Duality and sensitivity analysis

Lorenzo Castelli, Università degli Studi di Trieste.

UNIVERSITÀ
DEGLI STUDI DI TRIESTE

A relaxed problem

Consider the standard form problem

$$\begin{aligned} \min & \boldsymbol{c}^T \boldsymbol{x} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned}$$

which we call the primal problem, and let $\boldsymbol{x}^*$ be an optimal solution, assumed to exist. We introduce a relaxed problem in which the constraint $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is replaced by a penalty $\boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})$, where $\boldsymbol{p}$ is a vector of the same dimension as $\boldsymbol{b}$. We are then faced with the problem

$$\begin{aligned} \min & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

A relaxed problem

Let $\boldsymbol{g}(\boldsymbol{p})$ be the optimal cost for the relaxed problem, as a function of vector $\boldsymbol{p}$. The relaxed problem allows for more options than those present in the primal problem, and we expect $\boldsymbol{g}(\boldsymbol{p})$ to be no larger than the optimal cost $\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}^*$. Indeed,

$$\boldsymbol{g}(\boldsymbol{p}) = \min_{\boldsymbol{x}\geq\boldsymbol{0}} \left[\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} + \boldsymbol{p}^{\boldsymbol{T}}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\right] \leq \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}^* + \boldsymbol{p}^{\boldsymbol{T}}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*) = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}^*,$$

where the last equality follows from the fact that $\boldsymbol{x}^*$ is a feasible solution to the primal problem, and satisfies $\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{b}$. Thus, each $\boldsymbol{p}$ leads to a lower bound $\boldsymbol{g}(\boldsymbol{p})$ for the optimal cost $\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}^*$.

A tight bound

The problem

$$\begin{aligned} &\max \boldsymbol{g}(\boldsymbol{p}) \\ &\text{subject to} \;\; \text{no constraints} \end{aligned}$$

can be interpreted as a search for the tightest possible lower bound of this type, and is known as the dual problem.

The dual problem

Using the definition of $\boldsymbol{g}(\boldsymbol{p})$, we have

$$\boldsymbol{g}(\boldsymbol{p}) = \min_{\boldsymbol{x}\geq\boldsymbol{0}} \left[\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\right] = \boldsymbol{p}^T\boldsymbol{b} + \min_{\boldsymbol{x}\geq\boldsymbol{0}}(\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A})\boldsymbol{x}$$

Note that

$$\min_{\boldsymbol{x}\geq\boldsymbol{0}}(\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A})\boldsymbol{x} = \begin{cases} \boldsymbol{0} & \text{if } \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq \boldsymbol{0} \\ -\infty & \text{otherwise} \end{cases}$$

In maximising $\boldsymbol{g}(\boldsymbol{p})$ we only need to consider those values of $\boldsymbol{p}$ for which $\boldsymbol{g}(\boldsymbol{p})$ is not equal to $-\infty$. We therefore conclude that the dual problem is the same as

$$\begin{aligned} \max \boldsymbol{p}^T\boldsymbol{b} & \\ \boldsymbol{p}^T\boldsymbol{A} \leq \boldsymbol{c}^T. & \end{aligned}$$

The dual problem

In the example, we started with equality constraints $\boldsymbol{Ax} = \boldsymbol{b}$ and we ended up with no constraints on the sign of the vector $\boldsymbol{p}$. If the primal problem had instead inequality constraints of the form $\boldsymbol{Ax} \geq \boldsymbol{b}$, they could be replaced by $\boldsymbol{Ax} - \boldsymbol{s} = \boldsymbol{b}, \boldsymbol{s} \geq \boldsymbol{0}$. The equality constraints can be written in the form

$$[\boldsymbol{A}| - \boldsymbol{I}] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{b},$$

which leads to the dual constraints

$$\boldsymbol{p}^{\boldsymbol{T}}[\boldsymbol{a}| - \boldsymbol{I}] \leq [\boldsymbol{c}^{\boldsymbol{T}}|\boldsymbol{0}^{\boldsymbol{T}}],$$

or equivalently,

$$\boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{A} \leq \boldsymbol{c}^{\boldsymbol{T}}, \boldsymbol{p} \geq \boldsymbol{0}.$$

The dual problem

If the vector $\boldsymbol{x}$ is free rather sign-constrained, we use the fact

$$\min_{\boldsymbol{x}}(\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A})\boldsymbol{x} = \begin{cases} 0 & \text{if } \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} = \boldsymbol{0} \\ -\infty & \text{otherwise} \end{cases}$$

to end up with the constraints $\boldsymbol{p}^T\boldsymbol{A} = \boldsymbol{c}^T$ in the dual problem. These considerations motivate the primal-dual relationships.

PRIMAL	maximise	minimise	DUAL
constraints	$\leq \boldsymbol{b}_i$ $\geq \boldsymbol{b}_i$ $= \boldsymbol{b}_i$	$\geq \boldsymbol{0}$ $\leq \boldsymbol{0}$ free	variables
variables	$\geq \boldsymbol{0}$ $\leq \boldsymbol{0}$ free	$\geq \boldsymbol{c}_j$ $\leq \boldsymbol{c}_j$ $=$	constraints

Sensitivity analysis

The main purpose of sensitivity analysis is to identify the sensitive parameters (i.e., those that cannot be changed without changing the optimal solution). The sensitive parameters are the parameters that need to be estimated with special care to minimise the risk of obtaining an erroneous optimal solution.

The model parameters under study are the a_{ij}, b_i, c_j for $i = 1, \dots, m$ and $j = 1, \dots, n$.

Resource availability

- Linear programming problems often can be interpreted as allocating resources to activities.
- When the constraints are in $\leq$ form, we interpreted the $\boldsymbol{b_i}$ (the right-hand sides) as the amounts of the respective resources being made available for the activities under consideration.
- In many cases, the $\boldsymbol{b_i}$ values used in the initial model actually may represent management's tentative initial decision on how much of the organisation's resources will be provided to the activities considered in the model.
- From this broader perspective, some of the $\boldsymbol{b_i}$ values can be increased in a revised model, but only if a sufficiently strong case can be made to management that this revision would be beneficial.

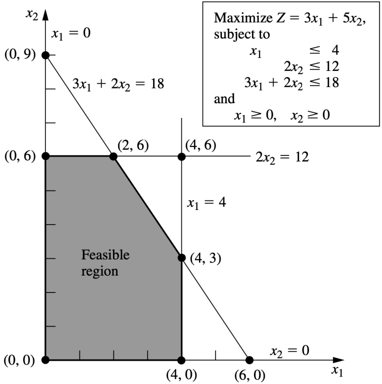
Shadow price

Definition

The shadow price for resource i (denoted by y_i^*) measures the marginal value of this resource, i.e., the rate at which Z could be increased by (slightly) increasing the amount of this resource (b_i) being made available.

In the case of a functional constraint in $=$ or $\geq$ form, its shadow price is again defined as the rate at which Z could be increased by (slightly) increasing the value of b_i, although the interpretation of b_i now would normally be something other than the amount of a resource being made available.

Shadow price – example



Here

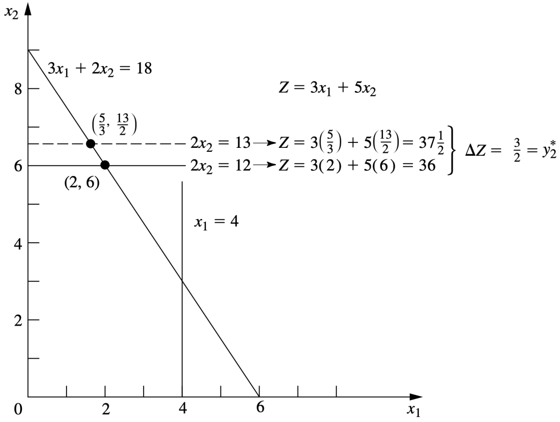
$$\boldsymbol{b_1} = \mathbf{4}$$

$$\boldsymbol{b_2} = \mathbf{12}$$

$$\boldsymbol{b_3} = \mathbf{18}$$

What if, for instance, $\boldsymbol{b_2}$ "slightly" changes, e.g., it increases by $\mathbf{1}$, that is $\boldsymbol{b_2} = \mathbf{13}$?

Shadow price – example



Shadow price - example

The graph shows that the shadow price is $y_2^* = \frac{3}{2}$ for resource 2. The two dots are the optimal solutions for $b_2 = 12$ or $b_2 = 13$, and plugging these solutions into the objective function reveals that increasing b_2 by 1 increases Z by $y_2^* = \frac{3}{2}$.

It demonstrates that $y_2^* = \frac{3}{2}$ is the rate at which Z could be increased by increasing b_2 "slightly". However, it also demonstrates the common phenomenon that this interpretation holds only for a small increase in b_2. Once b_2 is increased beyond 18, the optimal solution stays at $(0, 9)$ with no further increase in Z.

In other words, $Z = 45$ for any b_2 such that $b_2 \geq 18$ because the constraint $2x_2 = b_2$ becomes redundant.

Shadow price - example

Note that $y_1^* = 0$. Because the constraint on resource 1, $x_1 \leq 4$, is not binding on the optimal solution $(2, 6)$, there is a surplus of this resource. Therefore, increasing b_1 beyond 4 cannot yield a new optimal solution with a larger value of Z.

By contrast, the constraints on resources 2 and 3, $2x_2 \leq 12$ and $3x_1 + 2x_2 \leq 18$, are binding constraints (constraints that hold with equality at the optimal solution). Because the limited supply of these resources ($b_2 = 12, b_3 = 18$) binds Z from being increased further, they have positive shadow prices. We can easily show that $y_3^* = 1$.

Economists refer to such resources as scarce goods, whereas resources available in surplus (such as resource 1) are free goods (resources with a zero shadow price).

Primal and dual problems

Primal problem

$$
\begin{aligned}
\max z = {} & 3x_1 + 5x_2 \\
& x_1 \le 4 \\
& 2x_2 \le 12 \\
& 3x_1 + 2x_2 \le 18 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

The optimal solution is $x_1^* = 2$, $x_2^* = 6, z^* = 36$.

Dual problem

$$
\begin{aligned}
\min w = {} & 4\pi_1 + 12\pi_2 + 18\pi_3 \\
& \pi_1 + 3\pi_3 \ge 3 \\
& 2\pi_2 + 2\pi_3 \ge 5 \\
& \pi_1, \pi_2, \pi_3 \ge 0
\end{aligned}
$$

By strong duality we know that $w^* = 36$. How to find the optimal dual variables?

Complementary slackness

Complementary slackness conditions are

$$\pi_i^*(b_i - a_i^T x^*) = 0 \quad \forall i$$
$$(\pi^{T*} A_j - c_j) x_j^* = 0 \quad \forall j,$$

which in our case become

$$\begin{aligned} \pi_1^*(4 - x_1^*) &= 0 \\ \pi_2^*(12 - 2x_2^*) &= 0 \\ \pi_3^*(18 - 3x_1^* - 2x_2^*) &= 0 \\ (\pi_1^* + 3\pi_3^* - 3)x_1^* &= 0 \\ (2\pi_2^* + 2\pi_3^* - 5)x_2^* &= 0 \end{aligned}$$

Since $x_1^* = 2, x_2^* = 6$ then

$$\begin{aligned} 2 \times \pi_1^* &= 0 \\ 0 \times \pi_2^* &= 0 \\ 0 \times \pi_3^* &= 0 \\ 2 \times (\pi_1^* + 3\pi_3^* - 3) &= 0 \\ 6 \times (2\pi_2^* + 2\pi_3^* - 5) &= 0 \end{aligned}$$

Complementary slackness

It therefore follows that

$$\begin{aligned} \pi_1^* &= 0 \\ (\pi_1^* + 3\pi_3^* - 3) &= 0 \\ (2\pi_2^* + 2\pi_3^* - 5) &= 0 \end{aligned}$$

that is

$$\begin{aligned} \pi_1^* &= 0 \\ \pi_3^* &= 1 \\ \pi_2^* &= 3/2 \end{aligned}$$

In fact, $w^* = 4 \times 0 + 12 \times \frac{3}{2} + 18 \times 1 = 36$. In addition, we notice that

$$y_i^* = \pi_i^* \text{ for } i = 1, 2, 3.$$

The optimal dual variables are (equal to) the shadow prices

Dual variables and shadow prices

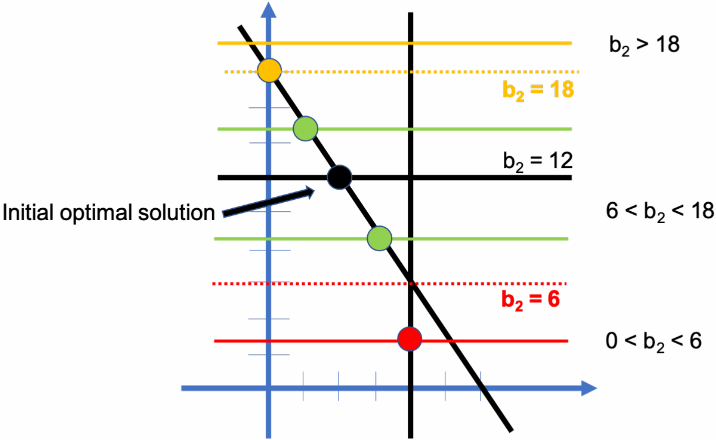
We have shown that each optimal dual variable represents the rate at which $\boldsymbol{Z}$ varies by varying the corresponding right-hand side value.

If we vary a right-hand side, the value of the optimal dual variables remains constant as long as the optimal solution lies on the intersection of the same constraint boundaries.

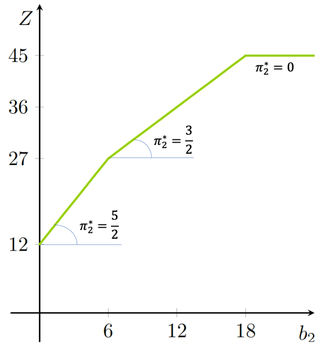
In our example,

- if $b_2 > 18$ the optimal solution is always $(0, 9)$. The optimal dual variables are $(0, 0, 5/2)$
- if b_2 varies in the interval $6 < b_2 < 18$ the optimal solution lies on the intersection between $2x_2 = b_2$ and $3x_1 + 2x_2 = 18$ and the optimal dual variables are $(0, 3/2, 1)$.
- if $0 < b_2 < 6$ the optimal solution lies on the intersection between $2x_2 = b_2$ and $x_1 = 4$. The optimal dual variables are $(3, 5/2, 0)$.
- if $b_2 = 6$ or $b_2 = 18$ the solution is called *degenerate*, which we don't address in this course.

Optimal solution's dependence on b_2

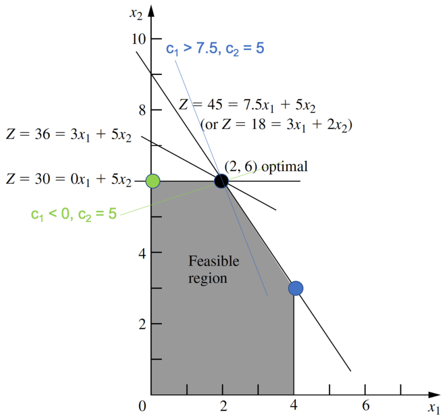


Optimal value's dependence on b_2



The optimal value $z(b_i)$ is a convex function of b_i.

Variation of obj. function's coefficients



Variation of obj. function's coefficients

The graph demonstrates the sensitivity analysis of c_1 and c_2 for our problem. Starting with the original objective function line [where $c_1 = 3, c_2 = 5$, and the optimal solution is $(2, 6)$], the other two black lines show the extremes of how much the slope of the objective function line can change and still retain $(2, 6)$ as an optimal solution. Thus,

with $c_2 = 5$, the allowable range for c_1 is $0 \leq c_1 \leq 7.5$,

with $c_1 = 3$, the allowable range for c_2 is $c_2 \geq 2$.