4.2.1 Shallow water gravity waves

The simplest case is to consider the shallow water equations in the absence of rotation. Consider small amplitude motions, linearised about a state of rest:

$$h_t(x,y,t) = H + h'(x,y,t)$$
$$\vec{v}_H(x,y,t) = \vec{v}_h'(x,y,t)$$

The linearised momentum equations in the absence of rotation ($f = 0$) therefore become

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \tag{7}$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} \tag{8}$$

and the linearised mass continuity equation becomes

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0. \tag{9}$$

Taking $\partial(7)/\partial x$ and $\partial(8)/\partial y$ and substituting into $\partial(9)/\partial t$ gives

$$\frac{\partial^2 h'}{\partial t^2} - gH(\nabla^2 h') = 0 \tag{10}$$

This is a wave equation for which we can obtain a solution of the form $h' = h_o exp(i(kx + ly - \omega t))$ where k and l represent the wavenumbers in the x and y directions respectively and ω is the wave frequency. Putting such a solution into 10 gives the following dispersion relation

$$\omega = \pm\sqrt{gH}K \qquad \text{where} \qquad K = \sqrt{k^2 + l^2} \tag{11}$$

K is the total wavenumber. The waves that obey this dispersion relation are known as shallow water gravity waves since the restoring force for the wave motion is gravity. If we were considering the 1D case of a wave propagating in the x direction we would have $\omega = \pm\sqrt{gH}k$ i.e. motion with a phase speed $\omega/k = \pm\sqrt{gH}$. The phase speed is independent of wavenuber and depends only on the depth of the fluid layer. Therefore all wavenumbers travel at the same phase speed (the waves are non-dispersive). An initial free surface height perturbation of the form $F(x)$ can be Fourier decomposed into many different wavenumber components and the general solution will be a superposition of all these different fourier components travelling in the $\pm x$ direction with the same phase speed $c = \sqrt{gH}$. Since all Fourier components travel at the same phase speed the initial perturbation will maintain the same shape over time but move to the left and right i.e. the general solution is given by

$$h'(x,t) = \frac{1}{2}\left[F(x - ct) + F(x + ct)\right] \tag{12}$$

These waves must have a wavelength that is larger than the vertical height scale in order for the hydrostatic balance appromation to hold. Therefore the above dynamics doesn't apply to short waves in a deep ocean but Tsunami's can be considered as shallow water waves due to their large length scales. For gravity waves in the atmosphere the stratification is important and they obey slightly different dynamics.

4.2.2 Poincaré (or Inertio-gravity) waves

The linearised shallow water equations in the presence of rotation on an f-plane (i.e. the coriolis parameter is constant $f_o = 2\Omega sin\phi_o$), are

$$\frac{\partial u'}{\partial t} - f_o v' = -g\frac{\partial h'}{\partial x} \tag{13}$$

$$\frac{\partial v'}{\partial t} + f_o u' = -g\frac{\partial h'}{\partial y} \tag{14}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{15}$$

These equations can be combined to give the following wave equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} + f_o^2 - c^2\nabla^2\right)h' = 0 \tag{16}$$

where $c = \sqrt{gH}$. This is again a wave equation with a plane wave solution of the form $h' = h_o exp(i(kx + ly - \omega t))$ which upon substitution into 16 gives

$$-i\omega\left(-\omega^2 + c^2(k^2 + l^2) + f_o^2\right) = 0. \tag{17}$$

which has solutions

$$\omega = 0 \qquad \text{or} \qquad \omega^2 = f_o^2 + c^2(k^2 + l^2) \tag{18}$$

The zero frequency case is the time independent flow that satisfies geostrophic balance. The waves that satisfy the other dispersion relation are known as Poincarè waves or (inertio-gravity waves). Both the effects of rotation and gravity are important for these waves. Considering the case of 1D propagation in the x direction, the phase speed is given by

$$c = \frac{\omega}{k} = \sqrt{\frac{f_o^2}{k^2} + c^2} \tag{19}$$

i.e. the phase speed is no longer independent of wavenumber. The waves are now dispersive and longer waves have larger phase speeds. The group velocity is given by

$$c_g = \frac{\partial\omega}{\partial k} = \frac{kc^2}{\sqrt{f_o^2 + c^2k^2}} \tag{20}$$

which is always less than 1 and is smallest for large wavelengths (small wavenumbers).

There are two interesting limits to consider:

- **The short wave limit** - If $(k^2 + l^2) >> f_o^2/gH$ then the dispersion relation can be approximated by

$$\omega^2 = c^2(k^2 + l^2) \tag{21}$$

  i.e. the shallow water gravity wave solution that occurs in the absence of rotation. Stating that $(k^2 + l^2) >> f_o^2/gH$ is equivalent to stating that $\lambda << \sqrt{gH}/f_o$ where $\lambda = 2\pi/\sqrt{k^2 + l^2}$ is the horizontal wavelength. So, if the wave disturbance is of sufficiently short length scale it will not be large enough to feel the effects of the rotation and the behaviour will approximately be that of shallow water gravity waves.