Linear programming

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Linear programming problem

A linear programming (LP) problem is a optimisation problem such that

$$z = \max\{c(x) : x \in X \subseteq \mathbb{R}^n\}$$

or

$$z = \min\{c(x) : x \in X \subseteq \mathbb{R}^n\}$$

where

- the objective function $c(x) : \mathbb{R}^n \to \mathbb{R}$ is linear, i.e., $c(0) = 0$ and $c(\alpha x + \beta y) = \alpha c(x) + \beta c(y)$. Therefore $c(x) = cx$ where c is a vector in $\mathbb{R}^n$.
- the set X of feasible solutions is defined by linear constraints such as $h(x) = \gamma$ and/or $h(x) \leq \gamma$ and/or $h(x) \geq \gamma$, where $h(x) : \mathbb{R}^n \to \mathbb{R}$ is a linear function and γ is scalar in $\mathbb{R}$.

Linear programming problem

Forms A LP problem can be formulated equivalently as

- Canonical form

$$\max \boldsymbol{cx}$$
$$\boldsymbol{Ax} \leq \boldsymbol{b}$$

- Standard form

$$\max \boldsymbol{cx}$$
$$\boldsymbol{Ax} = \boldsymbol{b} \ (\boldsymbol{b} \geq \boldsymbol{0})$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Terminology

$\boldsymbol{m}$ number of rows of matrix $\boldsymbol{A}$

$\boldsymbol{n}$ dimension of vector $\boldsymbol{x}$ and number of columns of matrix $\boldsymbol{A}$

$\boldsymbol{c}$ objective function vector

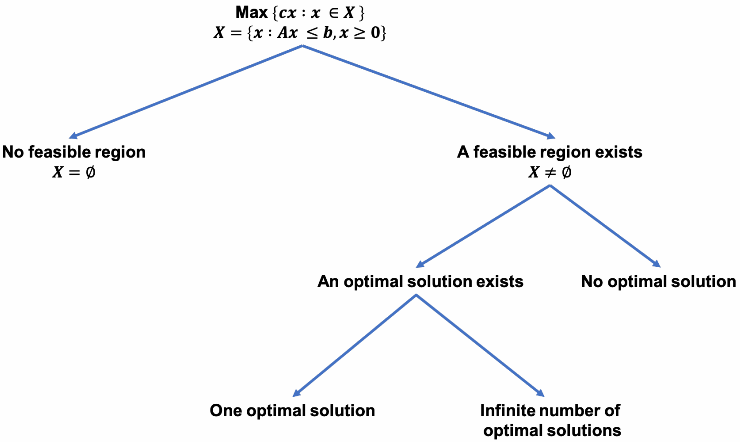
$\boldsymbol{A}$ technology matrix

$\boldsymbol{b}$ right-hand side vector ($\geq \boldsymbol{0}$ in the standard form)

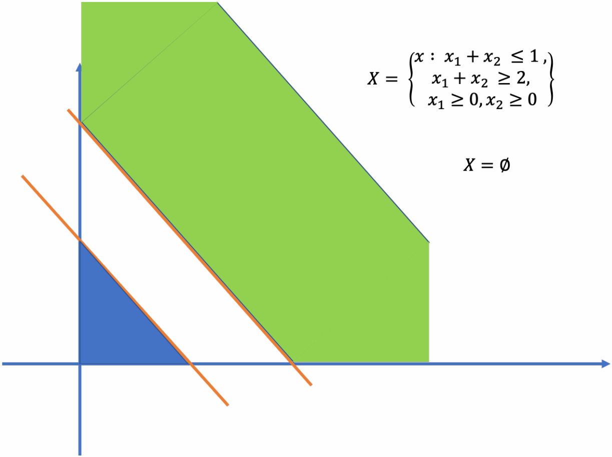
$\boldsymbol{x}$ decision variable vector

$\boldsymbol{X} = \{\boldsymbol{x} : \boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, the set of feasible solutions

Possible outcomes of a LP problem



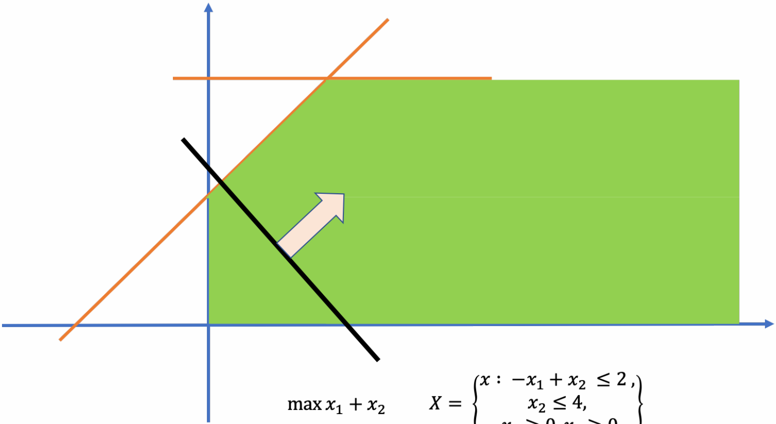
No feasible region



$$X = \left\{ \begin{array}{c} x: \; x_1 + x_2 \leq 1, \\ x_1 + x_2 \geq 2, \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right\}$$

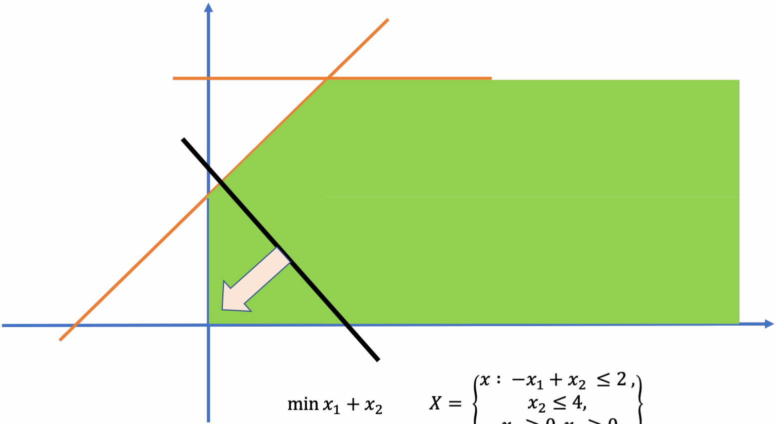
$$X = \emptyset$$

No optimal solution



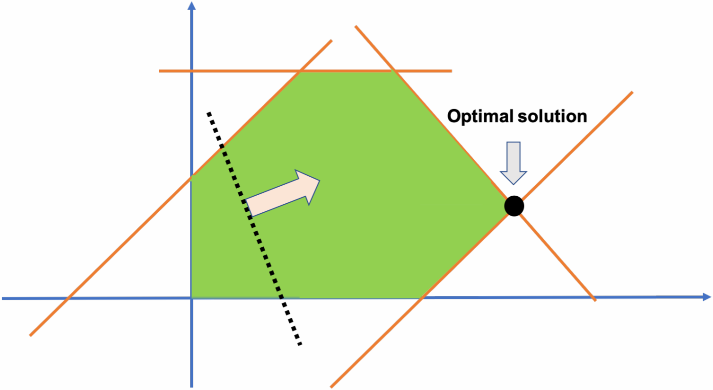
$$\max x_1 + x_2 \qquad X = \begin{Bmatrix} x : \ -x_1 + x_2 \leq 2\,, \\ x_2 \leq 4, \\ x_1 \geq 0, x_2 \geq 0 \end{Bmatrix}$$

Optimal solution!

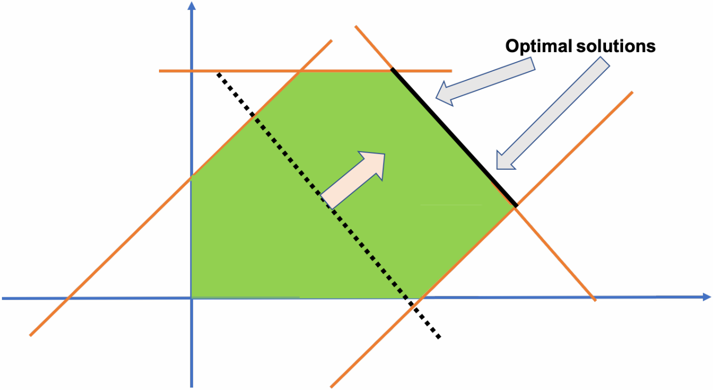


$$\min x_1 + x_2 \qquad X = \left\{ \begin{array}{c} x : -x_1 + x_2 \leq 2, \\ x_2 \leq 4, \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right\}$$

One optimal solution



Infinite number of optimal solutions



Key theorem

Theorem. The optimal solution is on a vertex

Given the LP problem

$$\max \boldsymbol{cx}$$
$$\boldsymbol{Ax} = \boldsymbol{b} \ (\boldsymbol{b} \geq \boldsymbol{0})$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

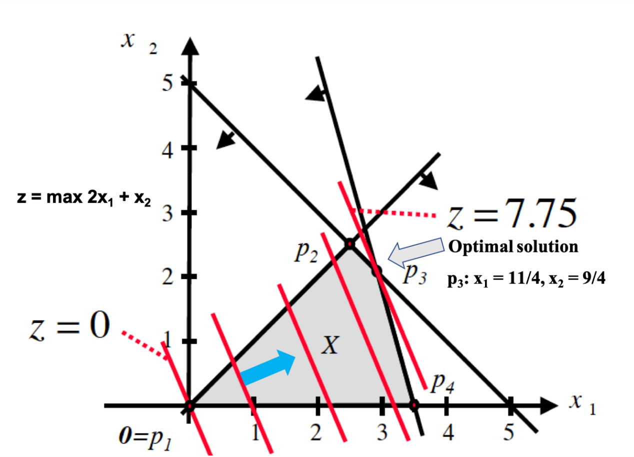
if $\boldsymbol{X} \neq \emptyset$ and it has an optimal and finite solution, then it exists a vertex of $\boldsymbol{X}$ which is an optimal solution.

The proof relies on the fact that $\boldsymbol{X}$ is a polyhedron and hence a convex set.

LP problem - example

$$
\begin{aligned}
\max z = & 2x_1 + x_2 \\
& x_1 + x_2 \leq 5 \\
& -x_1 + x_2 \leq 0 \\
& 6x_1 + 2x_2 \leq 21 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The optimal solution is on a vertex



The optimal solution is on a vertex

Given a LP problem that has an optimal and finite solution, since there is certainly one on a vertex, we can think of limiting the search for the optimal solution to the set of vertices.

The problem therefore arises on how to identify (potentially all) the vertices starting from a representation of the polyhedron of the feasible solutions.

Finding all vertices - example

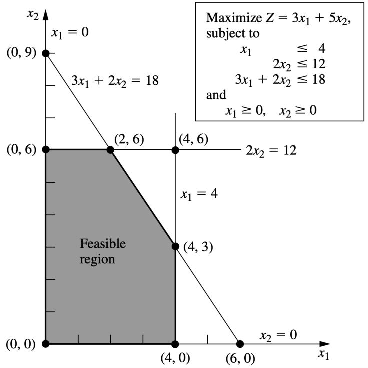


Figure: *Constraint boundaries and corner-point solutions*

Corner-point solutions

- A constraint boundary is a line that forms the boundary of what is permitted by the corresponding constraint.
- The points of intersection are the corner-point solutions of the problem. The five that lie on the corners of the feasible region—(0, 0), (0, 6), (2, 6), (4, 3), and (4, 0)—are the *corner-point feasible solutions (CPF solutions)*. The other three—(0, 9), (4, 6), and (6, 0)—are called *corner-point infeasible solutions*.

In this example, each corner-point solution lies at the intersection of two constraint boundaries. For a LP problem with $\boldsymbol{n}$ decision variables, each of its corner-point solutions lies at the intersection of $\boldsymbol{n}$ constraint boundaries.

Adjacent CPF solutions

Certain pairs of the CPF solutions in Fig. 1 share a constraint boundary, and other pairs do not. It will be important to distinguish between these cases by using the following general definitions.

Definitions

For any LP problem with n decision variables, two CPF solutions are adjacent to each other if they share $n-1$ constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line segment is referred to as an edge of the feasible region.

Adjacent CPF solutions

Since $n = 2$ in the example, two of its CPF solutions are adjacent if they share one constraint boundary; for example, (0, 0) and (0, 6) are adjacent because they share the $x_1 = 0$ constraint boundary. The feasible region in Fig. 1 has five edges, consisting of the five line segments forming the boundary of this region. Note that two edges emanate from each CPF solution. Thus, each CPF solution has two adjacent CPF solutions (each lying at the other end of one of the two edges), as enumerated in Table 1.

CPF solution	**Its adjacent CPF solutions**
(0,0)	(0,6) and (4,0)
(0,6)	(2,6) and (0,0)
(2,6)	(4,3) and (0,6)
(4,3)	(4,0) and (2,6)
(4,0)	(0,0) and (4,3)

Table: *Adjacent CPF solutions*

Optimality test

Optimality test

Consider any linear programming problem that possesses at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better (as measured by $\boldsymbol{Z}$), then it must be an optimal solution

Thus, for the example, $(\mathbf{2}, \mathbf{6})$ must be optimal simply because its $\boldsymbol{Z} = \mathbf{36}$ is larger than $\boldsymbol{Z} = \mathbf{30}$ for $(\mathbf{0}, \mathbf{6})$ and $\boldsymbol{Z} = \mathbf{27}$ for $(\mathbf{4}, \mathbf{3})$. This optimality test is the one used by the simplex method for determining when an optimal solution has been reached.

The simplex algorithm - i

Initialisation Choose $(\mathbf{0}, \mathbf{0})$ as the initial CPF solution to examine. (This is a convenient choice because no calculations are required to identify this CPF solution.)

Optimality Test Conclude that $(\mathbf{0}, \mathbf{0})$ is not an optimal solution. (Adjacent CPF solutions are better.)

The simplex algorithm - ii

Iteration 1 Move to a better adjacent CPF solution, $(\mathbf{0}, \mathbf{6})$, by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from $(\mathbf{0}, \mathbf{0})$, choose to move along the edge that leads up the x_2 axis. (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
2. Stop at the first new constraint boundary: $2x_2 = 12$. (Moving farther in the direction selected in step 1 leaves the feasible region; e.g., moving to the second new constraint boundary hit when moving in that direction gives $(\mathbf{0}, \mathbf{9})$, which is a corner-point infeasible solution.)
3. Solve for the intersection of the new set of constraint boundaries: $(\mathbf{0}, \mathbf{6})$. (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)

Optimality Test Conclude that $(\mathbf{0}, \mathbf{6})$ is not an optimal solution. (An adjacent CPF solution is better.)

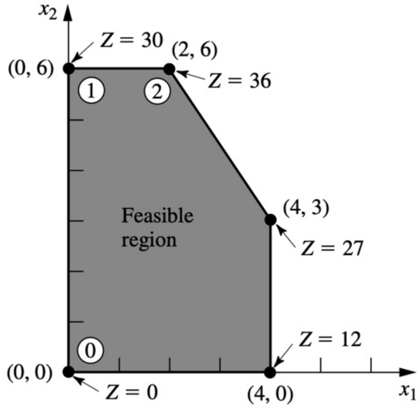
The simplex algorithm - iii

Iteration 2 Move to a better adjacent CPF solution, $(\mathbf{2}, \mathbf{6})$, by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from $(\mathbf{0}, \mathbf{6})$, choose to move along the edge that leads to the right. (Moving along this edge increases Z, whereas backtracking to move back down the x_2 axis decreases Z.)
2. Stop at the first new constraint boundary encountered when moving in that direction: $3x_1 + 2x_2 = 18$. (Moving farther in the direction selected in step 1 leaves the feasible region.)
3. Solve for the intersection of the new set of constraint boundaries: $(\mathbf{2}, \mathbf{6})$. (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)

Optimality Test Conclude that $(\mathbf{2}, \mathbf{6})$ is an optimal solution, so stop. (None of the adjacent CPF solutions are better.)

The simplex algorithm - iii



Concept 1

Relationships between optimal and CPF solutions

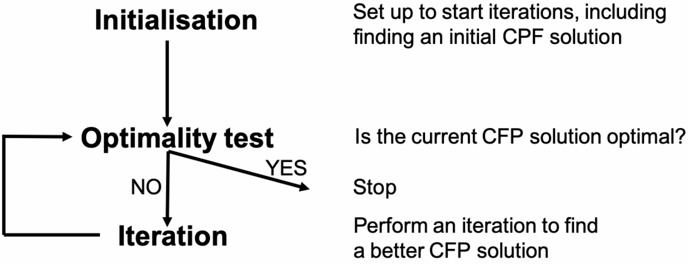
The simplex method focuses solely on CPF solutions. For any problem with at least one optimal solution, finding one requires only finding a best CFP solution.

Since the number of feasible solutions generally is infinite, reducing the number of solutions that need to be examined to a small finite number (just three in our example) is a tremendous simplification.

Concept 2

The flow of the simplex method

The simplex method is an iterative algorithm with the following structure



When the example was solved, this flow diagram was followed through two iterations until an optimal solution was found.

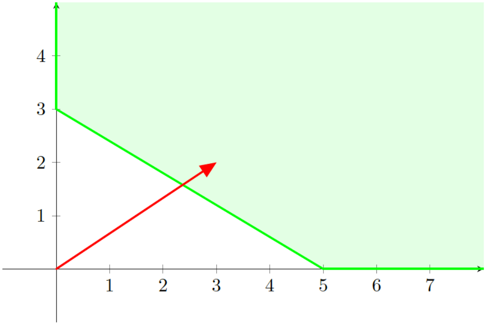
Concept 3

How to get started

Whenever possible, the initialisation of the simplex method chooses the origin (all decision variables equal to zero) to be the initial CPF solution. When there are too many decision variables to find an initial CPF solution graphically, this choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

Choosing the origin commonly is possible when all the decision variables have nonnegativity constraints, because the intersection of these constraint boundaries yields the origin as a corner-point solution. This solution then is a CPF solution unless it is infeasible because it violates one or more constraints. If it is infeasible, special procedures are needed to find the initial CPF solution.

Concept 3 - $(\mathbf{0}, \mathbf{0})$ is not a CPF

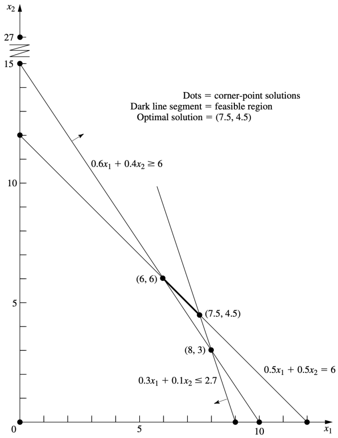


The corner-point $(\mathbf{0}, \mathbf{0})$ is not a feasible solution (i.e., a CPF)

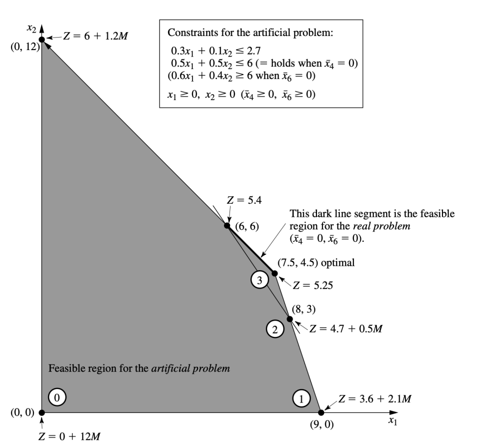
Concept 3 - $(0,0)$ is not a CPF

$$
\begin{aligned}
\min z = & 0.4x_1 + 0.5x_2 \\
& 0.3x_1 + 0.1x_2 \leq 2.7 \\
& 0.5x_1 + 0.5x_2 = 6 \\
& 0.6x_1 + 0.4x_2 \geq 6 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Concept 3 - $(0, 0)$ is not a CPF



Concept 3 - $(0, 0)$ is not a CPF



Concept 4

The choice of a better CPF soution at each iteration

Given a CPF solution, it is much quicker computationally to gather information about its *adjacent* CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is *adjacent* to the current one. No other CPF solutions are considered. Consequently, the entire path followed to eventually reach an optimal solution is along the *edges* of the feasible region.

Concept 5

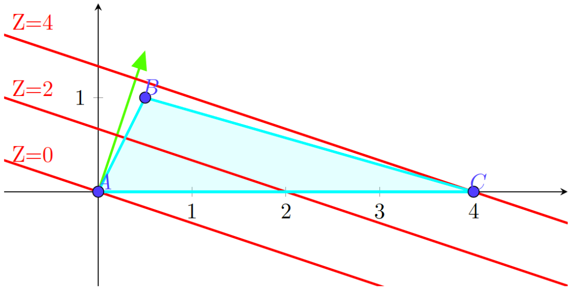
Which adjacent CFP solution to choose at each iteration

After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution and identifies the rate of improvement in Z that would be obtained by moving along the edge. Among the edges with a positive rate of improvement in Z, it then chooses to move along the one with the largest rate of improvement in Z. The iteration is completed by first solving for the adjacent CPF solution at the other end of this one edge and then relabelling this adjacent CPF solution as the current CPF solution for the optimality test and (if needed) the next iteration.

At the first iteration of the example, moving from $(0, 0)$ along the edge on the x_1 axis would give a rate of improvement in Z of 3 (Z increases by 3 per unit increase in x_1), whereas moving along the edge on the x_2 axis would give a rate of improvement in Z of 5 (Z increases by 5 per unit increase in x_2), so the decision is made to move along the latter edge. At the second iteration, the only edge emanating from $(0, 6)$ that would yield a positive rate of improvement in Z is the edge leading to $(2, 6)$, so the decision is made to move next along this edge.

Concept 5

In this case it is not a good idea to move along the edge with the largest rate of improvement in Z. In fact $Z = x_1 + 3x_2$. If we move along the edge on the x_2 axis we need two iterations ($A \to B \to C$), whereas if we move along the x_1 axis we just need one iteration ($A \to C$).



Concept 6

How the optimality test is performed efficiently

A positive rate of improvement in $\boldsymbol{Z}$ implies that the adjacent CPF solution is better than the current CPF solution, whereas a negative rate of improvement in $\boldsymbol{Z}$ implies that the adjacent CPF solution is worse. Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in $\boldsymbol{Z}$. If none do, then the current CPF solution is optimal.

In the example, moving along either edge from $(\mathbf{2}, \mathbf{6})$ decreases $\boldsymbol{Z}$. Since we want to maximise $\boldsymbol{Z}$, this fact immediately gives the conclusion that $(\mathbf{2}, \mathbf{6})$ is optimal.

Variation of Z

CPF $(2, 6)$ is the intersection point between $2x_2 = 12$ (i.e., $x_2 = 6$) and $3x_1 + 2x_2 = 18$. The latter constraint boundary can also be written as $x_1 = 6 - (2/3)x_2$. Hence, when $Z = 3x_1 + 5x_2$ moves along this constraint, we have that $Z = 3 * (6 - 2/3)x_2 + 5x_2$, i.e., $Z = 18 + 3x_2$. In $x_2 = 6, Z = 36$. Hence, if $x_2 < 6$, Z decreases and therefore this is not a viable option. Since also moving along $x_2 = 6$ does not allow to increase Z, it follows that $(2, 6)$ is the optimal solution.

Similarly, if we consider $x_2 = 9 - 3/2x_1$ then $Z = 45 - 9/2x_1$. In $x_1 = 2$ $Z = 36$ and this value decreases as long as x_1 increases.

Check that if $Z = 3x_1 + x_2$ then $(2, 6)$ is not optimal.