



$$x_\varphi = R \sin \varphi$$

$$y_\varphi = -R \cos \varphi$$

$$x_\theta = R \sin \theta$$

$$y_\theta = -R \cos \theta$$

$$d = 2R \sin \left( \frac{\theta - \varphi}{2} \right)$$

molle  $\rightarrow$

$$d^2 = 4R^2 \sin^2 \left( \frac{\theta - \varphi}{2} \right) = 2R^2 (1 - \cos(\theta - \varphi))$$

$$1) \quad T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \dot{\varphi}^2)$$

$$V = mg(y_\theta + y_\varphi) + \frac{1}{2} K d^2 = -mgR(\cos \theta + \cos \varphi) + KR^2 (1 - \cos(\theta - \varphi))$$

const.  
irrelevant

$$\stackrel{!}{=} -mgR(\cos \theta + \cos \varphi) - KR^2 \cos(\theta - \varphi)$$

$$L = \frac{1}{2} m R^2 (\dot{\theta}^2 + \dot{\varphi}^2) + mgR(\cos \theta + \cos \varphi) + KR^2 \cos(\theta - \varphi)$$

$$2) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m R^2 \ddot{\theta} \quad \frac{\partial L}{\partial \theta} = -mgR \sin \theta - KR^2 \sin(\theta - \varphi)$$

$$\ddot{\theta} = -\frac{g}{R} \sin \theta - \frac{K}{m} \sin(\theta - \varphi)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = m R^2 \ddot{\varphi} \quad \frac{\partial L}{\partial \varphi} = -mgR \sin \varphi + KR^2 \sin(\theta - \varphi)$$

$$\ddot{\varphi} = -\frac{g}{R} \sin \varphi + \frac{K}{m} \sin(\theta - \varphi)$$

$$3) \quad V = -mgR(\cos \theta + \cos \varphi) - KR^2 \cos(\theta - \varphi)$$

$$\begin{cases} \partial_\theta V = mgR \sin \theta + KR^2 \sin(\theta - \varphi) = 0 \\ \partial_\varphi V = mgR \sin \varphi - KR^2 \sin(\theta - \varphi) = 0 \end{cases}$$

$\downarrow$

Sommando le due eq.

$$\downarrow \begin{cases} \sin \theta + \sin \varphi = 0 \\ m_j R \sin \theta + k R^2 \sin(\theta - \varphi) = 0 \end{cases} \Rightarrow \varphi = -\theta, \theta + \pi$$

$$\varphi = -\theta \rightarrow 0 = m_j R \sin \theta + k R^2 \sin(2\theta) = m_j R \sin \theta + 2k R^2 \sin \theta \cos \theta$$

$$= 2k R^2 \sin \theta \left( \cos \theta + \frac{m_j}{2kR} \right)$$

$$\rightarrow \theta = 0, \pi \quad \cos \theta^* = -\frac{m_j}{2kR} \rightarrow 2 \text{ pts, esistono}$$

sob se  $m_j \leq 2kR$

$$\rightarrow (\theta, \varphi) = (0, 0), (\pi, \pi), \left( \theta_{1L}^*, -\theta_{1L}^* \right)$$

$$\varphi = \theta + \pi \rightarrow 0 = m_j R \sin \theta + k R^2 \sin \pi$$

$\hookrightarrow \theta = 0, \pi$

$$\rightarrow (\theta, \varphi) = (0, \pi), (\pi, 0)$$

$$\begin{aligned} \partial_\theta V &= m_j R \sin \theta + k R^2 \sin(\theta - \varphi) \\ \partial_\varphi V &= m_j R \sin \varphi - k R^2 \sin(\theta - \varphi) \end{aligned}$$

$$\partial^2 V = \begin{pmatrix} m_j R \cos \theta + k R^2 \cos(\theta - \varphi) & -k R^2 \cos(\theta - \varphi) \\ -k R^2 \cos(\theta - \varphi) & m_j R \cos \varphi + k R^2 \cos(\theta - \varphi) \end{pmatrix}$$

$$\partial^2 V_{(0,0)} = \begin{pmatrix} m_j R + k R^2 > 0 & -k R^2 \\ -k R^2 & m_j R + k R^2 > 0 \end{pmatrix} \rightarrow \det = m_j R (m_j R + 2k R^2) > 0$$

$\Rightarrow$  STABILE

$$\partial^2 V_{(\pi,\pi)} = \begin{pmatrix} -m_j R + k R^2 & -k R^2 \\ -k R^2 & -m_j R + k R^2 \end{pmatrix} \rightarrow \det = m_j R (m_j R - 2k R^2)$$

$$\det \geq 0 \text{ per } m_j R \geq 2k R^2,$$

ma in quest caso elem. sulla diagonale

sono negativi  $\Rightarrow$  INSTABILE

(si può dimostrare che entrambi gli autovalori sono negativi)

$$\partial^2 V_{\left(\begin{smallmatrix} 0, \pi \\ \pi, 0 \end{smallmatrix}\right)} = \begin{pmatrix} \pm \omega_j R - KR^2 & KR^2 \\ KR^2 & \mp \omega_j R - KR^2 \end{pmatrix}$$

INSTABILI

(uno degli elem. diag. è sempre negativo)

$$\omega_j \leq 2KR$$

$$\partial^2 V_{\left(\theta^*, 1-\theta^*\right)} = \begin{pmatrix} \omega_j R \cos \theta^* + KR^2 \cos 2\theta^* & -KR^2 \cos 2\theta^* \\ -KR^2 \cos 2\theta^* & \omega_j R \cos \theta^* + KR^2 \cos 2\theta^* \end{pmatrix}$$

$$\cos \theta^* = -\frac{\omega_j}{2KR} \quad \cos 2\theta^* = 2\cos^2 \theta^* - 1 = \frac{(\omega_j)^2}{2(KR)^2} - 1$$

$$\begin{aligned} \omega_j R \cos \theta^* + KR^2 \cos 2\theta^* &= -\frac{(\omega_j)^2}{2K} + \frac{2(KR)^2}{2K} \left( \frac{(\omega_j)^2}{2(KR)^2} - 1 \right) = \\ &= \frac{1}{2K} \left( -(\omega_j)^2 + (\omega_j)^2 - 2(KR)^2 \right) = -KR^2 < 0 \\ &\Rightarrow \text{INSTABILI} \end{aligned}$$

$$4) \quad B = \partial^2 V_{(0,0)} = \begin{pmatrix} \omega_j R + KR^2 & -KR^2 \\ -KR^2 & \omega_j R + KR^2 \end{pmatrix}$$

$$A = A_{(0,0)} = \begin{pmatrix} \omega R^2 & \\ & \omega R^2 \end{pmatrix}$$

$$\det(B - \lambda A) = \det \omega R^2 \begin{pmatrix} \frac{g}{R} + \frac{k}{\omega} - \lambda & -\frac{k}{\omega} \\ -\frac{k}{\omega} & \frac{g}{R} + \frac{k}{\omega} - \lambda \end{pmatrix} =$$

$$\downarrow = (\omega R^2)^2 \left( \lambda^2 - 2 \left( \frac{g}{R} + \frac{k}{\omega} \right) \lambda + \frac{g}{R} \left( \frac{g}{R} + \frac{2k}{\omega} \right) \right) = 0$$

$$\lambda_{1,2} = \left( \frac{g}{R} + \frac{k}{\omega} \right) \pm \sqrt{\left( \frac{g}{R} + \frac{k}{\omega} \right)^2 - \frac{g}{R} \left( \frac{g}{R} + \frac{2k}{\omega} \right)} =$$

$$= \left( \frac{g}{R} + \frac{k}{\omega} \right) \pm \sqrt{\left( \frac{k}{\omega} \right)^2} = \begin{matrix} \nearrow \frac{g}{R} \\ \searrow \frac{g}{R} + \frac{2k}{\omega} \end{matrix}$$

anche

$$\begin{aligned} \left( \frac{g}{R} + \frac{k}{\omega} - \lambda \right)^2 - \left( \frac{k}{\omega} \right)^2 &= \left( \frac{g}{R} + \frac{k}{\omega} - \lambda + \frac{k}{\omega} \right) \left( \frac{g}{R} + \frac{k}{\omega} - \lambda - \frac{k}{\omega} \right) = \\ &= \left( \frac{g}{R} + \frac{2k}{\omega} - \lambda \right) \left( \frac{g}{R} - \lambda \right) = 0 + \lambda_{1/2} = \begin{matrix} \nearrow \frac{g}{R} + \frac{2k}{\omega} \\ \searrow \frac{g}{R} \end{matrix} \end{aligned}$$

$$5) L = \frac{1}{2} \omega R^2 (\dot{\theta}^2 + \dot{\phi}^2) + kR^2 \cos(\theta - \phi)$$

$$\begin{aligned} \psi \equiv \theta - \phi &\rightarrow \theta = \frac{1}{2}(\psi + \phi) & \dot{\theta} &= \frac{1}{2}(\dot{\psi} + \dot{\phi}) \\ \phi = \theta + \psi &\rightarrow \phi = \frac{1}{2}(\phi - \psi) & \dot{\phi} &= \frac{1}{2}(\dot{\phi} - \dot{\psi}) \end{aligned}$$

$$\begin{aligned} L &= \frac{1}{2} \omega R^2 \left( \frac{1}{4} (\dot{\psi} + \dot{\phi})^2 + \frac{1}{4} (\dot{\psi} - \dot{\phi})^2 \right) + kR^2 \cos \psi \\ &= \frac{1}{4} \omega R^2 (\dot{\psi}^2 + \dot{\phi}^2) + kR^2 \cos \psi \end{aligned}$$

6) Coord. cilind.  $\phi$

costi. del moto:  $\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2} \omega R^2 \dot{\phi} \rightarrow$  momento angolare totale in direzione perpendicolare al foglio.

Sist. è inv. in rotazioni attorno asse  $\perp$  foglio.

$$7) \quad \dot{\phi} = \frac{2p_{\phi}}{\omega R^2} \quad L^* = \frac{1}{4} \omega R^2 (\dot{\psi}^2 + \frac{4p_{\phi}^2}{(\omega R^2)^2}) + KR^2 \cos\psi - \frac{2p_{\phi}^2}{\omega R^2}$$

$$\rightarrow L^* = \frac{1}{4} \omega R^2 \dot{\psi}^2 + KR^2 \cos\psi - \frac{p_{\phi}^2}{\omega R^2} \quad \text{const. irrelevant}$$

$$V_{\text{eff}} = -KR^2 \cos\psi \rightarrow \text{min in } \psi = 0$$

$$L^*_{\text{lin}} = \frac{1}{4} \omega R^2 \dot{\psi}^2 + KR^2 (1 - \frac{\psi^2}{2} + \dots) \quad \text{ord.} \quad \text{potentiell für alle}$$

$$= \frac{1}{4} \omega R^2 \dot{\psi}^2 - \frac{KR^2}{2} \psi^2$$