Pre-processing

Lorenzo Castelli, Università degli Studi di Trieste.

UNIVERSITÀ
DEGLI STUDI DI TRIESTE

Pre-processing

The basic idea is to try to quickly detect and eliminate redundant constraints and variables, and tighten bounds where possible.

If the resulting linear/integer program is smaller/tighter, it will typically be solved much more quickly. This is especially important in the case of branch-and-bound because tens or hundreds of thousands of linear programs may need to be solved.

Pre-processing - example (linear)

$$
\begin{aligned}
\max 2x_1 + x_2 - x_3 \\
5x_1 - 2x_2 + 8x_3 \leq 15 \\
8x_1 + 3x_2 - x_3 \geq 9 \\
x_1 + x_2 + x_3 \leq 6 \\
0 \leq x_1 \leq 3 \\
0 \leq x_2 \leq 1 \\
1 \leq x_3
\end{aligned}
$$

Tightening bounds - 1^{st} constraint, x_1

Isolating variable x_1 in the first constraint we obtain

$$5x_1 \leq 15 + 2x_2 - 8x_3 \leq 15 + 2 \times 1 - 8 \times 1 = 9$$

where we use bound inequalities $x_2 \leq 1$ and $-x_3 \leq -1$.

Thus we obtain the tightened bound $x_1 \leq \frac{9}{5}$.

Tightening bounds - 1^{st} constraint, x_3

Isolating variable x_3 in the first constraint we obtain

$$8x_3 \leq 15 + 2x_2 - 5x_1 \leq 15 + 2 \times 1 - 5 \times 0 = 17$$

where we use bound inequalities $x_2 \leq 1$ and $x_1 \geq 0$.

Thus we obtain the tightened bound $x_3 \leq \frac{17}{8}$.

Tightening bounds - 1^{st} constraint, x_2

Isolating variable x_2 in the first constraint we obtain

$$2x_2 \geq 5x_1 + 8x_3 - 15 \geq 5 \times 0 + 8 \times 1 - 15 = -7$$

where we use bound inequalities $x_1 \geq 0$ and $x_3 \geq 1$.

Thus we obtain the bound $x_2 \geq -\frac{7}{2}$. Here the existing bound $x_2 \geq 0$ has not changed.

Tightening bounds - 2^{nd} constraint, x_1

Isolating variable x_1 in the second constraint we obtain

$$8x_1 \geq 9 - 3x_2 + x_3 \geq 9 - 3 \times 1 + 1 \times 1 = 7$$

where we use bound inequalities $x_2 \leq 1$ and $x_3 \geq 1$.

Thus we obtain the tightened bound $x_1 \geq \frac{7}{8}$.

No more bounds are changed based on the second or third constraints.

Further tightening

However, as certain bounds have been tightened it is worth passing through the constraints again.

Constraint $\mathbf{1}$ for x_3 now gives

$$8x_3 \leq 15 + 2x_2 - 5x_1 \leq 15 + 2 \times 1 - 5 \times \frac{7}{8} = \frac{101}{8}.$$

where we use bound inequalities $x_2 \leq 1$ and $x_1 \geq \frac{7}{8}$.

Thus we obtain the new tightened bound $x_3 \leq \frac{101}{64} (< \frac{17}{8})$.

Redundant constraint

Using the latest upper bounds in constraint **3**, we see that

$$x_1 + x_2 + x_3 \leq \frac{9}{5} + 1 + \frac{101}{64} = 4.378125 < 6,$$

and so this constraint is redundant and can be discarded.

Reduced problem

$$
\begin{aligned}
\max\ & 2x_1 + x_2 - x_3 \\
& 5x_1 - 2x_2 + 8x_3 \leq 15 \\
& 8x_1 + 3x_2 - x_3 \geq 9 \\
& (0 <)\frac{7}{8} \leq x_1 \leq \frac{9}{5}(< 3) \\
& 0 \leq x_2 \leq 1 \\
& 1 \leq x_3 \leq \frac{101}{64}(< +\infty)
\end{aligned}
$$

Variable fixing x_2

Considering variable x_2, observe that increasing its value makes all constraints (other than its bound constraints) less tight.

$$
\begin{aligned}
\max & \; 2x_1 + x_2 - x_3 \\
& 5x_1 + 8x_3 \leq 15 + 2x_2 \\
& 8x_1 - x_3 \geq 9 - 3x_2 \\
& 0 \leq x_2 \leq 1
\end{aligned}
$$

As the variable has a positive objective coefficient, it is advantageous to make the variable as large as possible, and thus set it to its upper bound of 1.

Therefore we can set, before solving the problem, that $x_2 = 1$.

Variable fixing x_3

Considering variable x_3, observe that decreasing its value makes all constraints (other than its bound constraints) less tight.

$$\begin{aligned} \max & 2x_1 + x_2 - x_3 \\ & 5x_1 - 2x_2 \leq 15 - 8x_3 \\ & 8x_1 + 3x_2 \geq 9 + x_3 \\ & 1 \leq x_3 \leq \frac{101}{64} \end{aligned}$$

As the variable has a negative objective coefficient, it is advantageous to make the variable as small as possible, and thus set it to its lower bound of 1.

Therefore we can set, before solving the problem, that $x_3 = 1$.

Variable fixing, by duality

Let us write the dual of the reduced problem

$$\begin{aligned}
\min 15y_1 + 9y_2 + \frac{7}{8}y_3 + \frac{9}{5}y_4 + y_5 + y_6 + \frac{101}{64}y_7 \\
5y_1 + 8y_2 + y_3 + y_4 \geq 2 \\
-2y_1 + 3y_2 + y_5 \geq 1 \\
8y_1 - y_2 + y_6 + y_7 \geq -1 \\
y_1 \geq 0, y_2 \leq 0, y_3 \leq 0, y_4 \geq 0, y_5 \geq 0, y_6 \leq 0, y_7 \geq 0.
\end{aligned}$$

From the second constraint, we see that $y_5 \geq 1 + 2y_1 - 3y_2$. Therefore $y_5 > 0$ and for complementary slackness $x_2 = 1$.

The dual solution is $y_1 = y_2 = y_3 = y_7 = 0, y_4 = 2, y_5 = 1, y_6 = -1$.

Final problem

Since $x_2 = x_3 = 1$, the problem

$$\begin{aligned} \max\ & 2x_1 + x_2 - x_3 \\ & 5x_1 - 2x_2 + 8x_3 \leq 15 \\ & 8x_1 + 3x_2 - x_3 \geq 9 \\ & \frac{7}{8} \leq x_1 \leq \frac{9}{5} \\ & 0 \leq x_2 \leq 1 \\ & 1 \leq x_3 \leq \frac{101}{64} \end{aligned}$$

can be further reduced to

$$\max\{2x_1 : \frac{7}{8} \leq x_1 \leq \frac{9}{5}\},$$

which trivially leads to the optimal solution $x_1 = \frac{9}{5}$.

Pre-processing - example (binary)

Consider the set of constraints involving four $\mathbf{0-1}$ variables

$$\begin{aligned} 7x_1 + 3x_2 - 4x_3 - 2x_4 &\leq 1 \\ -2x_1 + 7x_2 + 3x_3 + x_4 &\leq 6 \\ -2x_2 - 3x_3 - 6x_4 &\leq -5 \\ 3x_1 - 2x_3 &\geq -1 \\ x &\in \{0,1\}^4 \end{aligned}$$

Generating logical inequalities, 1^{st} row

$$7x_1 + 3x_2 - 4x_3 - 2x_4 \leq 1$$

- If $x_1 = 1$, then necessarily $x_3 = 1$
- If $x_1 = 1$, then necessarily $x_4 = 1$
- This can be formulated with the linear inequalities $x_1 \leq x_3$ and $x_1 \leq x_4$
- The constraint is infeasible if both $x_1 = x_2 = 1$, leading to the constraint $x_1 + x_2 \leq 1$

Generating logical inequalities, 2^{nd} row 16 | 23

$$-2x_1 + 7x_2 + 3x_3 + x_4 \leq 6$$

- If $x_2 = 1$, then necessarily $x_1 = 1$
- This can be formulated with the linear inequality $x_2 \leq x_1$
- The constraint is infeasible if both $x_2 = x_3 = 1$, leading to the constraint $x_2 + x_3 \leq 1$

Generating logical inequalities, 3^{rd} row 17

$$-2x_2 - 3x_3 - 6x_4 \leq -5$$

- The constraint cannot be satisfied if both $x_2 = x_4 = 0$
- This leads to the constraint $x_2 + x_4 \geq 1$
- The constraint cannot be satisfied if both $x_3 = x_4 = 0$
- This leads to the constraint $x_3 + x_4 \geq 1$

Generating logical inequalities, 4^{th} row 18

$$3x_1 - 2x_3 \geq -1$$

- If $x_3 = 1$, then necessarily $x_1 = 1$
- This can be formulated with the linear inequality $x_3 \leq x_1$

Combining pairs of logical inequalities

1a) $x_1 \leq x_3$

1b) $x_1 \leq x_4$

1c) $x_1 + x_2 \leq 1$

2a) $x_2 \leq x_1$

2b) $x_2 + x_3 \leq 1$

3a) $x_2 + x_4 \geq 1$

3b) $x_3 + x_4 \geq 1$

4a) $x_1 \geq x_3$

- From (1a) and (4a): $x_1 = x_3$
- From (1c) and (2a): $x_2 = 0$
- From (3a) and $x_2 = 0$: $x_4 = 1$

Simplifying

Making the substitutions $x_2 = 0, x_4 = 1, x_1 = x_3$ the four constraints

$$\begin{aligned} 7x_1 + 3x_2 - 4x_3 - 2x_4 &\leq 1 \\ -2x_1 + 7x_2 + 3x_3 + x_4 &\leq 6 \\ -2x_2 - 3x_3 - 6x_4 &\leq -5 \\ 3x_1 - 2x_3 &\geq -1 \\ x &\in \{0, 1\}^4 \end{aligned}$$

become

$$3x_1 \leq 3; x_1 \leq 5, -3x_1 \leq 1, x_1 \geq -1, \text{ where } x_1 \in \{0, 1\}.$$

They are all redundant. Therefore the only feasible solutions are $(0, 0, 0, 1)$ and $(1, 0, 1, 1)$.

Exercise - B&B (integer)

Consider the two-variable integer program

$$\begin{aligned} \max \quad & 9x_1 + 5x_2 \\ & 4x_1 + 9x_2 \leq 35 \\ & x_1 \leq 6 \\ & x_1 - 3x_2 \geq 1 \\ & 3x_1 + 2x_2 \leq 19 \\ & x \in \mathbb{Z}_+^2 \end{aligned}$$

Solve by branch-and-bound graphically.

Exercise - B&B (knapsack)

Solve the instance

$$\begin{aligned} \max \quad & \mathbf{17}x_1 + \mathbf{10}x_2 + \mathbf{25}x_3 + \mathbf{17}x_4 \\ & \mathbf{5}x_1 + \mathbf{3}x_2 + \mathbf{8}x_3 + \mathbf{7}x_4 \quad \leq \mathbf{12} \\ & x \in \{0,1\}^4 \end{aligned}$$

by branch-and-bound.

Exercise - pre-processing (binary)

Consider the $\mathbf{0-1}$ problem

$$
\begin{aligned}
\max\, 5x_1 - 7x_2 - 10x_3 + 3x_4 - 5x_5 \\
x_1 + 3x_2 - 5x_3 + x_4 + 4x_5 \leq 0 \\
-2x_1 - 6x_2 + 3x_3 - 2x_4 - 2x_5 \leq -4 \\
2x_2 - 2x_3 - x_4 + x_5 \leq -2 \\
x \in \{0,1\}^5.
\end{aligned}
$$

Simplify using logical inequalities.