Course presentation

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UNIVERSITÀ
DEGLI STUDI DI TRIESTE

Who should attend

Code	Name	Master course	Curriculum
446SM	Mathematical optimisation	Data Science & Scientific Computing	Computational science and Engineering
578SM	Optimisation models	Matematica Matematica Data Science & Scientific Computing	Advanced mathematics Computational mathematics and modelling Data Science
269MI	Modelli di ottimizzazione	Ing. Elettronica e Informatica Ing. Elettronica e Informatica Ing. dell'Energia Elettrica e dei Sistemi	Apllicazioni informatiche Gestione dei dati e dei sistemi Sistemi

For everyone it is a 9 CFU course.

Lecturer, timetable and classroom

Lecturer	E-mail	Timetable		Classroom		
Lorenzo Castelli	castelli@units.it lorenzo.castelli@dia.units.it	MON	14.15 - 15.45	Aula I	Ed. C1	Third floor
		TUE	09.00 - 10.30	Aula I	Ed. C1	Third floor
		THU	11.15 - 12.45	Aula I	Ed. C1	Third floor
		FRI	11.15 - 12.45	Aula I	Ed. C1	Third floor

Exam rules

- Written exam only
  - If “in presenza”: formulate and implement an optimisation problem. 4 hours.
  - If “da remoto”: implement an optimisation model taken from a scientific paper. 12 hours + short oral exam the day after
  - Or.... make a project.
- On line registration through ESSE3 (no e-mails, etc.)
- 7 “appelli”

3 during the summer session	03.06.2021 - 31.07.2021
2 during the autumn session	01.09.2020 - 19.09.2020
2 during the winter session	17.01.2022 - 28.02.2022

Mathematical optimisation

Mathematical optimisation

Mathematical optimisation or (Mathematical Programming) is the discipline that studies problems in which we want to minimise or maximise a real function defined on $\mathbb{R}^n$ (the space of the real n-uples), and whose variables are bound to belong to a predefined set.

Optimisation problems

Problems in which we want to minimise or maximise a quantity that is expressed through a function.

Operations (operational) research

- "Operational research" (UK) or "Operations research" (US)
- 1937 RAF: Efficient use of radar: coverage and information management
  - 1938 "OR team" - 1939 last pre-war exercise: considerable improvement in air defence operations
  - 1941 "Operational Research Section"
- The Government Operational Research Service (GORS) `http://www.operational-research.gov.uk/`

Simply put, Operational Research brings intellectual rigour to the decision-making process

ETH-RAT

Swiss Data Science Center

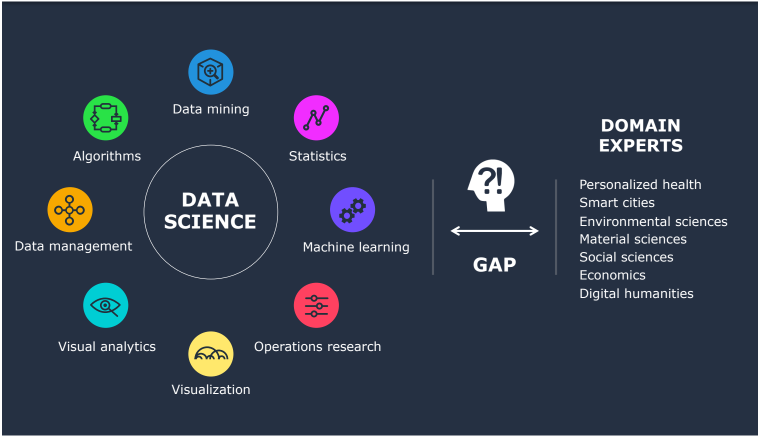
A joint center between EPFL and ETH Zürich

ETH zürich | ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE | PAUL SCHERRER INSTITUT PSI | WSL Eidg. Forschungsanstalt für Wald Schnee und Landschaft WSL | EMPA Materials Science & Technology | eawag aquatic research

Data science and OR

Data Science: a fragmented ecosystem

There is an inherent gap between data scientists,
domain experts and data providers



OR applications – Industrial area

Production planning Determination of the production levels and / or the use of resources; optimal resource allocation problems, i.e., problems concerning the distribution of resources limited between competing alternatives in order to minimise the overall cost or maximise total profit; these resources can be raw materials, labour, times of work on machines, capital invested.

Optimal stock management Warehouse organisation in the management of raw materials, products being processed etc., i.e., to decide when and how much, during a production process, they must store products in order to comply with deliveries while minimising costs, or if and when it is convenient to reorder materials in order to obtain the best compromise between purchase, production and storage costs.

Location and sizing of the plants Where to install production plants in order to optimally supply areas distributed over a territory, or decide where to build the base stations of a telecommunications network (4G, 5G) to cover the territory and with what power they must transmit.

OR applications - Optimal design

Network design and management Defining the connections and sizing the capabilities of a telecommunication, data transmission and circuit network, so as to guarantee traffic between the various origins and destinations and minimise the overall cost.

Structural Design These are problems that arise in civil, industrial, aeronautical mechanics, etc. and have the purpose of defining a design of a building, of a bridge so that they better resist stress coming from various agents (earthquakes, strong winds) or of the profile of a wing of an aircraft so that, for example, lift is maximised.

Design of optical systems, robot design The aim is to obtain a project that meets predetermined technical requirements by maximising some parameters linked, for example, to precision or performance

Optimal allocation of electronic components (VLSI design) Design of a motherboard so that, for example, the lengths of the electrical signal paths are minimised.

OR applications - Economics & finance

Choice of investments Choosing from a vast number of investment possibilities, such as achieving a realisation by respecting the constraints imposed by a financial budget and maximising earnings.

Composition of a portfolio It is the problem of deciding which securities and with which quotas to invest capital in order to maximise revenue or minimise risk

OR applications – Organisation and management

Project planning Deciding how to manage resources and how to sequence the multiple activities of a project.

Determination of staff shifts It is the problem of covering a series of services respecting the constraints of the company contract and minimising costs, such as, for example, the assignment of train staff to trains or crews to flights in order to minimise the number of trips needed to return the staff in its home base.

Maintenance planning It is the problem of deciding when and whether to carry out maintenance on some objects subject to wear over time, so as to minimise the overall cost.

Vehicle routing It is necessary to decide which routes the vehicles of a fleet must follow (e.g., vehicles used to collect waste or distribute products to a network of shops) in order to minimise the overall distance travelled.

OR applications – Other areas

- Systems biology
  - Studies of the structure of DNA
  - Computation of cellular metabolism
- Medical diagnostics
  - Interpretation and analysis of data obtainable from clinical analysis tools
  - Reconstruction of images
- Optimal control
  - Control of servomechanisms and driving systems
  - Trajectory management
- Social sciences, environmental sciences

Artificial Intelligence in Aviation

AI in Aviation & Optimisation (1)

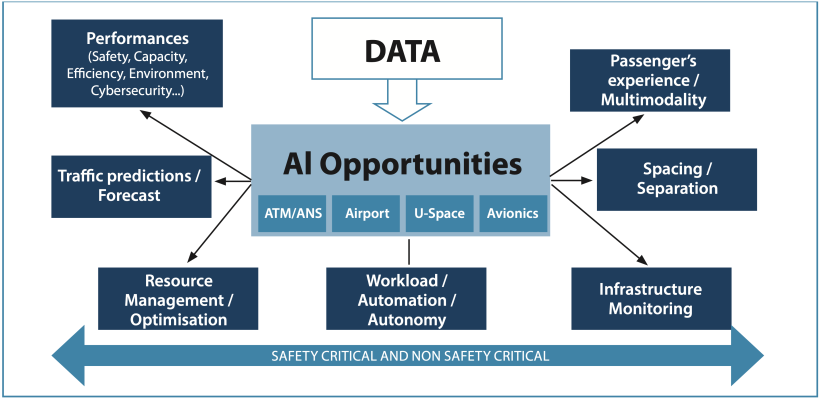


Figure 7: Key areas of AI opportunities for aviation/ATM

AI in Aviation & Optimisation (2)

AREA	BENEFITS	ANNEX 2 ITEM
Traffic predictions / forecasts/modeling	1. Improving predictions of aircraft trajectories, reducing uncertainty and increasing capacity	2,5,7,9,23
Resource management / Optimisation	1. Deploying the optimal configuration of sectors and thus optimising capacity with the available resources	10,11
	2. Supporting ATM demand and capacity balancing	15
Workload / Automation / Autonomy	1. Reducing ATCO workload (e.g. using speech recognition models for controller assistance)	14,17,19, 20,21
	2. Reducing risks with safety intelligence tools[11]	
Airport performance	1. Improving runway throughput (e.g. ROT prediction, improving spacing buffers)	6,12,13,14,16, 20,22
	2. Cutting airport delays	
Passenger experience	1. Improving passenger transfer/ customer satisfaction	
	2. Using biometrics to accelerate secure boarding	1,23
Infrastructure monitoring	1. Improving GNSS monitoring	4
	2. Cybersecurity monitoring	18
Airborne capabilities	1. Improving validation capabilities	3
	2. Generating environmental improvements	8
	3. Pilot and ATCO assistant through automatic speech recognition	17
	4. Enhancing safety with automatic taxi, take-off and landing enabled by computer vision	23
Airline performance	1. Optimising fuel usage	3
	2. Proposing better and more routes	5

Table 1: Aims and benefits of the AI applications of the catalogue

Software for mathematical optimistion

- Google https://developers.google.com/optimization/
- Matlab Optimization toolbox https://it.mathworks.com/products/optimization.html
- IBM Cplex https://www.ibm.com/it-it/marketplace/ibm-ilog-cplex
- FICO Xpress Optimization http://www.fico.com/en/products/fico-xpress-optimization
- Gurobi http://www.gurobi.com/
- Microsoft Excel https://support.office.com/it-it/article/definire-e-risolvere-un-problema-usando-il-risolutore-5

Optimisation problem - definition

An optimisation problem is defined by specifying

- A set $\boldsymbol{E}$. Its elements are called *solutions* (or decisions or alternatives).
- A subset $\boldsymbol{F} \subset \boldsymbol{E}$ (*feasible set*). Its elements are *feasible* solutions. On the contrary, elements in $\boldsymbol{E} \backslash \boldsymbol{F}$ are named *unfeasible*. The relationship $\boldsymbol{x} \in \boldsymbol{F}$ is called *constraint*.
- A function $\boldsymbol{f} : \boldsymbol{E} \to \mathbb{R}$ (*objective function*) to be *minimised* or *maximised* depending on the problem's purpose.

Optimisation problem's aim

Optimal solution

Each element $\boldsymbol{x}^* \in \boldsymbol{F}$ such that $\boldsymbol{f}(\boldsymbol{x}^*) \leq \boldsymbol{f}(\boldsymbol{y}), \forall \boldsymbol{y} \in \boldsymbol{F}$ for a minimisation problem, or $\boldsymbol{x}^* \in \boldsymbol{F}$ such that $\boldsymbol{f}(\boldsymbol{x}^*) \geq \boldsymbol{f}(\boldsymbol{y}), \forall \boldsymbol{y} \in \boldsymbol{F}$ for a maximisation problem is called *optimum* or *optimal solution*. The value $\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{x}^*)$ of the objective function in correspondence of the optimal solution is called *optimal value*.

Problem equivalence

A minimisation problem can be easily transformed into a maximisation one (and vice-versa) by just substituting $\boldsymbol{f}$ with $-\boldsymbol{f}$.

Optimisation problem's formulation

Minimisation

$$\boldsymbol{v} = \min \boldsymbol{f}(\boldsymbol{x})$$
$$\boldsymbol{x} \in \boldsymbol{F}$$

Maximisation

$$\boldsymbol{v} = \max \boldsymbol{f}(\boldsymbol{x})$$
$$\boldsymbol{x} \in \boldsymbol{F}$$

Classification of optimisation problems

- Continuous optimisation problems Variables can take all real values ($\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$). In addition, we distinguish
  - constrained optimisation if $\boldsymbol{F} \subset \mathbb{R}^{\boldsymbol{n}}$
  - unconstrained optimisation if $\boldsymbol{F} = \mathbb{R}^{\boldsymbol{n}}$
- Discrete optimisation problems Variables are constrained to be integer numbers ($\boldsymbol{x} \in \mathbb{Z}^{\boldsymbol{n}}$). We distinguish
  - integer programming if $\boldsymbol{F} \subseteq \mathbb{Z}^{\boldsymbol{n}}$
  - binary (or Boolean) programming if $\boldsymbol{F} \subseteq \{\mathbf{0}, \mathbf{1}\}^{\boldsymbol{n}}$
- Mixed optimisation problems. Only some variables are constrained to be integer.

Example 1 - Production planning

A chemical industry manufactures 4 types of fertilisers, Type 1, Type 2, Type 3, and Type 4, whose processing is carried out by two departments of the industry: the production department and the packaging department. In order to obtain ready-to-sell fertiliser, processing in both departments is necessary. The following table shows, for each type of fertiliser, the times (in hours) necessary for processing in each of the departments to have a ton of fertiliser ready for sale.

	Type 1	Type 2	Type 3	Type 4
production department	2	1.5	0.5	2.5
packing department	0.5	0.25	0.25	1

After deducting the cost of the raw material, each tonne of fertiliser gives the following profits (prices expressed in euros per tonne)

	Type 1	Type 2	Type 3	Type 4
profit	250	230	110	350

Determine the quantities that must be produced weekly of each type of fertiliser in order to maximise the overall profit, knowing that every week, the production department and the packaging department have a maximum working capacity of 100 and 50 hours, respectively.

Example 1 - Solution (1)

Decision variables The most natural choice is to introduce four real variables (x_1, x_2, x_3, x_4) representing the quantity of product of Type 1, Type 2, Type 3, and Type 4, respectively, to produce in one week.

Objective function Each ton of fertiliser contributes to the total profit, which can be expressed as

$$250x_1 + 230x_2 + 110x_3 + 350x_4 \tag{1}$$

The objective of the chemical industry is to choose the appropriate values of x_1, x_2, x_3, x_4 to maximise the profit as expressed in (1).

Example 1 - Solution (2)

- Constraints Obviously the production capacity of the factory limits the values that can take variables $x_j, j = 1, \dots, 4$. In fact there is a maximum working capacity in weekly hours of each Department. In particular, there are at most 100 hours per week for the production department and since each ton of Type 1 fertiliser uses the production department for 2 hours, each ton of Type 2 fertiliser uses the production department for 1.5 hours and so on for the others types of fertilisers you will have to have

$$2x_1 + 1.5x_2 + 0.5x_3 + 2.5x_4 \leq 100. \tag{2}$$

By reasoning in the same way for the packaging department we get

$$0.5x_1 + 0.25x_2 + 0.25x_3 + x_4 \leq 50 \tag{3}$$

Expressions (2) and (3) constitute the constraints of the model.
There is also the need to make explicit constraints due to the fact that variables $x_j, j = 1, \dots, 4$ representing quantities of product cannot be negative and therefore non-negative constraints must be added: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$.

Example 1 - Final formulation

$$
\begin{aligned}
\max\ & \mathbf{250x_1 + 230x_2 + 110x_3 + 350x_4} \\
& \mathbf{2x_1 + 1.5x_2 + 0.5x_3 + 2.5x_4 \leq 100} \\
& \mathbf{0.5x_1 + 0.25x_2 + 0.25x_3 + x_4 \leq 50} \\
& \mathbf{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0}
\end{aligned}
$$

The feasible set $\boldsymbol{F}$ is

$$
\begin{aligned}
\boldsymbol{F} = \{x \in \mathbb{R}^4 \mid \mathbf{2x_1 + 1.5x_2 + 0.5x_3 + 2.5x_4 \leq 100}, \\
\mathbf{0.5x_1 + 0.25x_2 + 0.25x_3 + x_4 \leq 50}, \\
\mathbf{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0}\}
\end{aligned}
$$

Example 2 - Capital budget

Suppose we have to invest €1000 on the financial market. We also assume that the market offers three different types of investments (A, B, C), each characterised by a purchase price and a net yield, which are summarised in the following table:

	A	B	C
Purchase price	750	200	800
Yield	20	5	10

You want to decide which of the investments to make to maximise the return, knowing that investments A, B, C cannot be carried out in a partial way, i.e., they are not divisible.

Example 2 - Solution

Decision variables The most natural choice is to introduce three binary variables (x_A, x_B, x_C) where

$$x_i = \begin{cases} 0 & \text{if investment } i \text{ is not made} \\ 1 & \text{if investment } i \text{ is made} \end{cases}$$

Objective function The goal is to maximise the total return, i.e,

$$20x_A + 5x_B + 10x_C$$

Constraints The total cost doesn't have to be above €1000, i.e.,

$$750x_A + 200x_B + 800x_C \leq 1000$$

Example 2 - Final formulation

$$
\begin{aligned}
\max\ & 20x_A + 5x_B + 10x_C \\
& 750x_A + 200x_B + 800x_C \leq 1000 \\
& x_i \in \{0, 1\}, i = A, B, C
\end{aligned}
$$

The feasible set F is

$$F = \{x \in \{0, 1\}^3 \mid 750x_A + 200x_B + 800x_C \leq 1000\}$$