Duality

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A production problem

A firm produces n different goods using m different raw materials.

- Let $b_i, i = 1, \ldots, m$ be the available amount of the ith raw material
- The jth good, $j = 1 \ldots, n$ requires a_{ij} units of the ith material and
- results in a revenue c_j per unit produced.

The firm faces the problem of deciding how much of each good to produce in order to maximise its total revenues.

The choice of the decision variable is simple. Let $x_j, j = 1 \ldots, n$ be the amount of the jth good to be produced.

A production problem

Primal formulation

$$\begin{aligned} \max\ & c_1 x_1 + \ldots + c_n x_n \\ & a_{i1} x_1 + \ldots + a_{in} x_n \leq b_i \quad & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{aligned}$$

Decision variables The quantity of goods produced: $x_j \in \mathbb{R}$ for $1, \ldots, n$. We assume these variables to be continuous.

Objective function Maximise the profit $\sum_{j=1}^{n} c_j x_j$

Constraints

- For each raw material the amount of material used to make the production cannot exceed the material availability.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, \ldots, m$$

- For each good, the quantity is always non-negative $x_j \geq 0, j = 1 \ldots, n$

A production problem - example

$$
\begin{aligned}
& \max 15x_1 + 10x_2 \\
(R_p) \quad & x_1 + x_2 \leq 2000 \\
(R_q) \quad & x_1 - 0.5x_2 \leq 1000 \\
(R_r) \quad & 2x_1 + x_2 \leq 3000 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Notice that R_q can also be obtained as a by-product of good **2**.

Comments to the primal problem

In addressing a production problem, even before deciding what to produce to obtain the maximum profit, one should ask him/herself if it is better to produce or if vice versa it is not convenient to sell (or use otherwise) the available resources.

The following question should be asked

What is the minimum price at which all the available resources should be sold rather than produced?

This question is answered by the dual problem.

The dual of the production problem

- in the hypothesis of linearity, the overall profit that can be obtained from the sale of the resources is equal to the sum of the profits that are obtained by selling the individual resources, the latter are equal to the unit sales price multiplied by the quantity of available resources.
- a good $\boldsymbol{P_1}$ gives a profit of **15** and consumes a unit of $\boldsymbol{R_p}$, a unit of $\boldsymbol{R_q}$ and 2 units of $\boldsymbol{R_r}$. Therefore, to make it convenient to sell the resources (or at least remain at par) instead of producing, the overall sale of a unit of $\boldsymbol{R_p}$, one unit of $\boldsymbol{R_q}$ and **2** units of $\boldsymbol{R_r}$ must provide a gain not lower than **15**.
- resource sales prices must be non-negative.

Dual variables and objective function

Decision variables The unit price of each resource

$$\pi_i \in \mathbb{R}, \text{ for } i = p, q, r$$

These are continuous variables.

Objective function The profit obtained by selling all the available resources

$$2000\pi_p + 1000\pi_q + 3000\pi_r$$

Dual constraints

For each good the gain obtained from selling the resources needed to produce it must not not lower than the profit obtainable from the sale of the good itself.

$$\sum_{i=1}^{m} a_{ij}\pi_i \geq c_j \text{ for } j = 1, 2$$

Resource selling price must be non-negative

$$\pi_i \geq 0, \text{ for } i = p, q, r$$

Dual formulation

$$
\begin{aligned}
\text{(profit)} \quad \min q = \quad & 2000\pi_p + 1000\pi_q + 3000\pi_r \\
\text{(product 1)} \quad & \pi_p + \pi_q + 2\pi_r \geq 15 \\
\text{(product 2)} \quad & \pi_p - 0.5\pi_q + \pi_r \geq 10 \\
& \pi_p, \pi_q, \pi_r \geq 0
\end{aligned}
$$

Dual formulation

$$\min \sum_{i=1}^{m} b_i \pi_i$$

$$\sum_{i=1}^{m} a_{ij} \pi_i \geq c_j \qquad \text{for } j = 1 \ldots, n$$

$$\pi_i \geq 0 \qquad \text{for } i = 1 \ldots, m$$

and in matrix form

$$\min b^T \pi$$

$$A^T \pi \geq c$$

$$\pi \geq 0$$

Comments to the dual problem

It is intuitively evident that

- each solution that is feasible for the dual allows to make a profit not lower than the maximum profit obtained by solving the primal problem (i.e., producing). Hence it is convenient to sell (or at worst you don't lose) if you find someone willing to buy all the resources and to pay them altogether so as to satisfy the dual constraints;
- the optimal solution of the dual cannot be lower than the optimal solution of the primal (otherwise there would be prices that, while satisfying the constraints, would make production still convenient).

Primal and dual problems

At each LP problem (primal) is associated a dual problem

Primal problem (P)

$$\begin{aligned} \max z = & c_1 x_1 + \ldots + c_n x_n \\ & a_{11} x_1 + \ldots + a_{1n} x_n \leq b_1 \\ & \vdots \\ & a_{m1} x_1 + \ldots + a_{mn} x_n \leq b_m \\ & x_1, \ldots, x_n \geq 0 \end{aligned}$$

n variables and m constraints

Dual problem (D)

$$\begin{aligned} \min w = & b_1 \pi_1 + \ldots + b_m \pi_m \\ & a_{11} \pi_1 + \ldots + a_{m1} \pi_m \geq c_1 \\ & \vdots \\ & a_{1n} \pi_1 + \ldots + a_{mn} \pi_m \geq c_n \\ & \pi_1, \ldots, \pi_m \geq 0 \end{aligned}$$

m variables and n columns

Property

Problem D has as many variables as there are constraints in P and as many constraints as there are variables in P.

Primal and dual - Matrix form

In matrix form, we see that vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ exchange their positions and the matrix of the coefficients $\boldsymbol{A}$ is transposed.

Primal problem (P)

$$\begin{aligned} \max \boldsymbol{z} = &\boldsymbol{c}^T \boldsymbol{x} \\ &\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ &\boldsymbol{x} \geq \boldsymbol{0} \\ &\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \min \boldsymbol{w} = &\boldsymbol{b}^T \boldsymbol{\pi} \\ &\boldsymbol{A}^T \boldsymbol{\pi} \geq \boldsymbol{c} \\ &\boldsymbol{\pi} \geq \boldsymbol{0} \\ &\boldsymbol{\pi} \in \mathbb{R}^{\boldsymbol{m}} \end{aligned}$$

Primal and dual problems - Comments

Duality arises not only from economic justifications, but also from the application of Kuhn-Tucker conditions to LP problems or from Lagrangian relaxation.

Duality is important because:

- the dual problem corresponds to a different view of the same problem (for which an economic interpretation of the formulation obtained must always be sought);
- on it are based algorithms, such as the Dual Simplex and the Primal-Dual Algorithm, alternative to the Simplex (Primal), which are useful for certain classes of problems;
- in some cases it may be convenient to solve $\boldsymbol{D}$ instead of $\boldsymbol{P}$ (it may be better to solve the problem with fewer constraints).

Primal-dual relationships

PRIMAL	maximise	minimise	DUAL
constraints	$\leq b_i$ $\geq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\geq c_j$ $\leq c_j$ $=$	constraints

Table: *Relation between primal and dual variables and constraints*

Primal and dual problems - example

Primal problem (P)

$$\max z = 2x_1 + 3x_2 - x_3 + 7x_4$$

$$\begin{array}{ll} (\pi_1) & 4x_1 + 3x_2 - x_3 + 2x_4 \leq 35 \\ (\pi_2) & x_1 + 5x_2 + 6x_3 + 10x_4 = 28 \\ (\pi_3) & 2x_1 + 7x_2 - 2x_3 + 4x_4 \geq 15 \\ & x_1 \geq 0, x_2 \text{ free }, x_3 \leq 0, x_4 \geq 0 \end{array}$$

Dual problem (D)

$$\min w = 35\pi_1 + 28\pi_2 + 15\pi_3$$

$$\begin{array}{ll} (x_1) & 4\pi_1 + \pi_2 + 2\pi_3 \geq 2 \\ (x_2) & 3\pi_1 + 5\pi_2 + 7\pi_3 = 3 \\ (x_3) & -\pi_1 + 6\pi_2 - 2\pi_3 \leq -1 \\ (x_4) & 2\pi_1 + 10\pi_2 + 4\pi_3 \geq 7 \\ & \pi_1 \geq 0, \pi_2 \text{ free }, \pi_3 \leq 0 \end{array}$$

The duality theorem (weak)

Weak duality

If $\boldsymbol{x}$ is a feasible solution to the primal problem and $\boldsymbol{\pi}$ is a feasible solution to the dual problem then

$$\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{b}^T\boldsymbol{\pi}$$

Proof In fact,

$$\boldsymbol{c}^T\boldsymbol{x} \leq (\boldsymbol{A}^T\pi)^T\boldsymbol{x} = \pi^T\boldsymbol{A}\boldsymbol{x} \leq \pi^T\boldsymbol{b} = \boldsymbol{b}^T\pi$$

Primal and dual problems

Corollaries

(a) If the optimal value in the primal is $+\infty$, then the dual problem must be infeasible

(b) If the optimal value in the dual is $-\infty$, then the primal problem must be infeasible

Proof Suppose that the optimal value in the primal problem is $+\infty$ and that the dual problem has a feasible solution $\boldsymbol{\pi}$. By weak duality, $\boldsymbol{\pi}$ satisfies $\boldsymbol{b}^T\boldsymbol{\pi} \geq \boldsymbol{c}^T\boldsymbol{x}$ for every primal feasible $\boldsymbol{x}$. Taking the maximum over all feasible $\boldsymbol{x}$, we conclude $\boldsymbol{b}^T\boldsymbol{\pi} \geq +\infty$. This is impossible and shows that the dual cannot have a feasible solution, thus establishing part (a). Part (b) follows by a symmetrical argument.

Primal and dual problems

Corollary

Let $\boldsymbol{x}$ and $\boldsymbol{\pi}$ be feasible solutions to the primal and the dual, respectively, and suppose that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\pi$. Then $\boldsymbol{x}$ and $\boldsymbol{\pi}$ are optimal solutions to the primal and the dual, respectively.

Proof Let $\boldsymbol{x}$ and $\boldsymbol{\pi}$ as in the statement of the corollary. For every primal feasible solution $\boldsymbol{y}$, the weak duality theorem yields $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\pi \geq \boldsymbol{c}^T\boldsymbol{y}$, which proves that $\boldsymbol{x}$ is optimal. The proof of the optimality of $\boldsymbol{\pi}$ is similar.

The duality theorem (strong)

Strong duality

If a linear programming problem has an optimal solution, so does it dual, and the respective optimal values are equal.

In other words, if $\boldsymbol{x}^*$ is a finite optimal solution for the primal also the dual has a finite optimal solution $\boldsymbol{\pi}^*$ and it is always true that

$$\boldsymbol{c}^T \boldsymbol{x}^* = \boldsymbol{b}^T \boldsymbol{\pi}^*.$$

Relations between Primal and Dual

Recall that in a linear programming problem, exactly one of the following three possibilities will occur

(a) There is an optimal solution

(b) The problem is “unbounded”, that is, the optimal value is $+\infty$ (for maximisation problems) or $-\infty$ (for minimisation problems)

(c) The problem is infeasible

This leads to nine possible combinations for the primal and the dual:

	Finite optimum	**Unbounded**	**Infeasible**
Finite optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Table: *The different possibilities for the primal (rows) and the dual (columns)*

Both problems are infeasible - example

Consider the infeasible primal

$$\begin{aligned} \min x_1 + 2x_2 \\ x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 3 \\ x_1, x_2 \text{ free} \end{aligned}$$

Its dual is

$$\begin{aligned} \max \pi_1 + 3\pi_2 \\ \pi_1 + 2\pi_2 = 1 \\ \pi_1 + 2\pi_2 = 2 \\ \pi_1, \pi_2 \text{ free} \end{aligned}$$

which is also infeasible.

Relations between Primal and Dual

Complementary slackness

Let $\boldsymbol{x}$ and $\boldsymbol{\pi}$ be feasible solution to the primal and the dual problem, respectively. The vectors $\boldsymbol{x}$ and $\boldsymbol{\pi}$ are optimal solutions for the two respective problems if and only if

$$\pi_i(b_i - a_i^T x) = 0 \quad \forall i$$
$$(\pi^T A_j - c_j)x_j = 0 \quad \forall j,$$

where $\boldsymbol{A_j}$ is the $\boldsymbol{j}$th column and $\boldsymbol{a_i}$ is the $\boldsymbol{i}$th row of martix $\boldsymbol{A}$.

Complementary slackness - Proof

if $(\boldsymbol{x}, \boldsymbol{\pi})$ are optimal Problems constraints impose that $\boldsymbol{\pi}^T\boldsymbol{b} \geq \boldsymbol{\pi}^T\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{c}^T\boldsymbol{x}$. Since $\boldsymbol{x}$ and $\boldsymbol{\pi}$ are optimal $\boldsymbol{\pi}^T\boldsymbol{b} = \boldsymbol{c}^T\boldsymbol{x}$, hence $\boldsymbol{\pi}^T\boldsymbol{b} = \boldsymbol{\pi}^T\boldsymbol{A}\boldsymbol{x}$ and therefore $\pi(\boldsymbol{b}^T - \boldsymbol{A}\boldsymbol{x}) = \boldsymbol{0}$. Similarly, it follows that $(\pi^T\boldsymbol{A} - \boldsymbol{c}^T)\boldsymbol{x} = \boldsymbol{0}$. Finally, since $\boldsymbol{\pi} \geq \boldsymbol{0}$, $\boldsymbol{b}^T - \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{0}$, $\pi^T\boldsymbol{A} - \boldsymbol{c}^T \geq \boldsymbol{0}$, $\boldsymbol{x} \geq \boldsymbol{0}$, it follows that each term of the scalar products must be equal to zero, i.e., $\boldsymbol{\pi}_i(\boldsymbol{a}_i^T\boldsymbol{x} - \boldsymbol{b}_i) = \boldsymbol{0}\ \forall \boldsymbol{i}$ and $(\boldsymbol{c}_j - \pi^T\boldsymbol{A}_j)\boldsymbol{x}_j = \boldsymbol{0}\ \forall \boldsymbol{j}$.

if the equations hold By writing the equations in compact form and for the pair $(\boldsymbol{x}, \boldsymbol{\pi})$ such that

$$\pi(\boldsymbol{b}^T - \boldsymbol{A}\boldsymbol{x}) = \boldsymbol{0} \Rightarrow \pi\boldsymbol{b}^T = \pi\boldsymbol{A}\boldsymbol{x}$$

and

$$(\pi^T\boldsymbol{A} - \boldsymbol{c}^T)\boldsymbol{x} = \boldsymbol{0} \Rightarrow \pi\boldsymbol{A}\boldsymbol{X} = \boldsymbol{c}^T\boldsymbol{x}$$

since $\boldsymbol{\pi}^T\boldsymbol{b} = \boldsymbol{\pi}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}^T\boldsymbol{x}$, then $(\boldsymbol{x}, \boldsymbol{\pi})$ is optimal.

Complementary slackness - Comments

- the complementary slackness theorem can be reformulated by stating that if a primal constraint is not strict (i.e., it is $<$) then the associated dual variable must be null, conversely if a dual variable is non-null the associated primal constraint must be strict (i.e., $=$).
- it may happen that the constraint is strict and the associated dual variable is null.

Exercise

Write the dual of the following problems

Ex. 1

$$\max 5x_1 + 2x_2 + 3x_3$$
$$x_1 - 2x_2 + 5x_3 = 4$$
$$2x_1 + 7x_2 + 2x_3 \geq 10$$
$$2x_1 + 4x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Ex. 2

$$\max 0$$
$$3x_1 - 2x_2 + 8x_3 \leq 4$$
$$7x_1 + 7x_2 + 9x_3 \leq 10$$
$$x_1 + 10x_3 \leq 5$$
$$x_1, x_2, x_3 \text{ free}$$

Solutions

Ex. 1

$$\begin{aligned} \min & 4\pi_1 + 10\pi_2 + 5\pi_3 \\ & \pi_1 + 2\pi_2 + 2\pi_3 \geq 5 \\ & -2\pi_1 + 7\pi_2 \geq 2 \\ & 5\pi_1 + 2\pi_2 + 4\pi_3 \geq 3 \\ & \pi_1 \text{ free }, \pi_2 \leq 0, \pi_3 \geq 0 \end{aligned}$$

Ex. 2

$$\begin{aligned} \min & 4\pi_1 + 10\pi_2 + 5\pi_3 \\ & 3\pi_1 + 7\pi_2 + \pi_3 = 0 \\ & -2\pi_1 + 7\pi_2 = 0 \\ & 8\pi_1 + 9\pi_2 + 10\pi_3 = 0 \\ & \pi_1, \pi_2, \pi_3 \geq 0 \end{aligned}$$