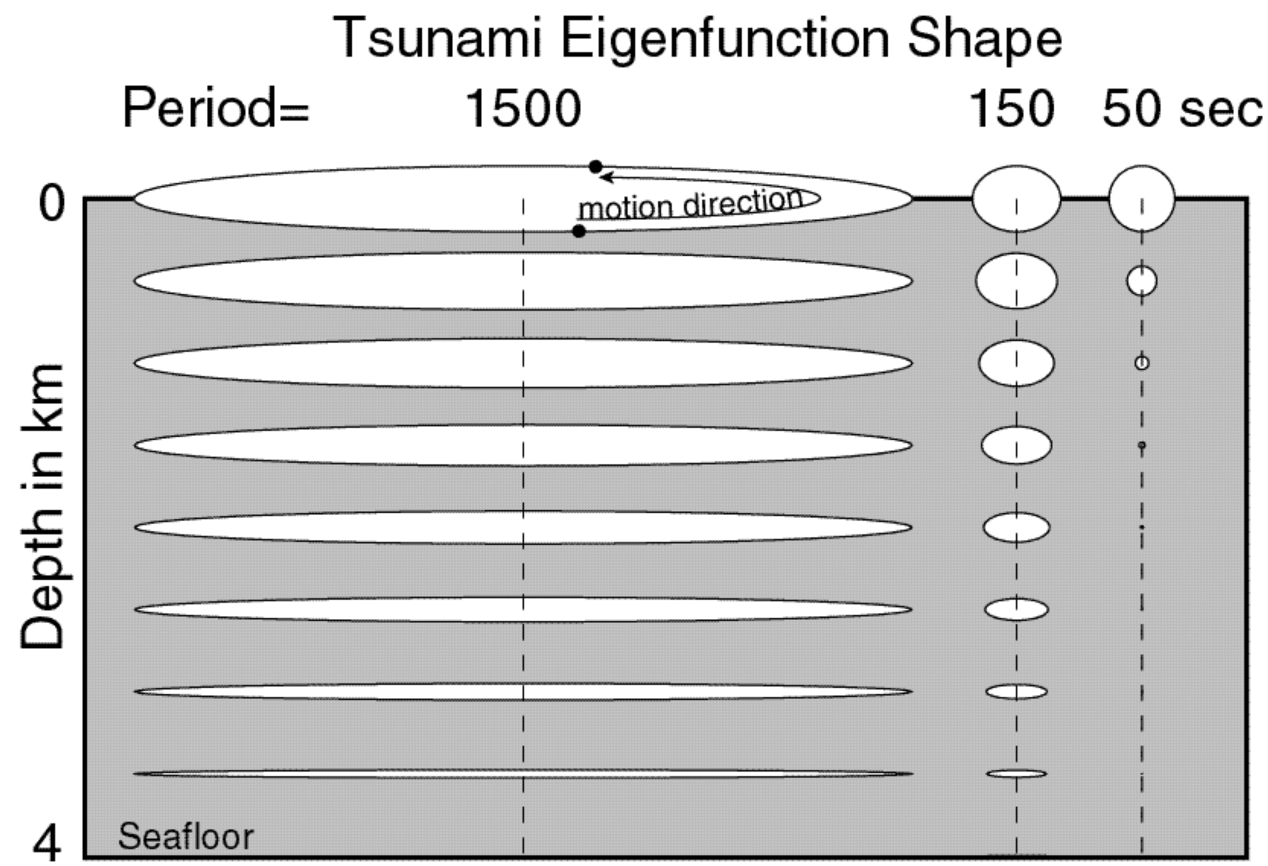
**shallow water** (kh goes to zero)

$$\omega^2 = k^2 gh$$

$$c = \sqrt{gh}$$

$$u = \frac{\partial\omega}{\partial k} = c = \sqrt{gh}$$


250
200
150
100
50
25
10
900
720
540
km/h
360
180
90
36
h=6 km
Ocean Depth
h=4 km
h=2 km
h=1 km
Phase
Velocity
Group
Velocity
Wave Velocity (m/s)
1000
100
10
1
0.1
0.01
h=6 km
h=1 km
Beach
Waves
Tsunami
Window
Wavelength (km)
10000
1000
100
10
1
Wave Period (seconds)


Tsunami Eigenfunction Shape
Period=
1500
150
50 sec
0
4
motion direction
Depth in km
Seafloor

# Gravity waves in deep water

The velocity distribution in the moving liquid is found by simply taking the space derivatives the velocity potential:

$$v_x = -Ake^{kz}\sin(kx-\omega t) \quad v_z = Ake^{kz}\cos(kx-\omega t)$$

We see that the velocity diminishes exponentially as we go into the liquid. At any given point in space (i.e. for given x, z) the velocity vector rotates uniformly in the xz-plane, its magnitude remaining constant.

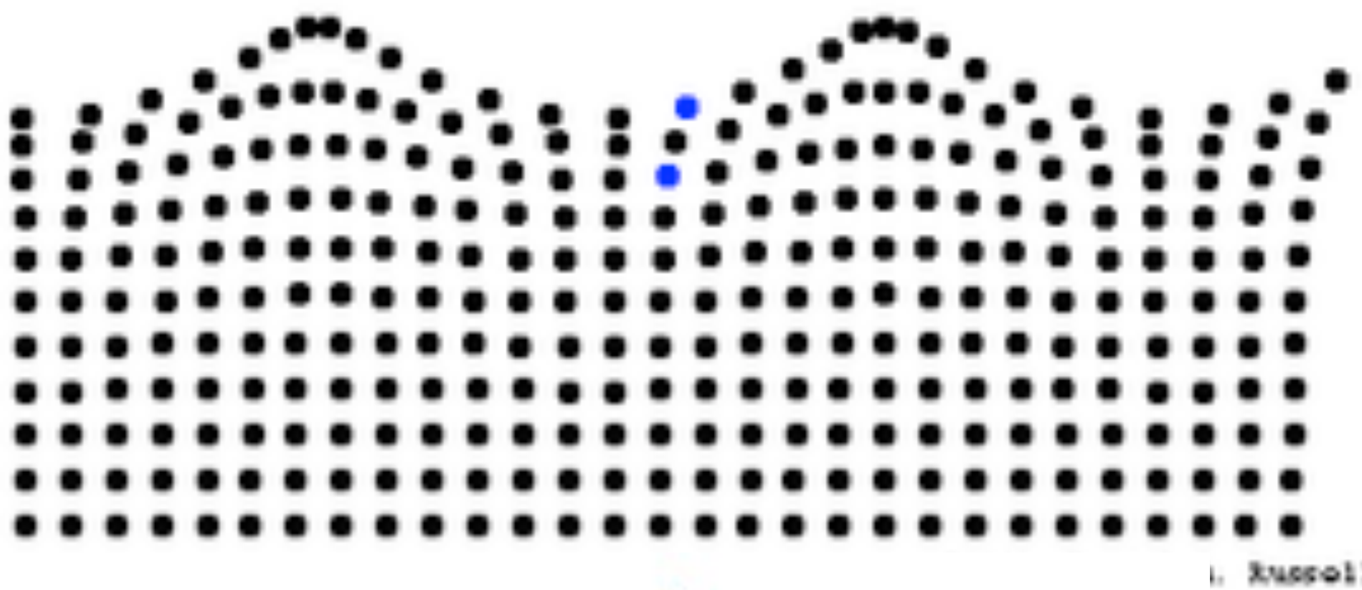
Let us also determine the paths of fluid particles in the wave. We temporarily denote by x, z the coordinates of a moving fluid particle (and not of a point fixed in space), and by $x_0$, $z_0$ the values of x and z at the equilibrium position of the particle. Then $v_x = dx/dt$, $v_z = dz/dt$, and on the right-hand side we may approximate by writing $x_0$, $z_0$ in place of x, z, since the oscillations are small.

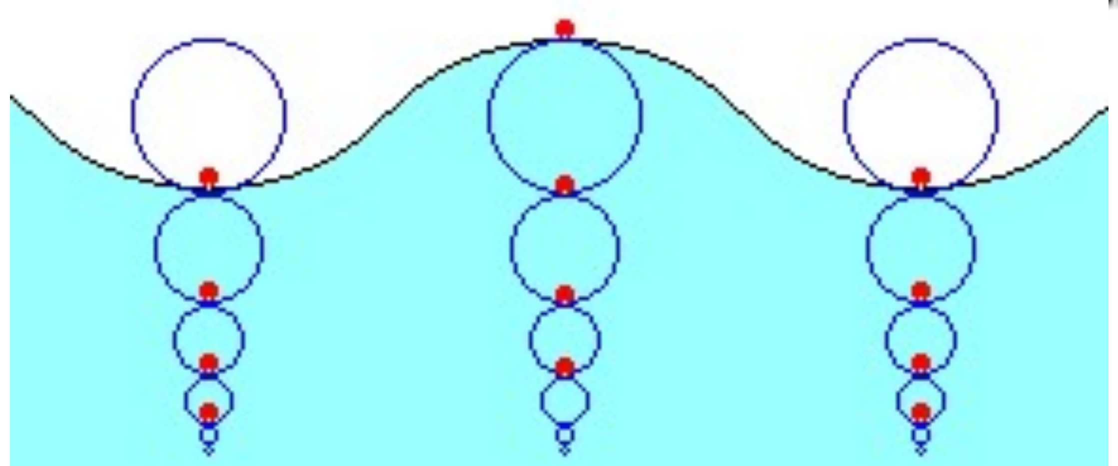
# Gravity waves in deep water

An integration with respect to time then gives:

$$x - x_0 = -A\frac{k}{\omega}e^{kz_0}\cos(kx_0 - \omega t) \quad z - z_0 = -A\frac{k}{\omega}e^{kz_0}\sin(kx_0 - \omega t)$$

Thus the fluid particles describe circles about the points ($x_0$, $z_0$) with a radius which diminishes exponentially with increasing depth.



# Long Gravity waves

Having considered gravity waves whose length is small compared with the depth of the liquid, let us now discuss the opposite limiting case of waves whose length is large compared with the depth. These are called **long waves**.

Let us examine the propagation of long waves in a channel that is supposed to be along the x-axis, and of infinite length. The cross-section of the channel may have any shape, and may vary along its length. We denote the cross-sectional area of the liquid in the channel by $S = S(x,t)$. The depth and width of the channel are supposed small in comparison with the wavelength.

We shall here consider longitudinal waves, in which the liquid moves along the channel. In such waves the velocity component $v_x$ along the channel is large compared with the components $v_y$, $v_z$. We denote $v_x$ by $v$ simply, and omit small terms.

# Long Gravity waves - alternative

The x-component of Euler's equation can then be written in the form:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and the z-component of Euler's equation can then be written in the form:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

we omit terms quadratic in the velocity, since the amplitude of the wave is again supposed small. From the second equation we have, since the pressure at the free surface (z =f) must be $p_0$:

$$p = p_0 + \rho g(f - z)$$

Substituting this expression in the first equation, we obtain

$$\frac{\partial v}{\partial t} = -g\frac{\partial f}{\partial x}$$

The second equation needed to determine the two unknowns v and f, that can be derived similarly to the equation of continuity; it is essentially the equation of continuity for the case in question. Let us consider a volume of liquid bounded by two plane cross-sections of the channel at a distance dx apart. In unit time a volume $(Sv)_x$ of liquid flows through one plane, and a volume $(Sv)_{x+dx}$ through the other. Hence the volume of liquid between the two planes changes, and since the liquid is incompressible, this change must be due simply to the change in the level of the liquid. The change per unit time is:

$$\frac{\partial S}{\partial t}dx = -\frac{\partial (Sv)}{\partial x}dx$$

Let $S_0$ be the equilibrium cross-sectional area of the liquid in the channel.
Then $S = S_0 + S'$, where $S'$ is the change in the cross-sectional area caused by the wave. Since the change in the liquid level is small, we can write $S'$ in the form bf, where b is the width of the channel at the surface of the liquid.

$$b\frac{\partial f}{\partial t} + \frac{\partial (S_0 v)}{\partial x} = 0$$

# Long Gravity waves

Differentiating it with respect to t and substituting ∂v/∂t we obtain

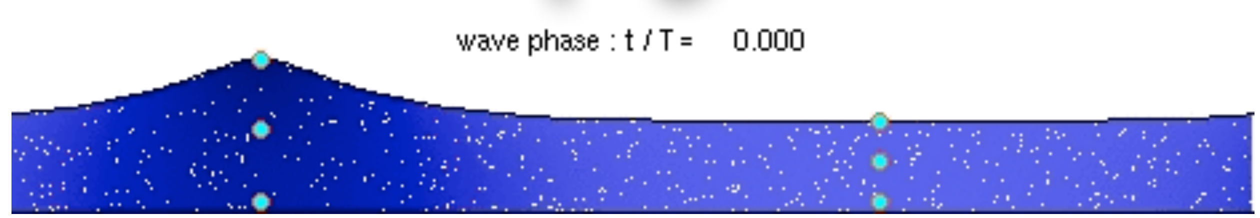
$$\frac{\partial^2 f}{\partial t^2} - \frac{g}{b}\frac{\partial}{\partial x}\left(S_0 \frac{\partial f}{\partial x}\right) = 0$$

If the channel cross-section is the same at all points, then $S_0$ = constant and

$$\frac{\partial^2 f}{\partial t^2} - \frac{gS_0}{b}\frac{\partial^2 f}{\partial x^2} = 0$$

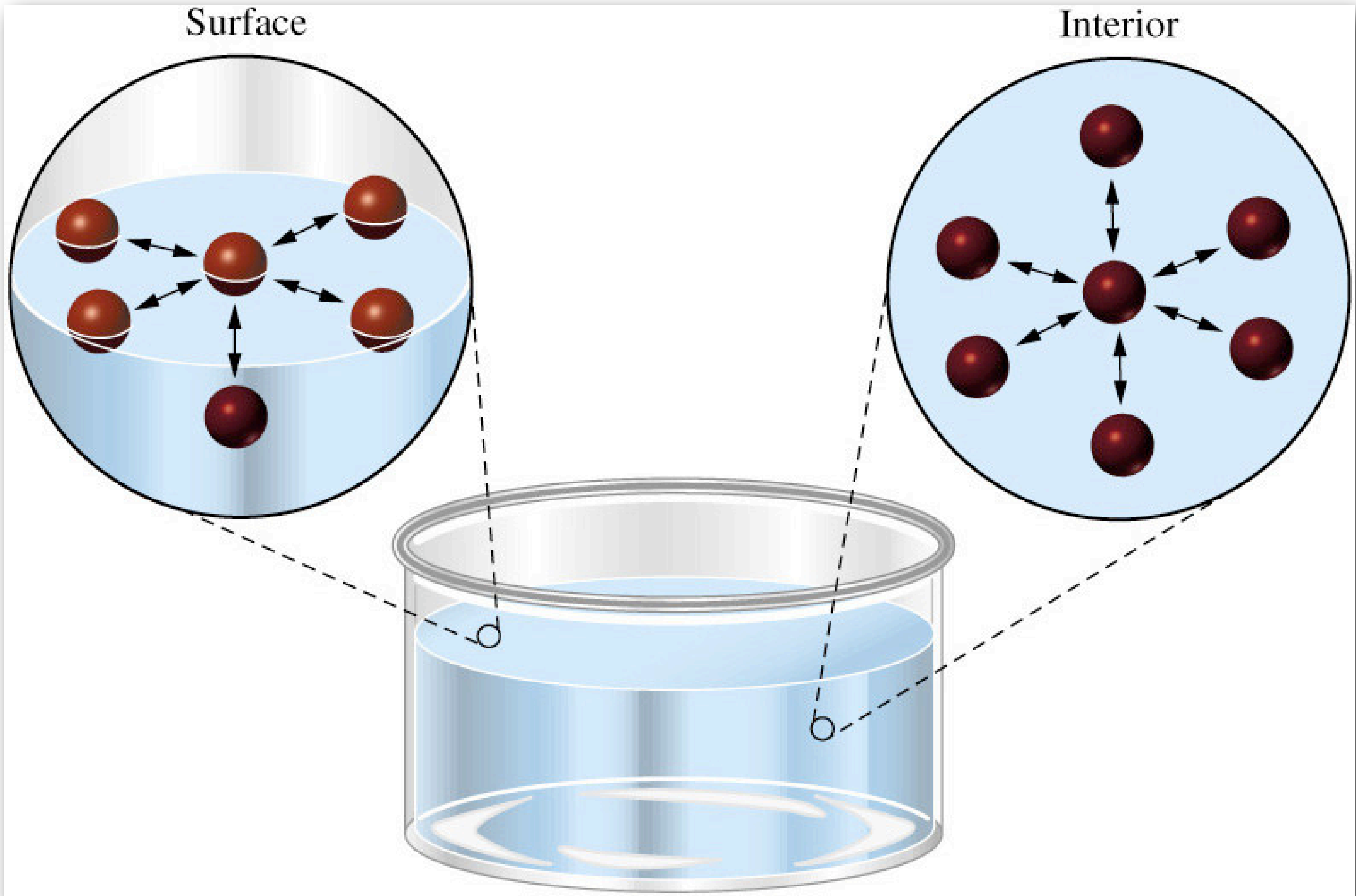
This is called a wave equation and corresponds to the propagation of waves with a velocity c(u) which is independent of frequency and is the square root of the coefficient :

$$c = u = \sqrt{\frac{gS_0}{b}} \approx \sqrt{gh}$$



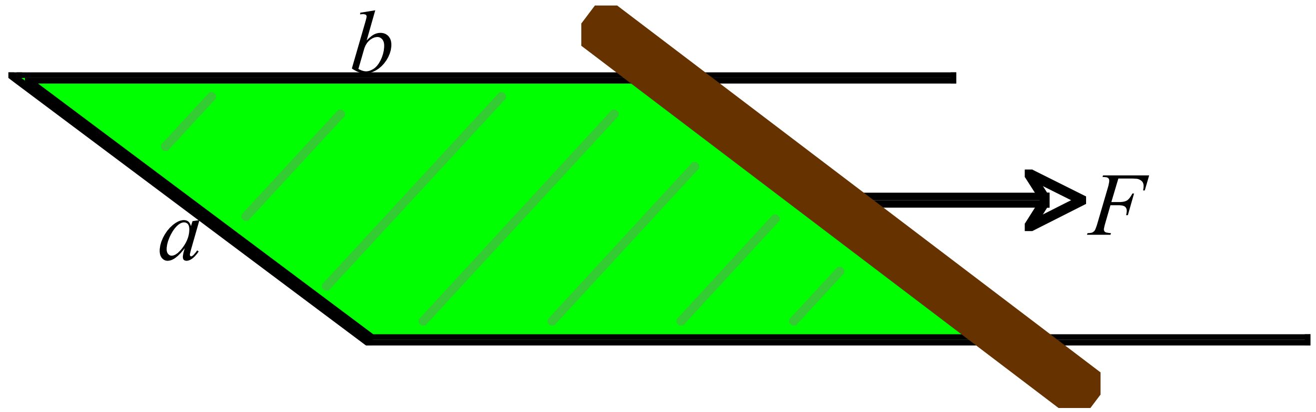
# Surface tension

# Surface tension



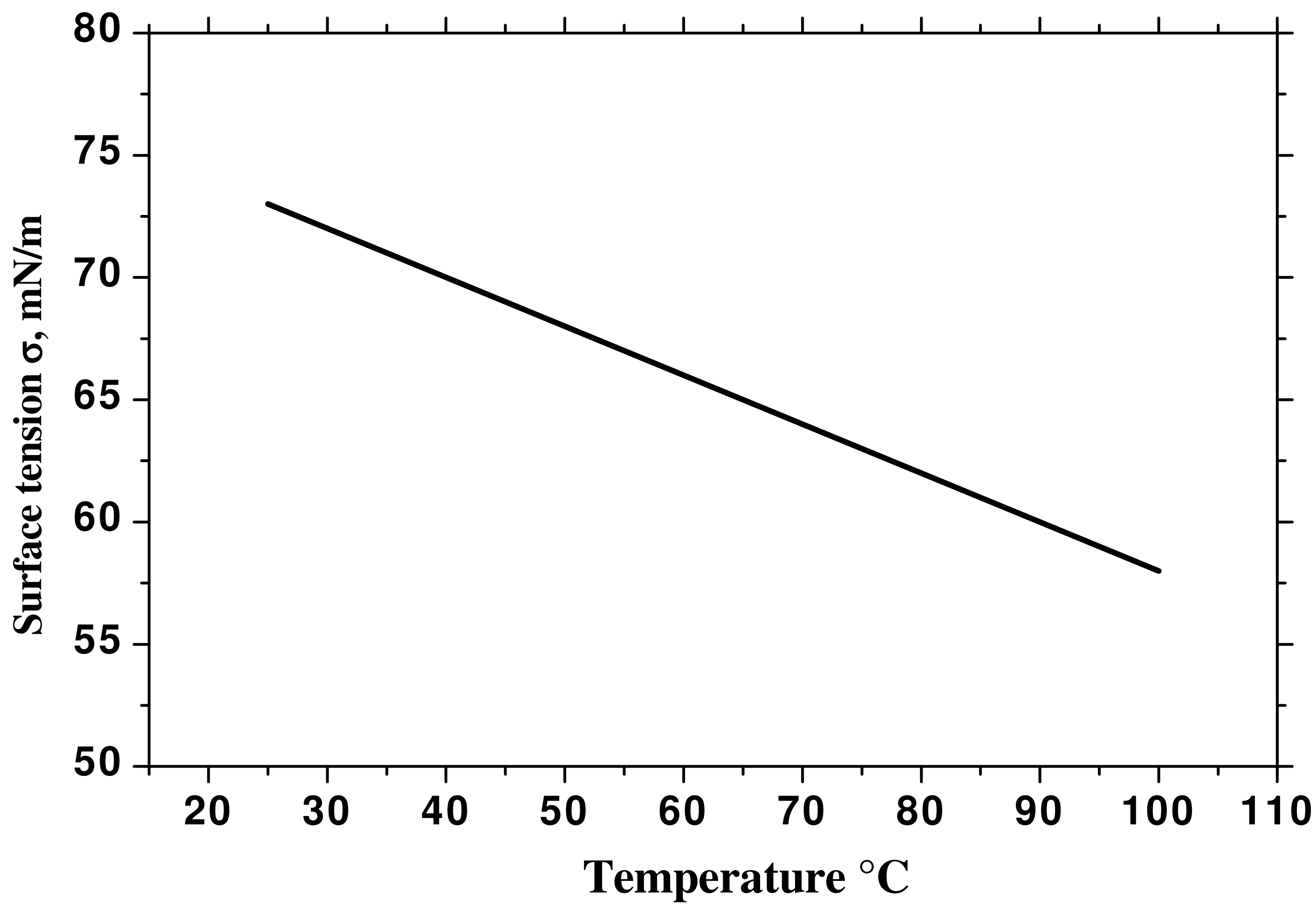

Molecules have a tendency to be drawn into the interior of a liquid to the greatest extent possible, leaving a minimum of surface area.
Because a sphere has a smaller ratio of surface area to volume than any other three-dimensional figure, free-falling liquids tend for form spherical drops.

# Measurement of surface tension




The work done to pull a thin film of fluid has to be equal to the increase in energy:

$$Fdx = 2\sigma a dx$$



# Capillary waves

When the surface of a liquid is curved, the surface tension is acting as a restoring force

# New BC at top

The condition that requires to be modified is the free-surface dynamic boundary condition: in the presence of surface tension, the gauge pressure on the free surface will be nonzero and will be balanced by surface tension.
After linearization, the new term, dependent of the radius of curvature at the surface, will be:

$$g f + \frac{\sigma}{\rho}\frac{\partial^2\theta}{\partial x^2}\bigg|_{z=f} + \frac{\partial\theta}{\partial t}\bigg|_{z=f} = 0$$

leading to the new dispersion relation:

$$\omega^2 = \left(kg + k^3\frac{\sigma}{\rho}\right)\tanh(kh)$$

that shows that surface tension is more significant for
**large k, i.e. wavelengths smaller than the capillary length $(\sigma/\rho g)^{1/2}$, that is 2-3 mm for water!**

# Gravity capillary waves dispersion

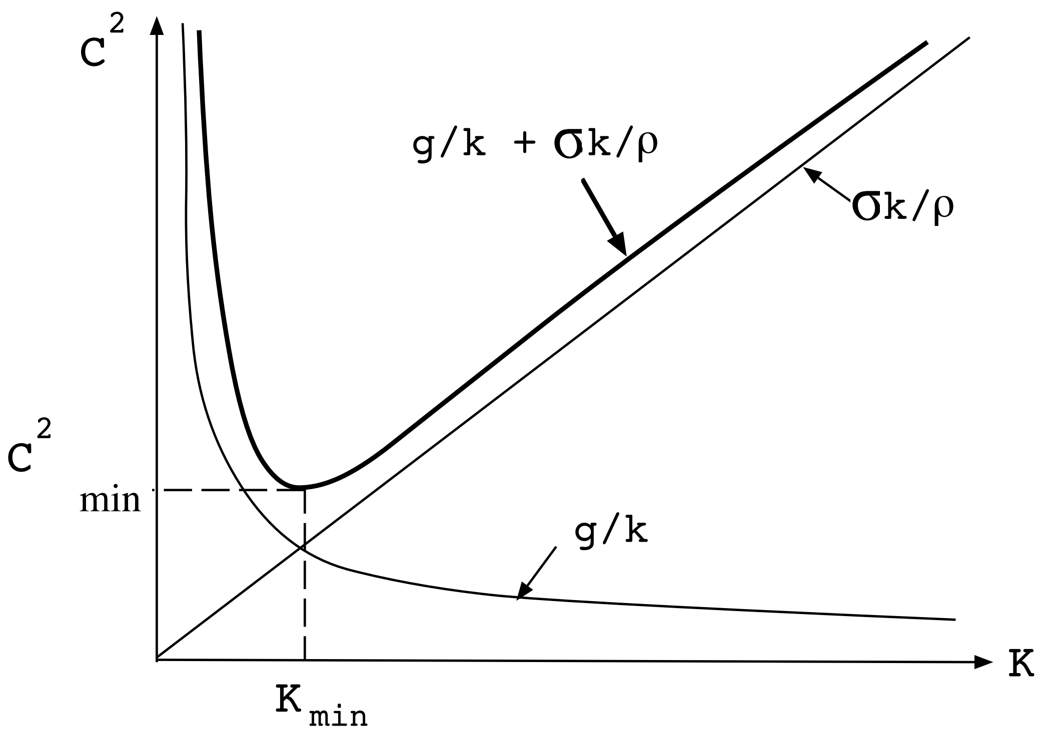
$$\omega^2 = \left( kg + k^3 \frac{\sigma}{\rho} \right) \tanh(kh)$$

**neglecting gravity in deep water**

$$\omega^2 = k^3 \frac{\sigma}{\rho}$$

$$c = \sqrt{\frac{\sigma}{\rho} k} \qquad u = \frac{\partial \omega}{\partial k} = \frac{3}{2} c$$

that shows that there is anomalous dispersion

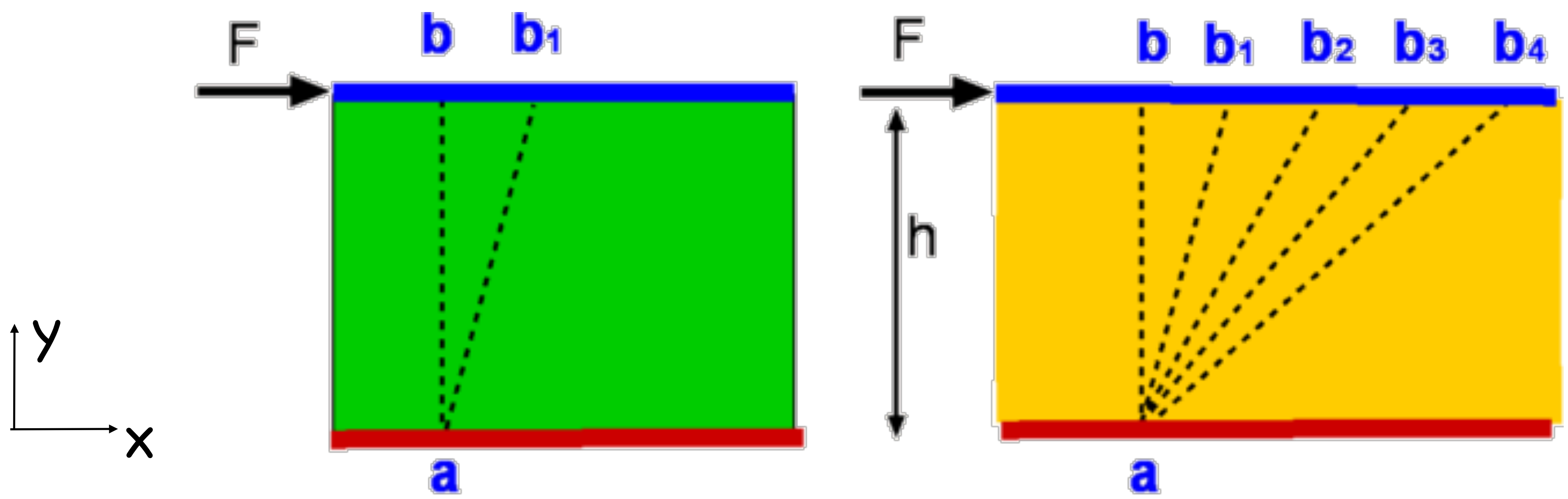



and the $k_{min} = (\rho g/\sigma)^{1/2}$, associated to a wavelength of 1.73 cm for the water, corresponds to a minimum for phase velocity (23.2 cm/s).

Capillary waves on water have usually wavelengths less than 4mm and frequencies higher than 70Hz, thus easily excited by a tuning fork

# Strain as a measure of Deformation

To understand deformation due to shear, picture two flat plates with a fixed spacing, h, between them:




a) Solid filled between plates

b) Fluid filled between plates

Fluids are qualitatively different from solids in their response to a shear stress. Ordinary fluids such as air and water have no intrinsic configuration, and hence fluids **do not develop a restoring force** that can provide a static balance to a shear stress. When the shear stress is held steady, and assuming that the geometry does not interfere, the **shear deformation rate**, may also be steady or have a meaningful time-average.

# Strain as a measure of Deformation

- A strain measure for simple shear can be obtained by dividing the displacement of the moving plate, ΔX, by the distance between the plates:

$$\gamma = \frac{\Delta x}{h} \simeq \frac{dx}{dy}$$

**Shear strain**

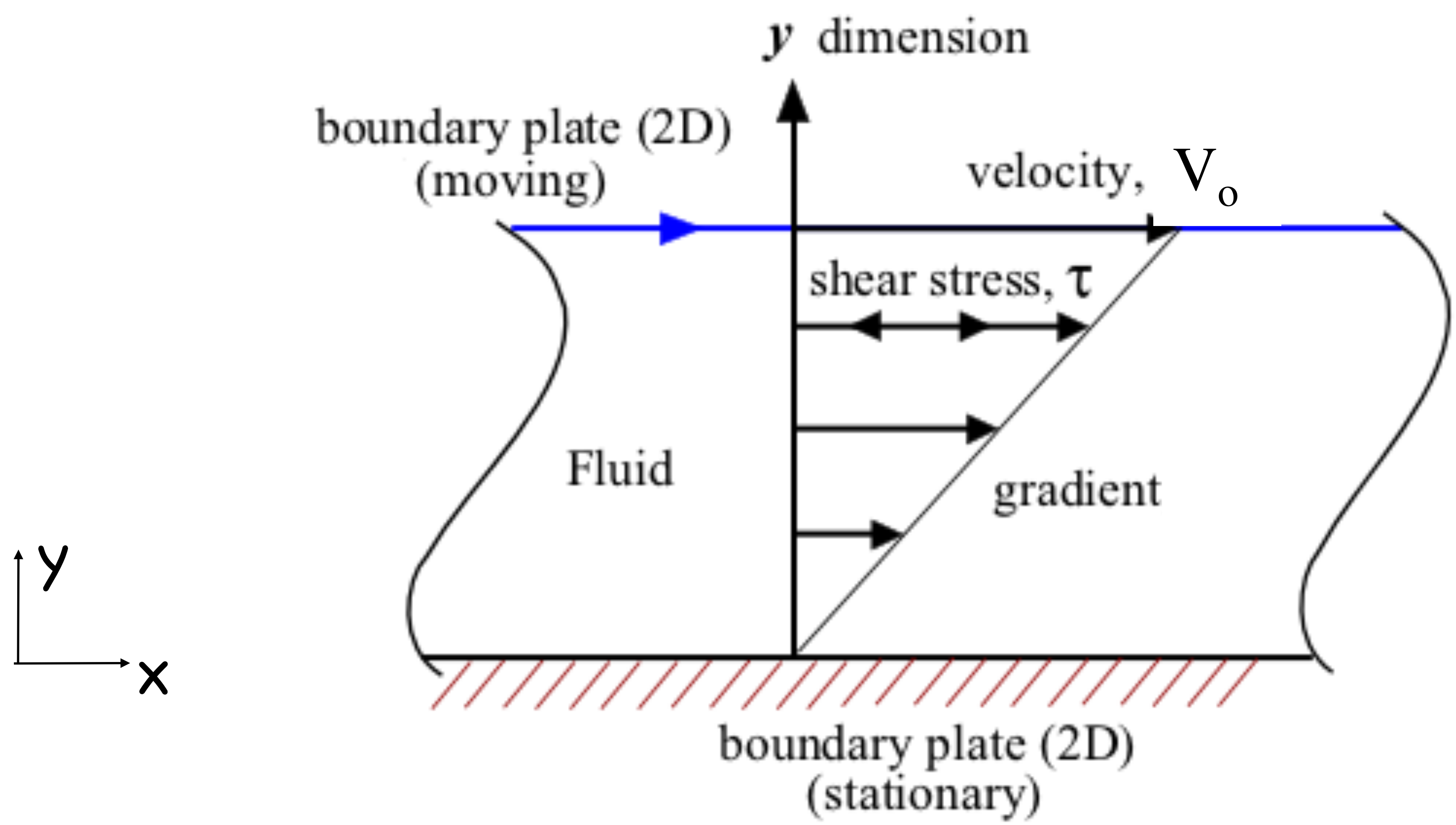
- The **shear rate**, or rate of shearing strain, is the rate of change of shear strain with time:

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{dx}{dt}\right)$$

$$\dot{\gamma} = \frac{dv}{dy}$$

**Shear strain rate**

# Simple Shear Flow

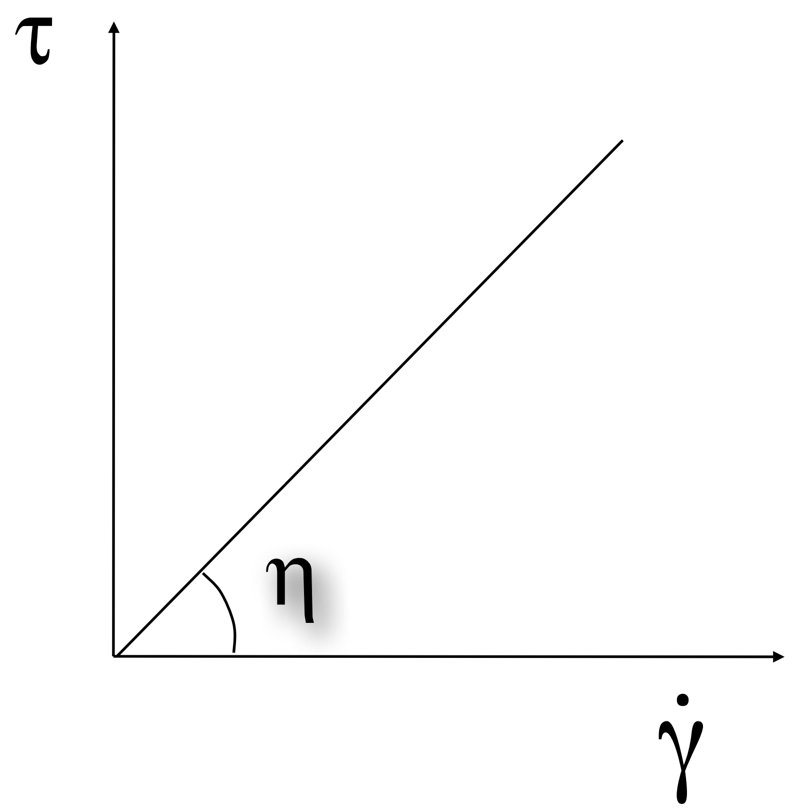



The velocity profile is a straight line:

the velocity varies uniformly from 0 to $V_o$

$$\dot{\gamma} = \frac{dv}{dy} = \frac{V_0}{h}$$

# Newton's Law of Viscosity




This is called a "**flow curve**"

The proportionality constant is the **viscosity**

**Newton's law of viscosity**

$$\tau = \eta \frac{dv}{dy} = \eta \frac{d\gamma}{dt}$$

$\therefore$ The deformation of a material is due to stresses imposed to it.

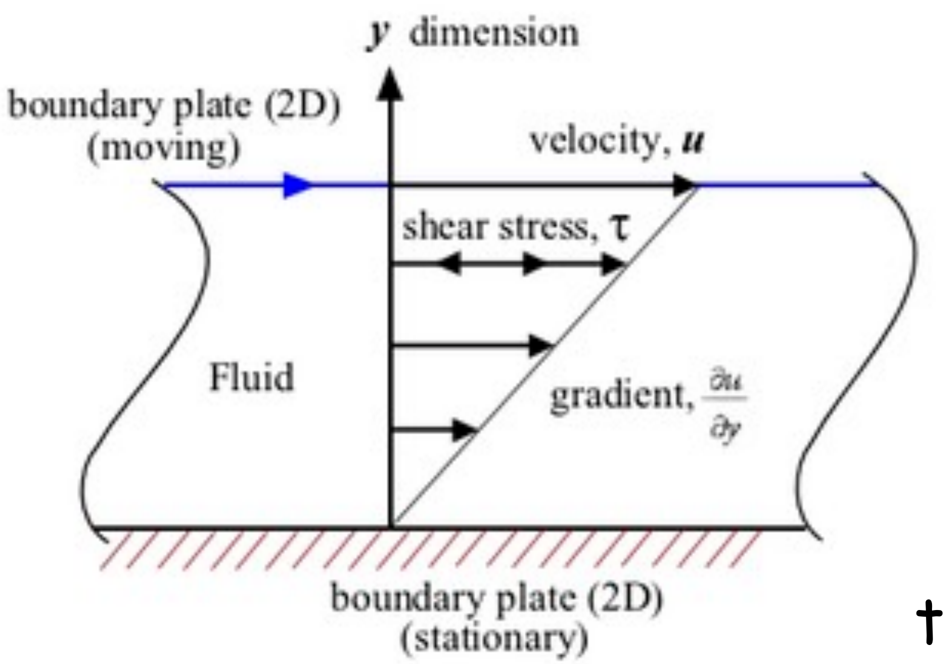
**Newtonian fluids** Fluids which obey Newton's law:

Shearing stress is linearly related to the rate of shearing strain.

The **viscosity** of a fluid measures its resistance to flow under an applied shear stress.

# Viscosity

**Viscosity** is a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to pouring. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Thus, water is "thin", having a lower viscosity, while vegetable oil is "thick" having a higher viscosity. All real fluids (except superfluids) have some resistance to shear stress, but a fluid which has no resistance to shear stress is known as an **ideal fluid**




$$\lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \eta \frac{\partial v_x}{\partial y}$$

thus the shear stress is proportional to the **rate of change of shear strain**

$$\eta\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)$$

In the general case there is another term that depends on other derivatives of the velocity and the general expression is:

$$\eta\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)+\eta'\delta_{ij}\left(\mathrm{div}(\mathbf{v})\right)$$

so two constants are required: the "first coefficient of viscosity" or "shear viscosity coefficient" and "second coefficient of viscosity" .

The component of the viscous force per unit volume in the direction of the rectangular coordinate $x_j$ is:

$$(f_{visc})_i=\sum_{j=1,3}\frac{\partial\tau_{ij}}{\partial x_j}=\sum_{j=1,3}\frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right]+\frac{\partial}{\partial x_i}\left(\eta'\mathrm{div}(\mathbf{v})\right)$$

$$\mathbf{f}_{visc}=\eta\Delta\mathbf{v}+(\eta+\eta')\mathrm{grad}\left(\mathrm{div}(\mathbf{v})\right)$$

# Navier-Stokes equations

Newton's law + Conservation of matter + **Viscosity**

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}\cdot \mathrm{grad})\mathbf{v}=-\mathrm{grad}(P)-\rho\,\mathrm{grad}(\phi)+$$
$$+\eta\Delta\mathbf{v}+(\eta+\eta')\mathrm{grad}\big(\mathrm{div}(\mathbf{v})\big)$$

and in the incompressible case...

$$\frac{\partial\Omega}{\partial t}+\mathrm{rot}(\Omega\times\mathbf{v})=\frac{\eta}{\rho}\Delta\Omega$$

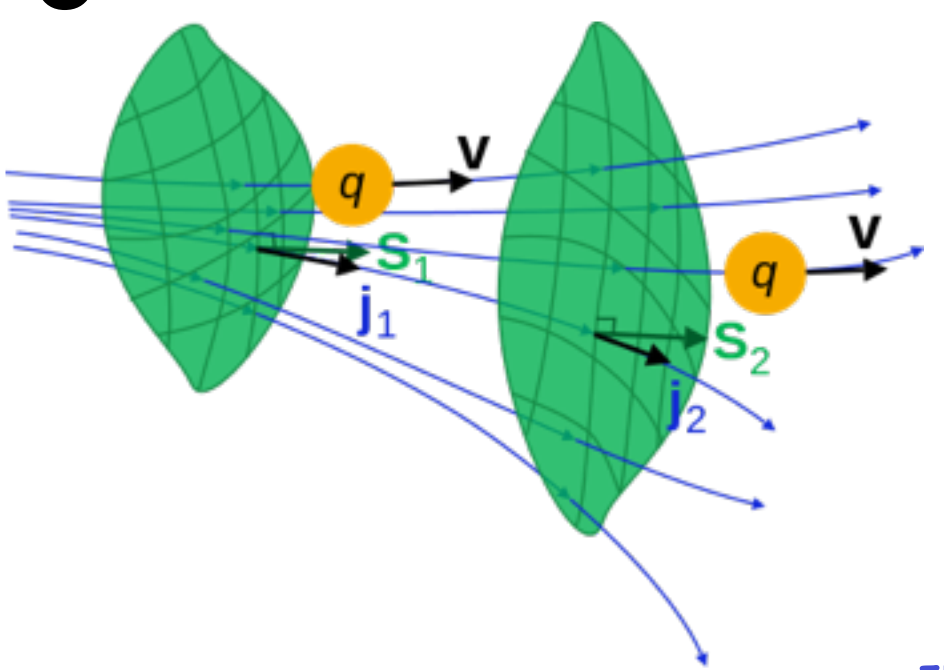
# Continuity equation

- General differential form: ρ is the density of a quantity q, j is the flux of q, σ is the generation of q per unit volume per unit time

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{j}) = \sigma$$

- In fluid dynamics, the continuity equation states that, in any steady state process, the rate at which mass enters a system is equal to the rate at which mass leaves the system:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0$$




- If the fluid is incompressible:

$$\mathrm{div}(\mathbf{v}) = 0$$

# Transport equation

- The convection–diffusion equation is a combination of the **diffusion** and convection (**advection**) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and advection.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{j} - D\,\mathrm{grad}(\rho)) = \sigma$$

- It can be derived in a straightforward way from the continuity equation, which states that the rate of change for a scalar quantity in a differential control volume is given by flow and diffusion into and out of that part of the system along with any generation or consumption inside the control volume

# Continuity and heat equation

- Conservation of energy says that energy cannot be created or destroyed: there is a continuity equation for energy U, is heat per unit volume, and its flow:

$$U = \rho C_p T$$

$$\frac{\partial U}{\partial t} + \mathrm{div}(\mathbf{q}) = 0$$

- When heat flows inside a solid, the continuity equation can be combined with Fourier's law, where **k** is **thermal diffusivity** (W/(m K))

$$\mathbf{q} = -k\ \mathrm{grad}(T)$$

# Continuity and heat equation

- When heat flows inside a solid, the continuity equation can be combined with Fourier's law to arrive at the heat equation, defining $\alpha(m^2/s)$ the heat diffusivity:

$$\frac{\partial T}{\partial t} - \frac{k}{\rho C_p}\Delta(T) = \frac{\partial T}{\partial t} - \alpha\Delta(T) = 0$$

- The equation of heat flow may also have source terms: Although energy cannot be created or destroyed, heat can be created from other types of energy, for example via friction or joule heating.

$$\frac{\partial T}{\partial t} - \alpha\Delta(T) = \sigma$$

# Continuity equation and moment

- Other than advecting momentum, the only other way to change the momentum in our representative volume is to exert forces on it. These forces come in two flavors: stress that acts on the surface of the volume (flux of force) and body forces (acting as a source of momentum):

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \mathrm{div}(\rho \mathbf{v}\mathbf{v}) = \mathrm{div}(\tau) + \mathrm{grad}(\rho\phi)$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \bullet \mathrm{grad})\mathbf{v} = \mathrm{div}(\tau) + \rho \mathbf{g}$$

# Navier-Stokes & Transport equations

advective inertial term — diffusion like viscosity term — buoyancy gravity term

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \mathrm{grad})\mathbf{v} = \eta \Delta \mathbf{v} - \mathrm{grad}(P) - \rho g \alpha T$$

$$\frac{\partial T}{\partial t} = \alpha \Delta(T) - \mathrm{div}(\mathbf{v}T) + \frac{H}{C_p}$$

conductive term — advective term — internal heating term

when the mass density difference is caused by temperature difference, Rayleigh number (Ra) is, the ratio of the time scale for diffusive thermal transport to the time scale for convective thermal transport

$$Ra = \frac{\Delta\rho \, l^3 g}{\eta \, \alpha}$$