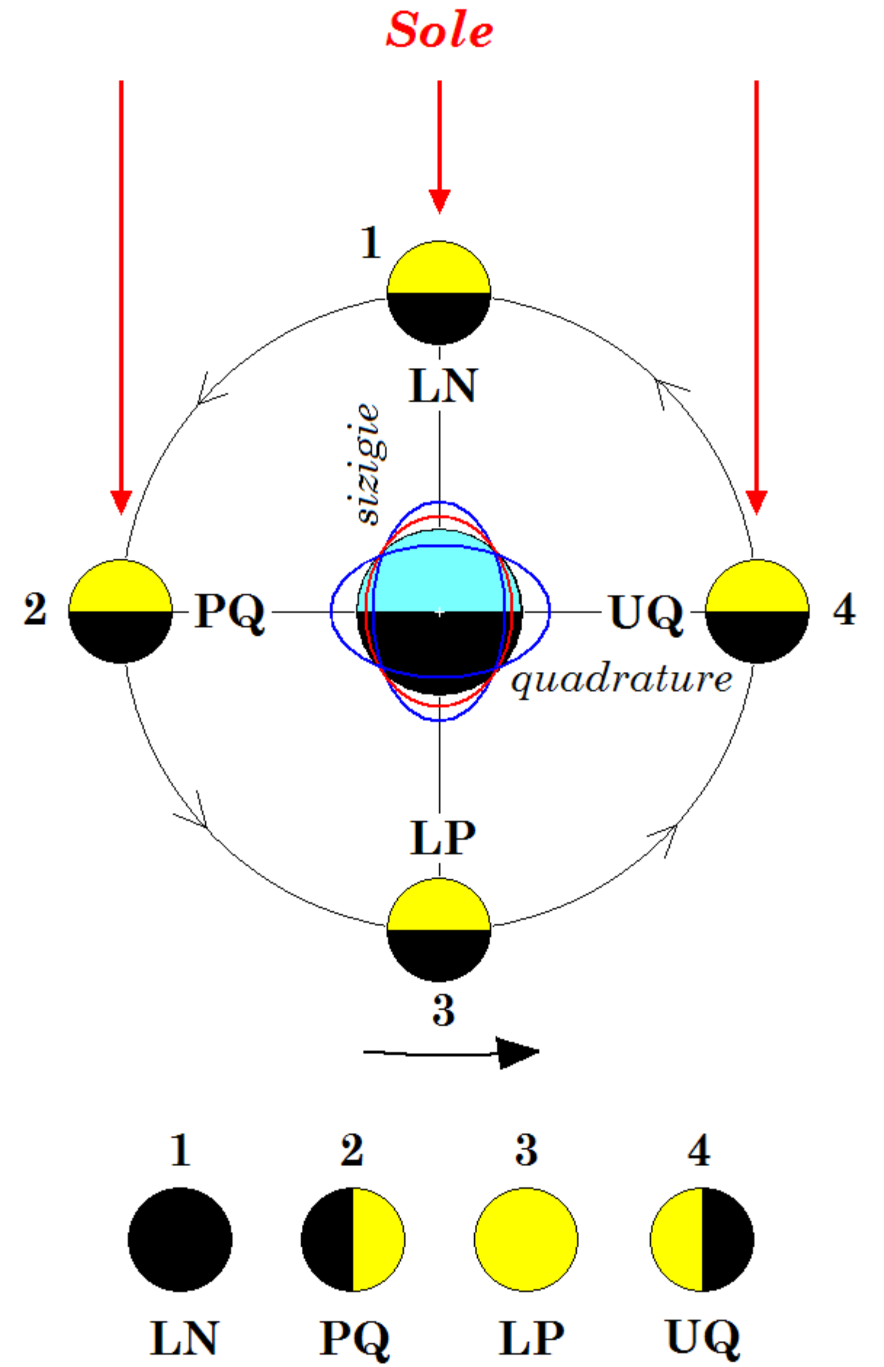
Harmonic Analysis and the Prediction of Tides

"THE SUBJECT on which I have to speak this evening is the tides, and at the outset I feel in a curiously difficult position. If I were asked to tell what I mean by the Tides I should feel it exceedingly difficult to answer the question. The tides have something to do with motion of the sea."

Lord Kelvin, 1882



Sole
1
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1
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LN
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Tidal Potential Periods

$$V = \frac{GMr^2}{4R^3}\left[\begin{array}{l}(3\sin^2\varphi_p - 1)(3\sin^2\delta - 1) \\ +3\sin 2\varphi_p \sin 2\delta \cos\tau \\ +3\cos^2\varphi_p \cos^2\delta \cos 2\tau\end{array}\right]$$

Lunar Tidal Potential - periods near 14 days, 24 hours, and 12 hours

Solar Tidal Potential - periods near 180 days, 24 hours, and 12 hours

Doodson (1922) - Fourier Series Expansion using 6 frequencies

Doodson’s Frequencies

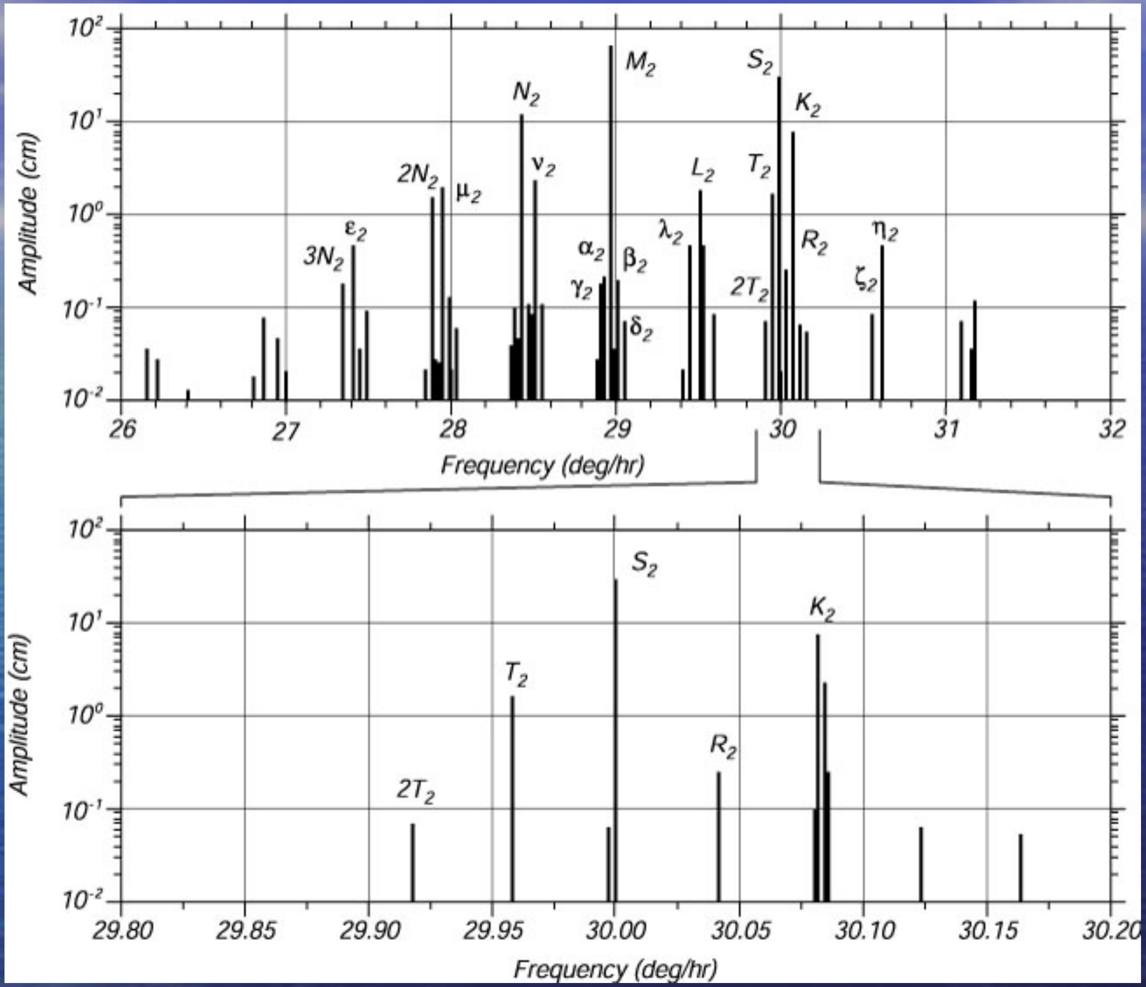
	Frequency (°/hour)		Period	Source
f_1	14.49205211	1	lunar day	Local mean lunar time
f_2	0.54901653	1	month	Moon's mean longitude
f_3	0.04106864	1	year	Sun's mean longitude
f_4	>0.00464184	8.847	years	Longitude of Moon's perigee
f_5	-0.00220641	18.613	years	Longitude of Moon's ascending node
f_6	0.00000196	20,940	years	Longitude of sun's perigee

$$f = n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4 + n_5 f_5 + n_6 f_6$$

The Tidal Constituents

Tidal Species	Name	n_1	n_2	n_3	n_4	n_5	Equilibrium Amplitude* (m)	Period (hr)
Semidiurnal	**$n_1 = 2$**							
Principal lunar	M_2	2	0	0	0	0	0.242334	12.4206
Principal solar	S_2	2	2	-2	0	0	0.112841	12.0000
Lunar elliptic	N_2	2	-1	0	1	0	0.046398	12.6584
Lunisolar	K_2	2	2	0	0	0	0.030704	11.9673
Diurnal	**$n_1 = 1$**							
Lunisolar	K_1	1	1	0	0	0	0.141565	23.9344
Principal lunar	O_1	1	-1	0	0	0	0.100514	25.8194
Principal solar	P_1	1	1	-2	0	0	0.046843	24.0659
Elliptic lunar	>Q_1	1	-2	0	1	0	0.019256	26.8684
Long Period	**$n_1 = 0$**							
Fortnightly	M_f	0	2	0	0	0	0.041742	327.85
Monthly	M_m	0	1	0	-1	0	0.022026	661.31
Semiannual	S_{sa}	0	0	2	0	0	0.019446	4383.05

Constituent Splitting



Doodson's expansion:399 constituents,

100 are long period, 160 are daily, 115 are twice per day, and 14 are thrice per day. Most have very small amplitudes.

Sir George Darwin named the largest tides.

How to Obtain Constituents

- Fourier (Spectral) Analysis
- Harmonic Analysis

Fourier Analysis … In the beginning …

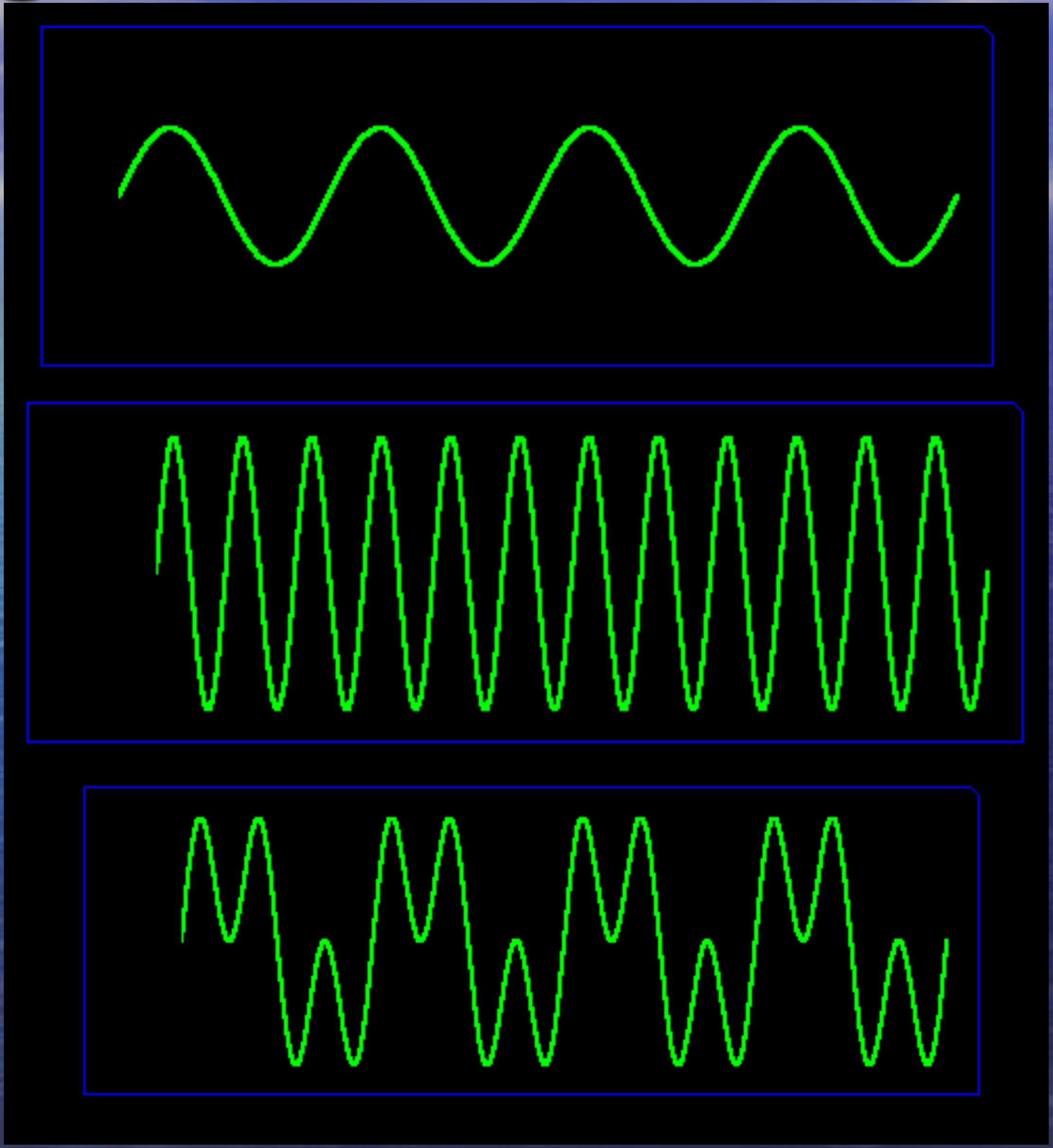
- 1742 – d'Alembert – solved wave equation
- 1749 – Leonhard Euler – plucked string
- 1753 – Daniel Bernoulli – solutions are superpositions of harmonics
- 1807 - Joseph Fourier solved heat equation

Problems – lead to modern analysis!

$$y(x,t) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L}$$

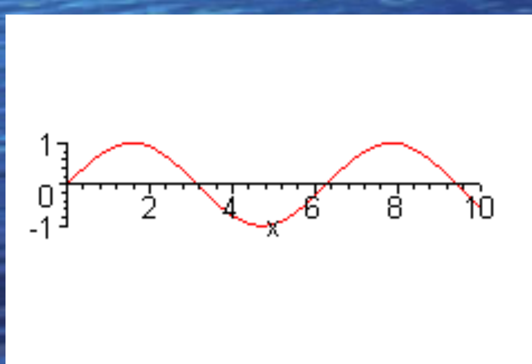
$$y(x,0) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{L}$$

Adding Sine Waves



Spectral Theory

- Fourier Series
  - Sum of Sinusoidal Functions
- Fourier Analysis
  - Spectrum Analysis
  - Harmonic Analysis



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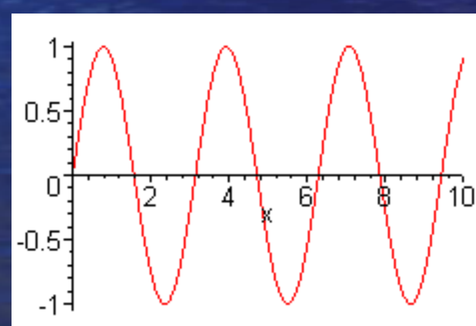

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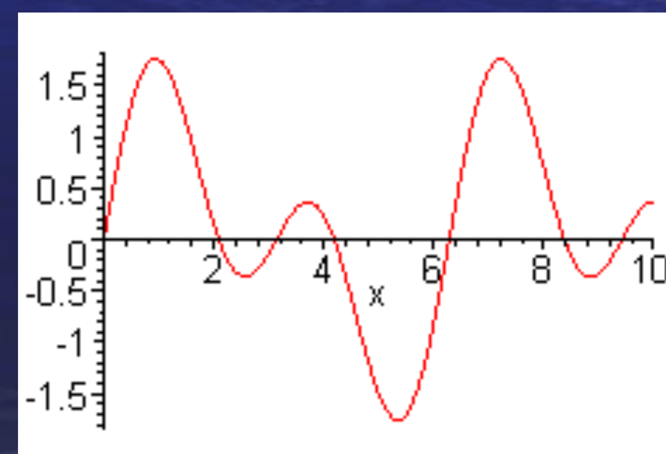


Fourier Series

Fourier Series

Eigenfunctions:

$$L := 2\pi \quad c(x,n) := \cos\left(\frac{n\cdot\pi\cdot x}{L}\right) \quad s(x,n) := \sin\left(\frac{n\cdot\pi\cdot x}{L}\right)$$

Function: $f(x) := \sin(x) + 2\cos(2x) \quad x := 0, .1 .. L$

Fourier Coefficients:

$$N := 8 \quad n := 1 .. N \qquad a_0 := \frac{1}{2L}\cdot\int_{-L}^{L} f(x)\,dx$$

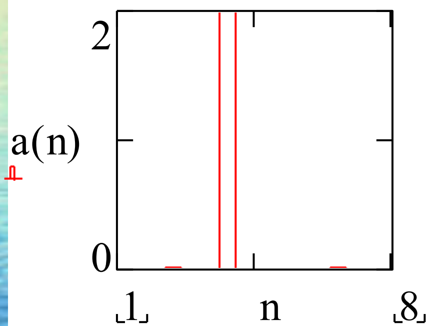
$$a(n) := \frac{1}{L}\cdot\int_{-L}^{L} f(x)\cdot c(x,n)\,dx$$

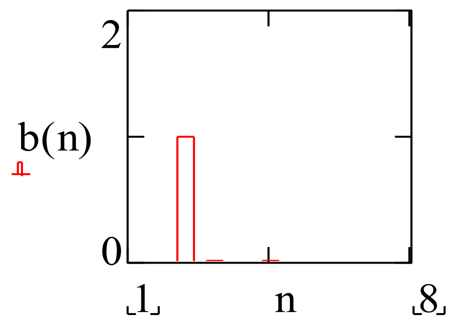
$$b(n) := \frac{1}{L}\cdot\int_{-L}^{L} f(x)\cdot s(x,n)\,dx$$

Reconstruction

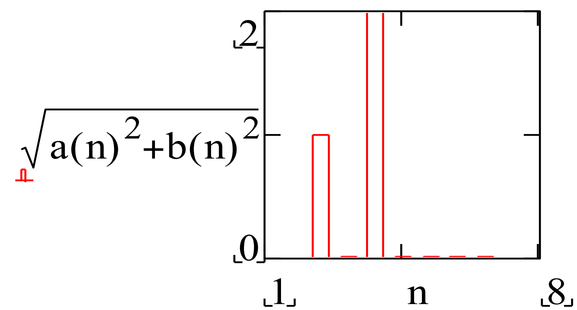
Fourier Expansion

$$F(x) := a_0 + \sum_n (a(n) \cdot c(x,n) + b(n) \cdot s(x,n))$$

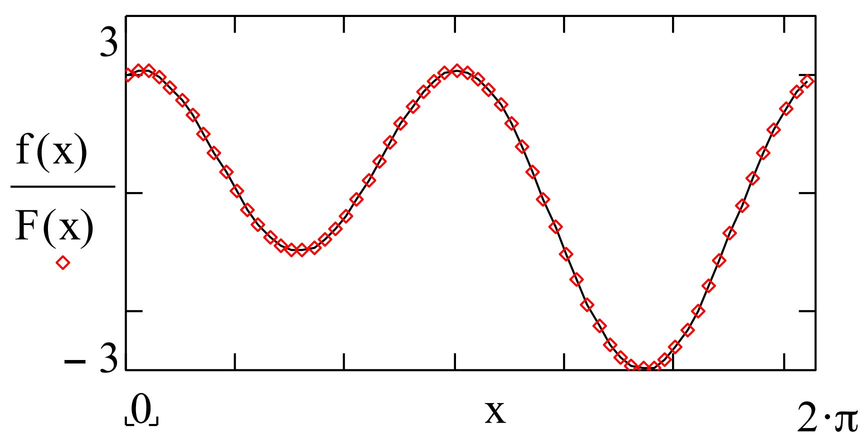




Power Spectrum



Comparison
between
f(x) and F(x)



Analog Signals

- Analog Signals
  - Continuous in time and frequency
  - Infinite time and frequency domains
  - Described by Fourier Transform
- Real Signals
  - Sampled at discrete times
  - Finite length records
  - Leads to discrete frequencies on finite interval
  - Described by Discrete Fourier Transform

Harmonic Analysis

- Consider a set of data consisting of *N* values at equally spaced times,
- We seek the best approximation using *M* given frequencies.
- The unknown parameters in this case are the A's and B's.

$$y(t) = A_0 + \sum_{k=1}^{M} [A_k \cos(2\pi f_k t) + B_k \sin(2\pi f_k t)]$$

Linear Regression

- Minimize

$$e^2 = \sum_{n=1}^{N} [y(t_n) - (A_0 + \sum_{k=1}^{M} [A_k \cos(2\pi f_k t_n) + B_k \sin(2\pi f_k t_n)])]^2$$

- Normal Equations

$$0 = \frac{\partial e^2}{\partial A_q} = 2\sum_{n=1}^{N} [y(t_n) - (A_0 + \sum_{k=1}^{M} [A_k \cos(2\pi f_k \frac{n}{N}) + B_k \sin(2\pi f_k \frac{n}{N})])](-\cos(2\pi f_q \frac{n}{N})),\ q = 1,\ldots,M$$

$$0 = \frac{\partial e^2}{\partial B_q} = 2\sum_{n=1}^{N} [y(t_n) - (A_0 + \sum_{k=1}^{M} [A_k \cos(2\pi f_k \frac{n}{N}) + B_k \sin(2\pi f_k \frac{n}{N})])](-\sin(2\pi f_q \frac{n}{N})),\ k = 1,\ldots,M$$

System of Equations – *DZ=Y*

$$Y = \begin{pmatrix} \bar{y} \\ Cy \\ Sy \end{pmatrix}, \quad Z = \begin{pmatrix} A \\ B \end{pmatrix}, \qquad D = \begin{pmatrix} N & c^T & s^T \\ c & CC & CS \\ s & CS & SS \end{pmatrix}$$

$$y = \begin{pmatrix} y(t_1) \\ \vdots \\ y(t_N) \end{pmatrix}$$

$$\left(CC^T\right)_{qk} = \left(\sum_{n=1}^{N} \cos(2\pi f_k t_n) \cos(2\pi f_q t_n) \right)$$

$$c_q = \sum_{n=1}^{N} C_{qn}, \quad s_q = \sum_{n=1}^{N} S_{qn}$$

$$\left(CS^T\right)_{qk} = \left(\sum_{n=1}^{N} \sin(2\pi f_k t_n) \cos(2\pi f_q t_n) \right)$$

$$S_{qn} = \sin(2\pi f_k t_n),\ q = 1, \ldots, M,\ n = 1, \ldots, N$$

$$C_{qn} = \cos(2\pi f_k t_n),\ q = 1, \ldots, M,\ n = 1, \ldots, N$$

$$\left(SS^T\right)_{qk} = \left(\sum_{n=1}^{N} \sin(2\pi f_k t_n) \sin(2\pi f_q t_n) \right).$$