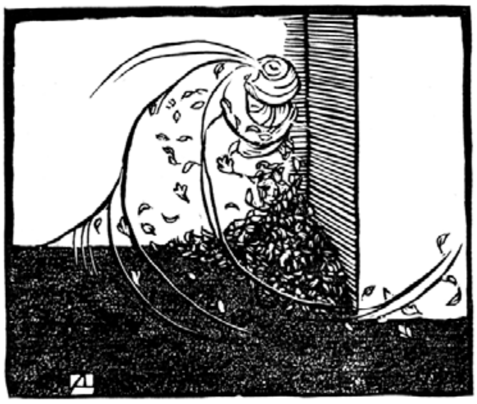
Vortices occur on small scales

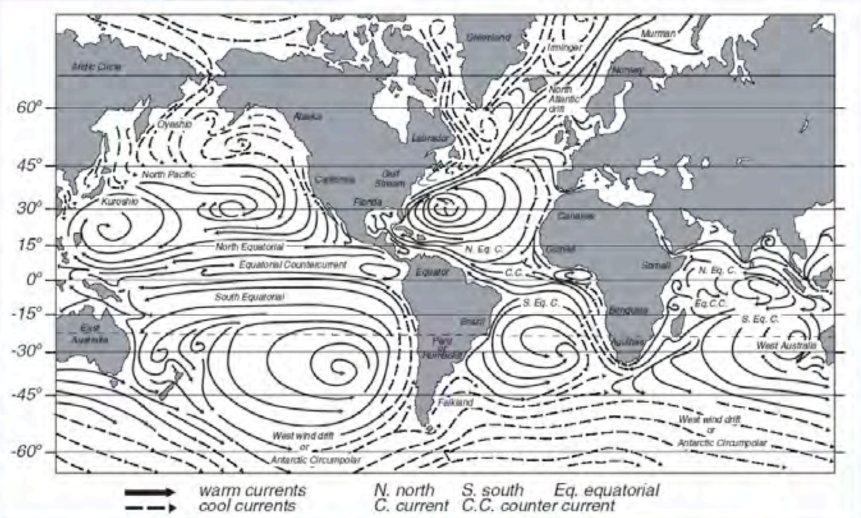
Leaves whirling in the wind



Von Karman street behind bridge pillars

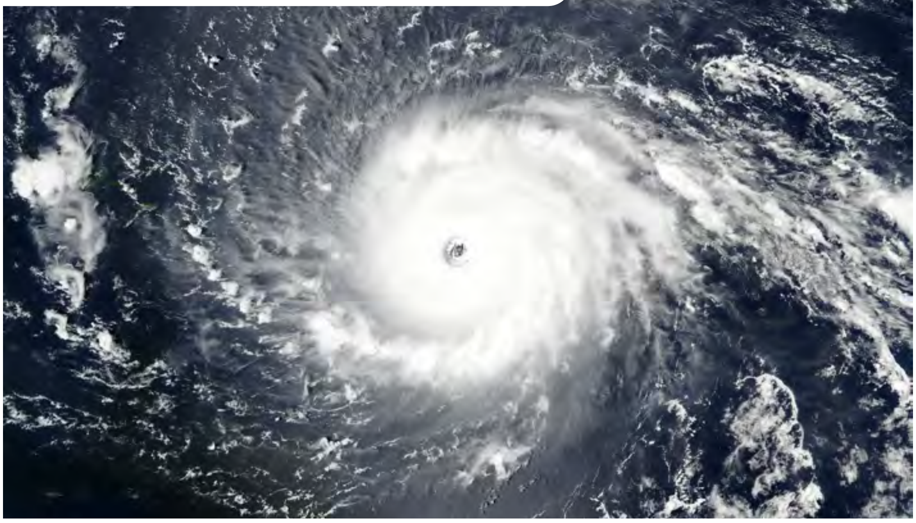
Vortices occur in the oceans

Large scale wind-driven circulation in the upper ocean



Vortices occur in the atmosphere

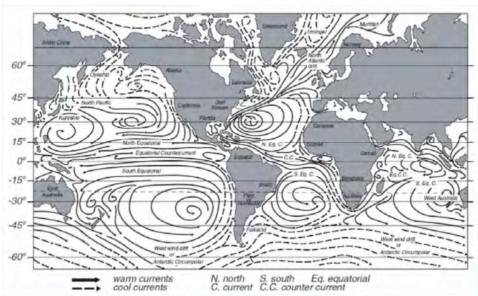
Satellite image of tropical cyclone Irma (2017)

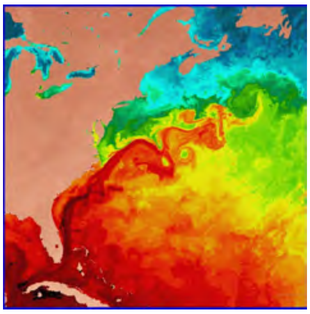


Vortices are omni-present in geophysical flows

Because of:

- (two-dimensional) turbulence
- Rotation of the Earth
  - In particular its latitudinal variation





Definition of vorticity

3D:

$$\vec{\zeta} \equiv \nabla \times \vec{u}$$

2D:

$$\zeta = \vec{e}_3 \cdot \nabla \times \vec{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- In 3D: curl of velocity field.
- In 2D: vertical component only.

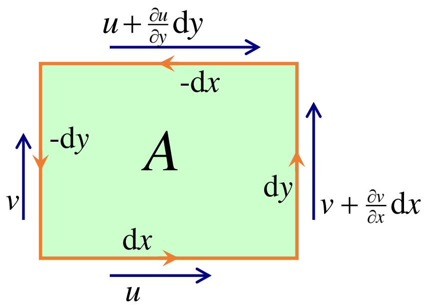
Interpretation:

- Local measure of rotation:
  spin of infinitesimal fluid parcel around its own axis.
- positive: counterclockwise, negative: clockwise, cf. cork screw rule
- Local density of circulation per unit infinitesimal area:
  Stokes theorem (Green in 2D):

$$\oint_{\partial A} \vec{u} \cdot \mathrm{d}l = \iint_A \zeta \, \mathrm{d}x \, \mathrm{d}y$$

Local density of circulation per unit infinitesimal area

Circulation along the red path around infinitesimal area A:



$$\oint_{\partial A} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathrm{d}l = u \, \mathrm{d}x + \\ + (v + \tfrac{\partial v}{\partial x} \mathrm{d}x) \mathrm{d}y \\ -(u + \tfrac{\partial u}{\partial y} \mathrm{d}y) \mathrm{d}x - v \, \mathrm{d}y = \\ = (\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}) \mathrm{d}x \, \mathrm{d}y = \iint_A \zeta \, \mathrm{d}x \mathrm{d}y$$

Dividing by the area $A = \mathrm{d}x \; \mathrm{d}y$, one finds: vorticity is the local circulation per unit area.

Vorticity in the geophysical context

On the Earth, even fluids at rest have vorticity:

Planetary vorticity due to rigid body rotation of the Earth:

$$\zeta_p = 2\Omega \sin(\theta) = f$$

Fluid motion is measured relative to the Earth' rotation, so
Relative vorticity is calculated from velocities relative to the Earth:

$$\zeta_r = \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

In an inertial frame, fluid motion is both with and relative to the Earth,
Absolute vorticity is the sum of planetary and relative vorticity:

$$\zeta_a = \zeta + f$$

Potential vorticity is a conserved quantity following fluid parcels:

$$\Pi = \frac{\zeta + f}{h}$$

It will be derived later.

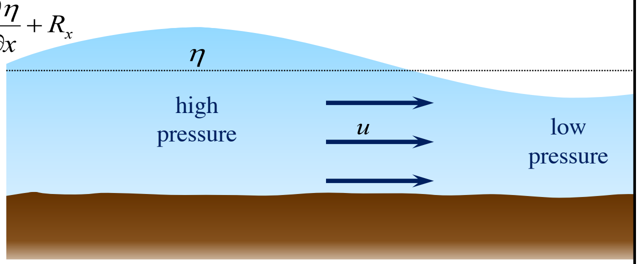
Depth-averaged 2D shallow water equations

Momentum equation: $\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - f\,v\right) = -\frac{\partial p}{\partial x} + r_x$

Main assumption: vertical scale << horizontal scales of motion

- Velocities (almost) independent of z
- Hydrostatic balance: $p = \rho\, g\, (\eta - z)$

Depth-averaged: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f\,v = -g\frac{\partial \eta}{\partial x} + R_x$



Derivation of the vorticity equation

(starting from the depth-averaged 2D shallow water equations)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-g\frac{\partial \eta}{\partial x}+R_x$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-g\frac{\partial \eta}{\partial y}+R_y$$

Now differentiate v-momentum equation with respect to x, u-momentum equation with respect to y and substract:

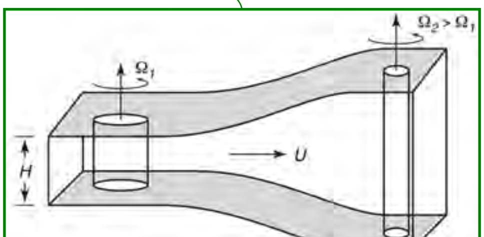
Remember: $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

Vorticity equation

$$\frac{D\zeta}{Dt}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\beta v=\frac{\partial R_y}{\partial x}-\frac{\partial R_x}{\partial y}$$

Local derivative + advection

Stretching or ballerina effect



From Stewart(2005)

Vorticity equation

$$\frac{D\zeta}{Dt} + (\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\beta v = \frac{\partial R_y}{\partial x} - \frac{\partial R_x}{\partial y}$$

Local derivative + advection

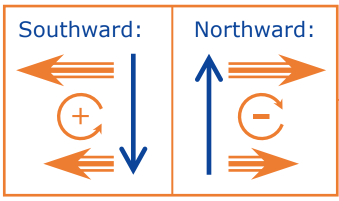
Stretching or ballerina effect

Beta-effect

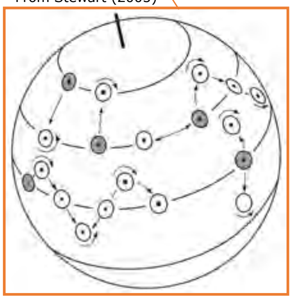
Frictional torque

Important contributions in depth-integrated equations due to wind stress and bottom drag:

$$\frac{\partial \frac{1}{\rho}(\tau_y^w - \tau_y^b)}{\partial x} - \frac{\partial \frac{1}{\rho}(\tau_x^w - \tau_x^b)}{\partial y}$$



From Stewart (2005)



Intermezzo: Depth-integrated continuity equation

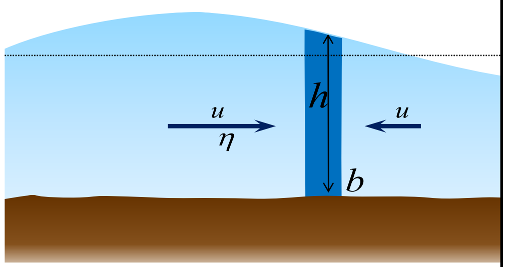
Continuity equation (incompressible flow): $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

Integrate over depth from bottom ($z = b$) to surface ($z = \eta$):

$$0 = h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+w|_{z=\eta} - w|_{z=b} = h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{D\eta}{Dt}-\frac{Db}{Dt}$$

(remember: u,v do not depend on z in shallow water)

Conclusion: $\frac{Dh}{Dt}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$



Derivation of potential vorticity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{1}{h}\frac{\mathrm{D}h}{Dt}$$

to eliminate the convergence term in the vorticity equation:

$$\frac{\mathrm{D}\zeta}{Dt}-\frac{\zeta+f}{h}\frac{\mathrm{D}h}{Dt}+\frac{\mathrm{D}f}{Dt}=\frac{\partial R_y}{\partial x}-\frac{\partial R_x}{\partial y}$$

$$\boxed{\frac{\mathrm{D}}{Dt}\left(\frac{\zeta+f}{h}\right)=\frac{1}{h}\left(\frac{\partial R_y}{\partial x}-\frac{\partial R_x}{\partial y}\right)}$$

So in the absence of friction, potential vorticity is conserved along trajectories!

Vorticity-driven flows

Introductory:

a) Zonal flow

b) Topographic steering

c) Interaction between vortices

More advanced:

d) Moving vortices due to β-effect

e) Large-scale wind driven circulation

a) Zonal flow in the atmosphere

Remind: Potential vorticity = $\Pi = \dfrac{\zeta + f}{h}$

Conserving potential vorticity when:

- velocities are relatively small $(\zeta << f)$,
- depth h does not change much,

implies that f does not change much,
i.e. flow is zonal…

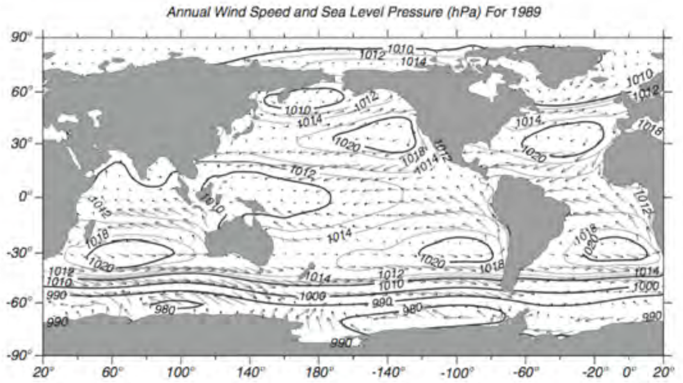
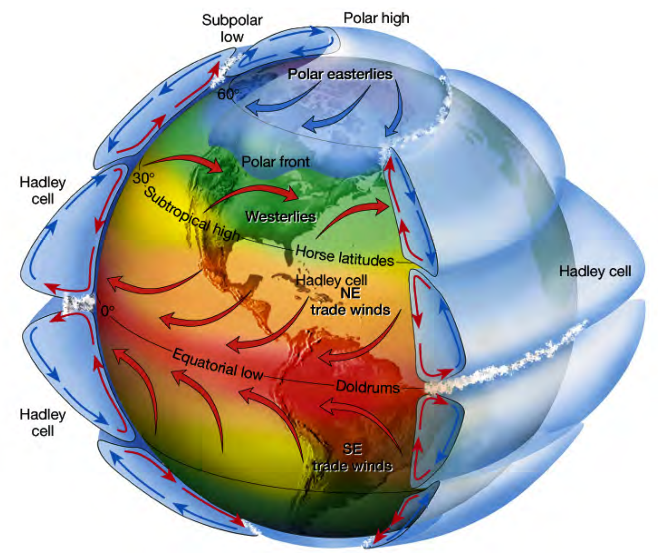


Figure 4.2 Map of mean annual wind velocity calculated from Trenberth et al. (1990) and sea-level pressure for 1989 from the NASA Goddard Space Flight Center's Data Assimilation Office (Schubert et al. 1993). The winds near 140°W in the equatorial Pacific are about 8 m/s.

a) Zonal flow in the atmosphere

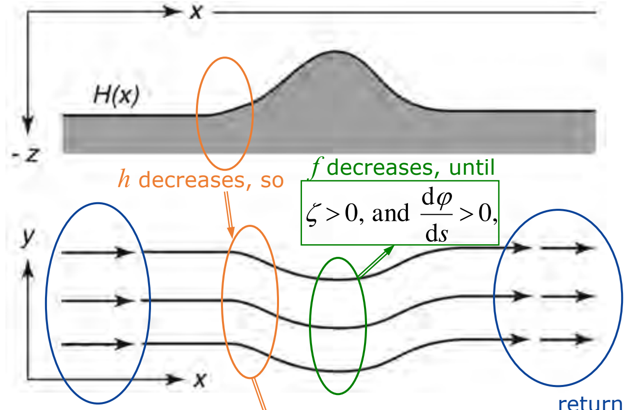


Warm air rises along equator, cold air sinks air at poles, inducing meridional flow;
Angular momentum conservation requires huge westerly winds
⇒ instabilities develop and flow will be zonal.

b) Topographic steering

A northern hemisphere example.

Potential vorticity: $\Pi = \dfrac{\zeta + f}{h}$



h decreases, so

f decreases, until $\zeta > 0$, and $\dfrac{d\varphi}{ds} > 0$,

return to normal by reverse process.

laminar flow enters from the left: $\zeta = 0$, so $\Pi = \dfrac{f}{h}$

$\zeta < 0$, i.e. $\dfrac{d\varphi}{ds} < 0$,

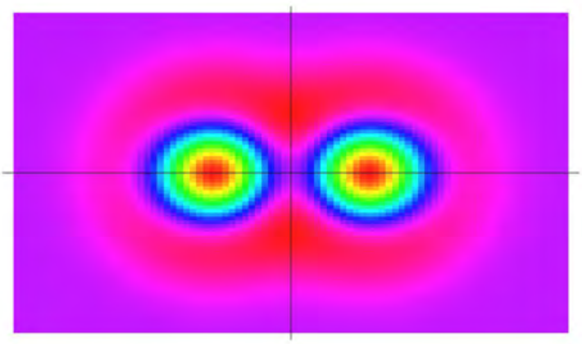
turning right,

c) Interacting vortices

Suppose:

- No friction,
- constant depth (i.e. no stretching),
- Length scale small compared to Earth' radius (i.e. no beta).

And we have a vorticity field like this (two positive vortices):



First question: what is the corresponding velocity field?

Vorticity dynamics: $0 = \dfrac{D\xi}{Dt} = \dfrac{\partial \xi}{\partial t} + u\dfrac{\partial \xi}{\partial x} + v\dfrac{\partial \xi}{\partial y}$

Vorticity is just advected with the flow!