Valid inequalities

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Introduction

We consider the general integer program

$$\max\{\boldsymbol{cx} : \boldsymbol{x} \in \boldsymbol{X}\}$$

where $\boldsymbol{X} = \{\boldsymbol{x} : \boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}^{\boldsymbol{n}}_{+}\}$. Remember that

$$\boldsymbol{conv}(\boldsymbol{X}) = \{\boldsymbol{x} : \tilde{\boldsymbol{A}}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}, \boldsymbol{x} \geq \boldsymbol{0}\} \text{ is a polyhedron.}$$

This result tells us that we can, in theory, reformulate problem ***IP*** as the linear program:

$$\max\{\boldsymbol{cx} : \tilde{\boldsymbol{A}}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

and then for any value of $\boldsymbol{c}$, an optimal extreme point solution of ***LP*** is an optimal solution of ***IP***.

Introduction

- We have seen that there are problems for which we have given an explicit description of $\boldsymbol{conv}(\boldsymbol{X})$.
- However, unfortunately for $\mathcal{NP}$-hard problems, there is almost no hope of finding "good" description.
- Given an instance of an $\mathcal{NP}$-hard problem, the goal is to find effective ways to try to approximate $\boldsymbol{conv}(\boldsymbol{X})$ for a given instance

Valid inequality - Definition

Definition

An inequality $\pi x \leq \pi_0$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\pi x \leq \pi_0$ for all $x \in X$

If $X = \{x \in \mathbb{Z}^n : Ax \leq b\}$ and $conv(X) = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}\}$, the constraints $a^i x \leq b_i$ and $\tilde{a}^i x \leq \tilde{b}_i$ are clearly valid inequalities for X.

Two questions immediately come to mind:

- Which are the "good" or useful valid inequalities?
- If we know a set of family of valid inequalities for a problem, how can we use them in trying to solve a particular instance?

Valid inequalities - Example 1

A Pure $\mathbf{0-1}$ set. Consider the $\mathbf{0-1}$ knapsack set:

$$X = \{x \in \{0,1\}^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

- If $x_2 = x_4 = 0$ the lhs (left-hand-side) is $3x_1 + 2x_3 + x_5 \geq 0$, whereas the rhs (right-hand-side) is equal to -2, which is impossible. So all feasible solutions satisfy the valid inequality $x_2 + x_4 \geq 1$
- If $x_1 = 1$ and $x_2 = 0$, the lhs is $3 + 2x_3 - 3x_4 + x_5$ whose minimum value is $3 - 3 = 0$, whereas the rhs is -2, and this is impossible. Hence $x_1 \leq x_2$ is a valid inequality.

Valid inequalities - Example 2

A Mixed $\mathbf{0-1}$ set. Consider the set:

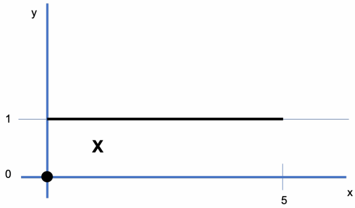
$$X = \{(x, y) : x \leq 9999y, 0 \leq x \leq 5, y \in \{0, 1\}\}$$

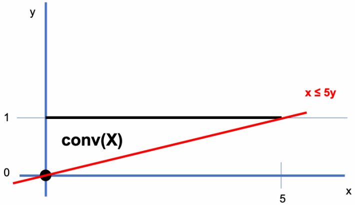
It is easily checked that the inequality

$$x \leq 5y$$

is valid because $X = \{(0, 0), (x, 1)$ with $0 \leq x \leq 5\}$. Graphically, it is also easy to check that the addition of the inequality $x \leq 5y$ gives us the convex hull of X.

Valid inequalities - Example 2





Valid inequalities - Example 2

Such constraints arise often. For instance in the capacitated facility location problem one has the feasible region

$$\sum_{i \in M} x_{ij} \leq b_j y_j \text{ for } j \in N$$
$$\sum_{j \in N} x_{ij} = a_i \text{ for } i \in M$$
$$x_{ij} \geq 0 \text{ for } i \in M, j \in N, y_j \in \{0, 1\} \text{ for } j \in N.$$

All feasible solutions satisfy $x_{ij} \leq b_j y_j$ and $x_{ij} \leq a_i$ with $y_j \in \{0, 1\}$. This leads to the family of valid inequalities $x_{ij} \leq \min\{a_i, b_j\} y_j$.

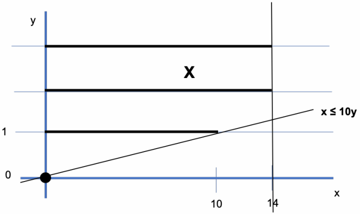
Valid inequalities - Example 3

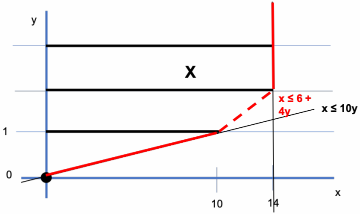
A Mixed integer set. Consider the set

$$\boldsymbol{X} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} \leq \mathbf{10}\boldsymbol{y}, \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{14}, \boldsymbol{y} \in \mathbb{Z}^{\mathbf{1}}_{+}\}$$

It is not difficult to verify the validity of the inequality $\boldsymbol{x} \leq \mathbf{6} + \mathbf{4}\boldsymbol{y}$, or written another way, $\boldsymbol{x} \leq \mathbf{14} - \mathbf{4}(\mathbf{2} - \boldsymbol{y})$. In the next slide, we see that the addition of the inequality $\boldsymbol{x} \leq \mathbf{6} + \mathbf{4}\boldsymbol{y}$ gives the convex hull of $\boldsymbol{X}$.

Valid inequalities - Example 3





Valid inequalities - Example 3

For the general case, when $\boldsymbol{C}$ does not divide $\boldsymbol{b}$, and

$$\boldsymbol{X} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} \leq \boldsymbol{C}\boldsymbol{y}, \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{y} \in \mathbb{Z}_+^{\boldsymbol{1}}\}$$

one obtains the valid inequality $\boldsymbol{x} \leq \boldsymbol{b} - \gamma(\boldsymbol{K} - \boldsymbol{y})$ where $\boldsymbol{K} = \lceil \frac{\boldsymbol{b}}{\boldsymbol{C}} \rceil$ and $\gamma = \boldsymbol{b} - (\lceil \frac{\boldsymbol{b}}{\boldsymbol{C}} \rceil - \boldsymbol{1})\boldsymbol{C}$.

Valid inequalities - Example 4

A combinatorial set: matching. Consider the set X of incidence vectors of matching

$$\sum_{e \in \delta(i)} x_e \leq 1 \text{ for } i \in V$$

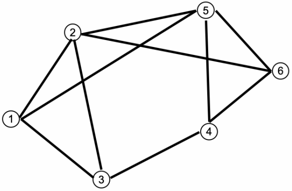
$$x \in \mathbb{Z}_+^{|E|}$$

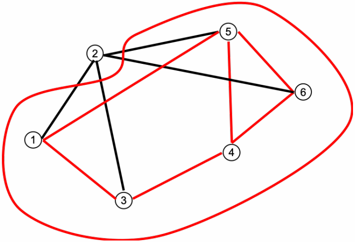
where $\delta(i) = \{e \in E : e = (i,j) \text{ for some } j \in V\}$. Take a set $T \subseteq V$ of nodes of odd cardinality. As the edges of a matching are disjoint, the number of edges of a matching having both endpoints in T is at most $\frac{|T|-1}{2}$. Therefore

$$\sum_{e \in E(T)} x_e \leq \frac{|T|-1}{2}$$

is a valid inequality for X if $|T| \geq 3$ and $|T|$ is odd.

Valid inequalities - Example 4





Valid inequalities - Example 4

$$
\begin{aligned}
x_{12} + x_{13} + x_{15} &\leq 1 \\
x_{12} + x_{23} + x_{25} + x_{26} &\leq 1 \\
x_{13} + x_{23} + x_{34} &\leq 1 \\
x_{34} + x_{45} + x_{46} &\leq 1 \\
x_{15} + x_{25} + x_{45} + x_{56} &\leq 1 \\
x_{26} + x_{46} + x_{56} &\leq 1
\end{aligned}
$$

If $T = \{1, 3, 4, 5, 6\}$ we add the valid inequality

$$x_{15} + x_{13} + x_{34} + x_{45} + x_{46} + x_{56} \leq 2$$

Valid inequalities - Example 5

Integer rounding. Consider the integer region $\boldsymbol{X} = \boldsymbol{P} \cap \mathbb{Z}^4$ where

$$\boldsymbol{P} = \{x \in \mathbb{R}^4_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$$

Dividing by 11 gives the valid inequality for $\boldsymbol{P}$

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11}$$

As $x \geq 0$, rounding up the coefficients on the left to the nearest integer gives

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11}$$

Valid inequalities - Example 5

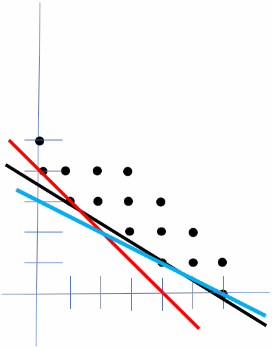
and so we get a new valid inequality for P

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{72}{11}$$

As x is integer and all the coefficients are integer, the lhs must be integer. An integer that is greater than or equal to $\frac{72}{11}$ must be at least 7, and so we can round the rhs up to the nearest integer giving the valid inequality for X:

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

Valid inequalities - Example 5



$13x_1 + 20x_2 \geq 72$ (original constraint)

$x_1 + 2x_2 \geq 6$ (divide by 13)

$x_1 + x_2 \geq 4$ (divide by 20)

Valid inequalities - Example 5

The Generalised Transportation Problem is to satisfy demand d_j of client j using trucks of different types. A truck of type i has capacity C_i, there are a_i of them available, and the cost if truck of type i is sent to client j is c_{ij}. Decision variables x_{ij} represent the number of trucks of type i that go to client j.

$$
\begin{aligned}
\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
\sum_{i=1}^{m} C_i x_{ij} \geq d_j & \text{ for } j = 1, \dots, n \\
\sum_{j=1}^{n} x_{ij} \leq a_i & \text{ for } i = 1, \dots, m \\
x \in \mathbb{Z}_+^{mn} &
\end{aligned}
$$

where each demand constraint gives rise to the set of the form X.

Valid inequalities - Example 6

Mixed Integer Rounding. Consider the same example as above with the addition of a continuous variable. Let $X = P \cap (\mathbb{Z}^4 \times \mathbb{R}^1)$ where

$$P = \{(y, s) \in \mathbb{R}^4_+ \times \mathbb{R}^1 : 13y_1 + 20y_2 + 11y_3 + 6y_4 + s \geq 72\}.$$

Dividing by 11 gives

$$\frac{13}{11}y_1 + \frac{20}{11}y_2 + y_3 + \frac{6}{11}y_4 \geq \frac{72 - s}{11}$$

suggesting there is a valid inequality

$$2y_1 + 2y_2 + y_3 + y_4 + \alpha s \geq 7 \text{ for some } \alpha \tag{1}$$

Looking at the rhs term $\frac{72-s}{11}$, we see that $\lceil \frac{72-s}{11} \rceil$ decreases from 7 to 6 at the critical value $s = 6$, indicating the value $\alpha = \frac{1}{6}$. Inequality (1) turns out to be valid for values of $\alpha \geq \frac{1}{6}$.

Valid inequalities for Linear Programs

When is the inequality $\pi x \leq \pi_0$ valid for $P = \{x : Ax \leq b, x \geq 0\}$.

Proposition

$\pi x \leq \pi_0$ is valid for $P = \{x : Ax \leq b, x \geq 0\} \neq \emptyset$ if and only if

- there exists $u \geq 0, v \geq 0$ such that $uA - v = \pi$ and $ub \leq \pi_0$, or alternatively
- there exists $u \geq 0$ such that $uA \geq \pi$ and $ub \leq \pi_0$

Proof By linear programming duality, $\max\{\pi x : x \in P\} \leq \pi_0$ if and only if $\min\{ub : uA - v = \pi, u \geq 0, v \geq 0\} \leq \pi_0$. (or $\min\{ub : uA \geq \pi, u \geq 0\} \leq \pi_0$).

Valid inequalities for Integer Programs

Proposition

Let $\boldsymbol{X} = \{\boldsymbol{y} \in \mathbb{Z}^{\mathbf{1}} : \boldsymbol{y} \leq \boldsymbol{b}\}$, then the inequality $\boldsymbol{y} \leq \lfloor \boldsymbol{b} \rfloor$ is valid for $\boldsymbol{X}$.

Valid inequalities for Integer Programs

Let $\boldsymbol{X} = \boldsymbol{P} \cap \mathbb{Z}^{\boldsymbol{n}}$ be the set of integer points $\boldsymbol{P}$ where $\boldsymbol{P}$ is given by

$$\begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x &\geq 0 \end{aligned}$$

Valid inequalities for Integer Programs

(i) First combining the constraints with non-negative weights $\boldsymbol{u} = (\frac{2}{7}, \frac{37}{63}, \mathbf{0})$, we obtain the valid inequality for $\boldsymbol{P}$

$$2x_1 + \frac{1}{63}x_2 \leq \frac{121}{21}$$

(ii) Reducing the coefficients on the lhs to the nearest integer gives the valid inequality for $\boldsymbol{P}$

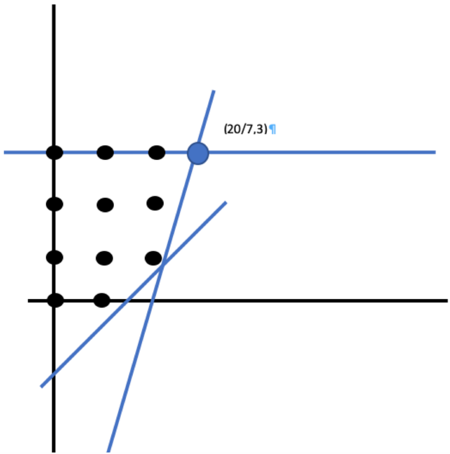
$$2x_1 + 0x_2 \leq \frac{121}{21}$$

(iii) Now, as the lhs is integral for all points of $\boldsymbol{X}$, we can reduce the rhs to the nearest integer, and we obtain the valid inequality for $\boldsymbol{X}$

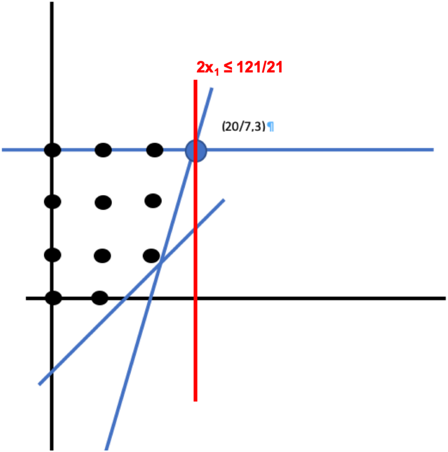
$$2x_1 \leq \lfloor \frac{121}{21} \rfloor = 5$$

If we repeat the procedure, and we use a weight of $\frac{1}{2}$ on this last constraint, we obtain the tighter inequality $x_1 \leq \lfloor \frac{5}{2} \rfloor = 2$.

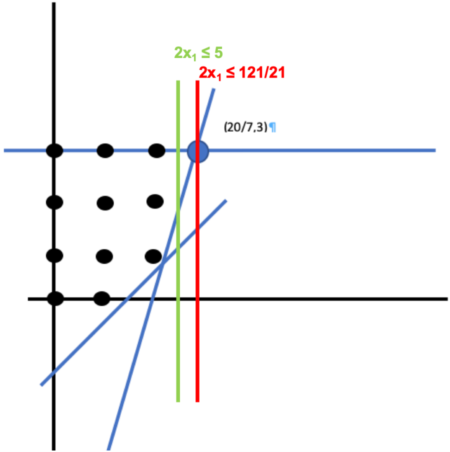
Valid inequalities for Integer Programs



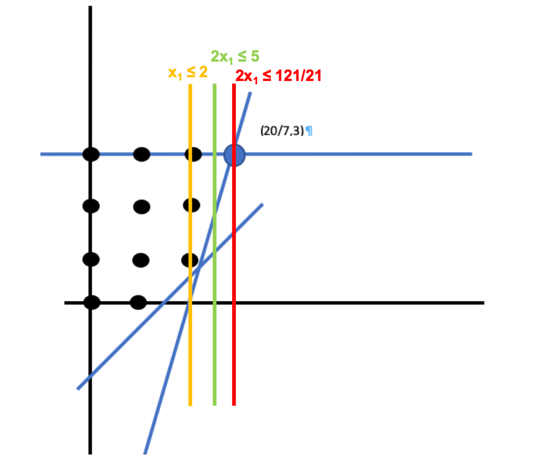
Valid inequalities for Integer Programs



Valid inequalities for Integer Programs



Valid inequalities for Integer Programs



Chvátal–Gomory procedure

Chvátal-Gomory procedure to construct a valid inequality for the set $X = P \cap \mathbb{Z}^n$, where $P = \{x \in \mathbb{R}^n_+ : AX \leq b\}$, A is an $m \times n$ matrix with columns $\{a_1, \ldots, a_n\}$, and $u \in \mathbb{R}^m_+$:

(i) the inequality

$$\sum_{j=1}^{n} ua_j x_j \leq ub$$

is valid for P as $u \geq 0$ and $\sum_{j=1}^{n} a_j x_j \leq b$

(ii) the inequality

$$\sum_{j=1}^{n} \lfloor ua_j \rfloor x_j \leq ub$$

is valid for P as $x \geq 0$

(iii) the inequality

$$\sum_{j=1}^{n} \lfloor ua_j \rfloor x_j \leq \lfloor ub \rfloor$$

is valid for X as x is integer, and thus $\sum_{j=1}^{n} \lfloor ua_j \rfloor x_j$ is integer.

Chvátal-Gomory procedure

This simple procedure is sufficient to generate all valid inequalities for an integer program.

Theorem

Every valid inequality for $\boldsymbol{X}$ can be obtained by applying the Chvátal-Gomory procedure a finite number of times.

Discussion

In discussing B&B we saw that pre-processing was a first step in tightening a formulation. Here the idea is to

- Examine the initial formulation $\boldsymbol{P} = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ with $\boldsymbol{X} = \boldsymbol{P} \cap \mathbb{Z}^{\boldsymbol{n}}$
- Find a set of valid inequalities $\boldsymbol{Q}\boldsymbol{x} \leq \boldsymbol{q}$ for $\boldsymbol{X}$
- Add these to the formulation immediately giving a new formulation $\boldsymbol{P'} = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{Q}\boldsymbol{x} \leq \boldsymbol{q}, \boldsymbol{x} \geq \boldsymbol{0}\}$ with $\boldsymbol{X} = \boldsymbol{P'} \cap \mathbb{Z}^{\boldsymbol{n}}$
- Then one can apply one's favourite algorithm, B&B or whatever, to formulation $\boldsymbol{P'}$

Pros and cons

Advantages. When using a standard B&B software, if the valid inequalities are well chosen so that formulation $\boldsymbol{P}'$ is significantly smaller than $\boldsymbol{P}$, the bounds should be improved and hence the B&B algorithm should be more effective. In addition, the chances of finding feasible integer solutions in the course of the algorithm should increase.

Disadvantages. Often the family of valid inequalities one would like to add is enormous. In such cases either the linear programs become very big and take a long time to solve, or it becomes impossible to use standard B&B software because there are too many constraints.