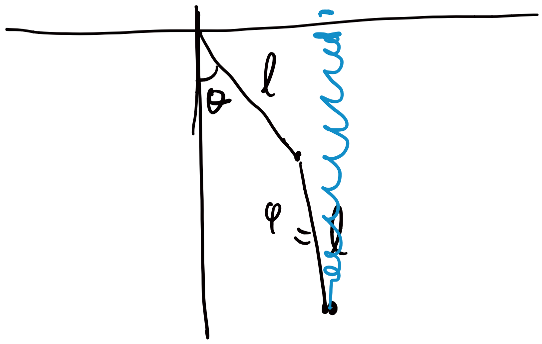

$$x = l(\operatorname{sen}\theta + \operatorname{sen}\varphi) \qquad \dot{x} = l\dot{\theta}\cos\theta + l\dot{\varphi}\cos\varphi$$

$$y = -l(\cos\theta + \cos\varphi) \qquad \dot{y} = l\dot{\theta}\operatorname{sen}\theta + l\dot{\varphi}\operatorname{sen}\varphi$$

$$T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\overbrace{(\cos\theta\cos\varphi + \operatorname{sen}\theta\operatorname{sen}\varphi)}^{\cos(\theta-\varphi)}\right) = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)\right)$$

$$V = -mgl(\cos\theta + \cos\varphi) + \frac{1}{2}kl^2(\cos\theta + \cos\varphi)^2$$

1) LAGRANGIANA e MATRICE CINETICA

$$L = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)\right) + mgl(\cos\theta + \cos\varphi) - \frac{1}{2}kl^2(\cos\theta+\cos\varphi)^2$$

$$A = ml^2\begin{pmatrix} 1 & \cos(\theta-\varphi) \\ \cos(\theta-\varphi) & 1 \end{pmatrix}$$

2) Quante costanti del moto ci sono? Quali? C'è un'unica costante del moto, l'ENERGIA (perché la Lagrangiana non dip. esplicit. dal tempo)

3) Calcolare i momenti coniugati alle coordinate θ e φ

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml^2\dot{\varphi}\cos(\theta-\varphi)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = ml^2\dot{\varphi} + ml^2\dot{\theta}\cos(\theta-\varphi)$$

4) Eq. di Lagrange

$$\dot{p}_\theta = ml^2\ddot{\theta} + ml^2\ddot{\varphi}\cos(\theta-\varphi) - ml^2\dot{\varphi}(\cancel{\dot{\theta}} - \dot{\varphi})\operatorname{sen}(\theta-\varphi)$$

$$\frac{\partial L}{\partial \theta} = -\cancel{ml^2\dot{\theta}\dot{\varphi}}\operatorname{sen}(\theta-\varphi) - mgl\operatorname{sen}\theta + kl^2(\cos\theta + \cos\varphi)\operatorname{sen}\theta$$

$$\ddot{\theta} + \ddot{\varphi}\cos(\theta-\varphi) + \dot{\varphi}^2\operatorname{sen}(\theta-\varphi) + \frac{g}{l}\operatorname{sen}\theta - \frac{k}{m}\cos\theta\operatorname{sen}\theta - \frac{k}{m}\cos\varphi\operatorname{sen}\theta = 0$$

$p_\varphi = ml^2\ddot{\varphi} + ml^2\ddot{\theta}\cos(\theta-\varphi) - ml^2\dot{\theta}(\dot{\theta}-\cancel{\dot{\varphi}})\sin(\theta-\varphi)$

$\frac{\partial L}{\partial \varphi} = \cancel{ml^2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi)} - mgl\,\mathrm{sen}\,\varphi + kl^2(\cos\theta+\cos\varphi)\sin\varphi$

$\ddot{\varphi} + \ddot{\theta}\cos(\theta-\varphi) - \dot{\theta}^2\sin(\theta-\varphi) + \frac{g}{l}\sin\varphi - \frac{k}{m}\cos\theta\sin\varphi - \frac{k}{m}\cos\varphi\sin\varphi = 0$

5) PUNTI DI EQUIL. e LORO STABILITA'

$V = -mgl(\cos\theta+\cos\varphi) + \frac{1}{2}kl^2(\cos\theta+\cos\varphi)^2$

$\partial_\theta V = mgl\sin\theta - kl^2(\cos\theta+\cos\varphi)\sin\theta$

$\partial_\varphi V = mgl\sin\varphi - kl^2(\cos\theta+\cos\varphi)\sin\varphi$

$$\left.\begin{aligned} &mgl\sin\theta\left(1 - \frac{kl}{mg}(\cos\theta+\cos\varphi)\right) = 0 \\ &mgl\sin\varphi\left(1 - \frac{kl}{mg}(\cos\theta+\cos\varphi)\right) = 0 \end{aligned}\right\} \quad \begin{cases} \sin\theta = 0 \\ \sin\varphi = 0 \end{cases} \quad \vee \quad \cos\theta+\cos\varphi = \frac{mg}{kl}$$

$\theta = 0, \pi$
$\varphi = 0, \pi$

questo esiste solo se $\frac{mg}{kl} \leq 2$

questo è una curva che è un continuo di pti di equilibrio

$\partial_\theta V = mgl\sin\theta - kl^2(\cos\theta+\cos\varphi)\sin\theta$

$\partial_\varphi V = mgl\sin\varphi - kl^2(\cos\theta+\cos\varphi)\sin\varphi$

$$\partial^2 V = \begin{pmatrix} mgl\cos\theta - kl^2(\cos\theta+\cos\varphi)\cos\theta + kl^2\sin^2\theta & kl^2\sin\varphi\sin\theta \\ kl^2\sin\varphi\sin\theta & mgl\cos\varphi - kl^2(\cos\theta+\cos\varphi)\cos\varphi + kl^2\sin^2\varphi \end{pmatrix}$$

$$\partial^2 V_{(0,0)} = \begin{pmatrix} mgl - 2kl^2 & 0 \\ 0 & mgl - 2kl^2 \end{pmatrix}$$

stab. $mg > 2kl$

$$\partial^2 V_{\substack{(0,\pi) \\ (\pi,0)}} = \begin{pmatrix} \pm mgl & 0 \\ 0 & \mp mgl \end{pmatrix}$$

instabile

$$\partial^2 V_{(\pi,\pi)} = \begin{pmatrix} -mgl-2kl^2 & \\ & -mgl-2kl^2 \end{pmatrix} \quad \text{instabile}$$

Il continuo di pti è dato da (θ^*,φ^*) t.c. $\cos\theta^* + \cos\varphi^* = \dfrac{mg}{kl}$

$$\partial^2 V_{(\theta^*,\varphi^*)} = l \begin{pmatrix} \cancel{mg\cos\theta^*} - \cancel{mg\cos\theta^*} + kl\sin^2\theta^* & kl\sin\theta^*\sin\varphi^* \\ kl\sin\theta^*\sin\varphi^* & \cancel{mg\cos\varphi^*} - \cancel{mg\cos\varphi^*} + kl\sin^2\varphi^* \end{pmatrix}$$

$$= kl^2 \begin{pmatrix} \sin^2\theta^* & \sin\theta^*\sin\varphi^* \\ \sin\theta^*\sin\varphi^* & \sin^2\varphi^* \end{pmatrix} \rightarrow$$

c'è un autovalore positivo e uno nullo

Continuo di pti equil. stabili. t.c. $\cos\theta^* + \cos\varphi^* = \dfrac{mg}{kl}$

7) $\dfrac{mg}{kl} = \dfrac{1}{2} \Rightarrow kl^2 = 2mgl$

↳ pt di equil. stab. sono (θ^*,φ^*)

$$A = ml^2 \begin{pmatrix} 1 & \cos(\theta^*-\varphi^*) \\ \cos(\theta^*-\varphi^*) & 1 \end{pmatrix}$$

$$\cos(\theta-\varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$$

$$B - \lambda A = ml^2 \begin{pmatrix} \frac{2g}{l}\sin^2\theta^* - \lambda & \sin\theta^*\sin\varphi^* - \lambda(\ ) \\ & \frac{2g}{l}\sin^2\varphi^* - \lambda \end{pmatrix}$$

ci aspettiamo un autovalore positivo e uno nullo

$$\downarrow \qquad \left(\frac{mg}{kl} = \frac{1}{2} \Rightarrow \cos\theta^* + \cos\varphi^* = \frac{1}{2} \right)$$

$$B = 2mgl \begin{pmatrix} \sin^2\theta & \sin\theta \sin\varphi \\ \sin\theta\sin\varphi & \sin^2\varphi \end{pmatrix} \qquad A = ml^2 \begin{pmatrix} 1 & \cos\theta\cos\varphi + \sin\theta\sin\varphi \\ * & 1 \end{pmatrix}$$

$$B - \lambda A = ml^2 \begin{pmatrix} \frac{2g}{l}\sin^2\theta - \lambda & * \\ \frac{2g}{l}\sin\theta\sin\varphi - \lambda(\cos\theta\cos\varphi + \sin\theta\sin\varphi) & \frac{2g}{l}\sin^2\varphi - \lambda \end{pmatrix}$$

$$\cos^2\theta\cos^2\varphi - \cos^2\theta - \cos^2\varphi + 1$$

$$\lambda^2\left(1 - \left(\cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + 2\cos\theta\sin\theta\cos\varphi\sin\varphi\right)\right) +$$

$$-\lambda\frac{2g}{l}\left(\sin^2\theta + \sin^2\varphi - \sin\theta\sin\varphi\left(\cos\theta\cos\varphi + \sin\theta\sin\varphi\right)\right)$$

$$+\left(\frac{2g}{l}\right)^2\left(\cancel{\sin^2\theta\sin^2\varphi} - \cancel{\sin^2\theta\sin^2\varphi}\right) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = \frac{2g}{l} f(\theta^*, \varphi^*)$$

$\frac{mg}{kl} = 2$, $\cos\theta + \cos\varphi = 2 \Rightarrow \theta = \varphi = 0$ pto di diramazione

$\partial^2 V = 0 \longrightarrow$ non c'è moto armonico qui attorno. Il potenziale è quartico.