Linearisation and other modelling issues

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Minimax objective functions

The non-linear objective function

$$\min \max(x_1, x_2, \ldots, x_n)$$

can be linearised by introducing a decision variable y and n linear constraints

$$\begin{aligned} \min\ & y \\ y &\geq x_1 \\ y &\geq x_2 \\ &\vdots \\ y &\geq x_n \end{aligned}$$

You can see that y has to be no less than each of the x's, and the minimisation objective will force it down to take the value of the largest x. Similarly for the maxmin case. Unfortunately the same trick cannot be applied where we want to $\min \min(x_1, x_2, \ldots, x_n)$ or $\max \max(x_1, x_2, \ldots, x_n)$

Absolute value

Since

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

it follows that $|x| = \max\{x, -x\}$. Then $\min |x|$ can be also written as

$$\begin{aligned} \min y \\ y \geq x \\ y \geq -x \end{aligned}$$

Ratio objective function

The non-linear problem

$$\begin{aligned} \max f(x) = \frac{cx + c_0}{dx + d_0} \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

can be linearised by introducing $t = \frac{1}{dx+d_0}$. If we assume $dx + d_0 > 0$ (and hence $t > 0$) - if $t < 0$ the inequality signs must be reversed when we multiply by t

we can write the non-linear problem

$$\begin{aligned} \max f(x,t) = ctx + c_0 t \\ Atx \leq bt \\ dtx + d_0 t = 1 \\ x, t \geq 0, \end{aligned}$$

which becomes linear by introducing $y = tx$

$$\begin{aligned} \max f(y,t) = cy + c_0 t \\ Ay - bt \leq 0 \\ dy + d_0 t = 1 \\ y, t \geq 0, \end{aligned}$$

Logical relations - NOT

The use of **binary variables** allows us to model logical relations as linear constraints

NOT

A	B
0	1
1	0

It can be easily modelled by constraint

$$x_A = 1 - x_B, \text{ where } x_A, x_B \in \{0, 1\}$$

.

Logical relations – OR

A	B	C
0	0	0
0	1	1
1	0	1
1	1	1

It can be modelled by constraints

$$x_C \geq x_A$$
$$x_C \geq x_B$$
$$x_C \leq x_A + x_B$$
$$x_A, x_B, x_C \in \{0, 1\}$$

Logical relations - AND

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

It can be modelled by constraints

$$\begin{aligned} x_C &\leq x_A \\ x_C &\leq x_B \\ x_C &\geq x_A + x_B - 1 \\ x_A, x_B, x_C &\in \{0, 1\} \end{aligned}$$

Logical relations – XOR

A	B	C
0	0	0
0	1	1
1	0	1
1	1	0

It can be modelled by constraints

$$x_C \geq x_A - x_B$$
$$x_C \geq x_B - x_A$$
$$x_C \leq x_A + x_B$$
$$x_C \leq 2 - x_A - x_B$$
$$x_A, x_B, x_C \in \{0, 1\}$$

The Bin packing problem

In the bin packing problem, items of different volumes must be packed into a finite number of bins or containers each of a fixed given volume in a way that minimises the number of bins used.

If there are m items ($i = 1, \ldots, m$), each of volume d_i, and n bins ($j = 1, \ldots, n$), each of capacity K_j, some obvious decision variables are

$$x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is inserted in bin } j \\ 0 & \text{otherwise.} \end{cases}$$

There are two alternative formulations

The Bin packing problem

$$\min \sum_{y=1}^{n} y_j$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \dots, m$$

$$\sum_{i=1}^{m} d_i x_{ij} \leq K_j y_j \text{ for } j = 1, \dots, n$$

$$x_{ij} \in \{0,1\}, y_j \in \{0,1\}$$

$y_j = 1$ if the jth bin used, 0 otherwise.

$$\min w$$

$$w \geq \sum_{j=1}^{n} j x_{ij} \text{ for } i = 1, \dots, m$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \dots, m$$

$$\sum_{i=1}^{m} d_i x_{ij} \leq K_j \text{ for } j = 1, \dots, n$$

$$x_{ij} \in \{0,1\}, w \geq 0$$

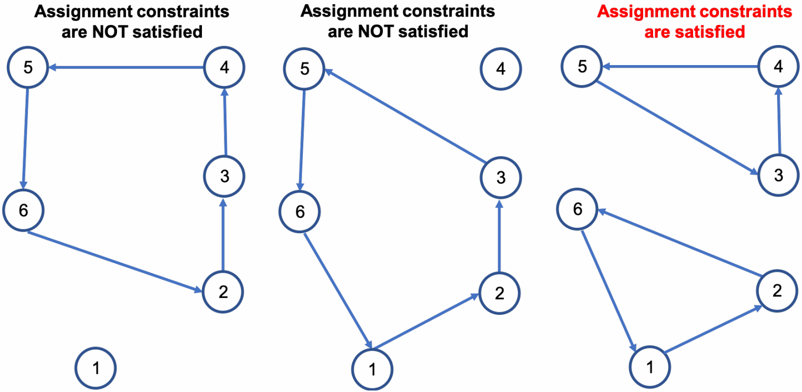
The value of w must be at least as large as the highest index number of the bins that are used.

Eliminating sub-tours

As we might remember, in the TSP formulation sub-tours can be eliminated only by adding an exponential number of constraints. A alternative reasoning follows.

- Let arbitrarily fix one node as node **1**.
- If there is a sub-tour that does not contain node **1**, there must be another sub-tour that contains node **1**.
- The idea is not to allow the existence of sub-tours that do not contain node **1**.
- In this way, we only allow complete tours.

Eliminating sub-tours



Eliminating sub-tours

For a direct graph $\mathcal{G} = (N, A)$, let's introduce variable u_i, where $u_1 = 1$ and $u_i \geq 1$ for each $i \in N$ and such that $u_i = k$ if i is the kth node visited in the cycle. If $x_{ij} = 1$ if arc (i, j) belongs to the Hamiltonian tour and 0 otherwise, a formulation is

$$
\begin{aligned}
\min \sum_{(i,j)\in A} & c_{ij} x_{ij} \\
\sum_{i \in N \setminus \{j\}} & x_{ij} = 1 && \forall j \in N \\
\sum_{j \in N \setminus \{i\}} & x_{ij} = 1 && \forall i \in N \\
& |N| x_{ij} + u_i - u_j \leq |N| - 1 && \forall (i,j) \in A, j \neq 1 \\
& x_{ij} \in \{0, 1\} && \forall (i,j) \in A \\
& u_1 = 1 \\
& 2 \leq u_i \leq |N| && \forall i \in N, i \neq 1
\end{aligned}
$$

Eliminating sub-tours

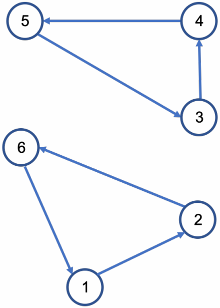
Why the following inequalities allow to eliminate sub-tours?

$$|N|x_{ij} + u_i - u_j \leq |N| - 1 \quad \forall (i,j) \in A, j \neq 1$$

- If x satisfy the assignment constraints but it is not a Hamiltonian tour, then there are some sub-tours and one of them does not contain node 1. In these case the above inequalities are not satisfied.
- Instead, the constraints are satisfied by each Hamiltonian tour.

These constraints are on the order of $|N|^2$, and are not exponential in number($2^{|N|}$).

Eliminating sub-tours



$6 + u_3 - u_4 \leq 5$
$6 + u_4 - u_5 \leq 5$
$6 + u_5 - u_3 \leq 5$
then
$18 \leq 15$!!!
Impossible

$6 + u_1 - u_6 \leq 5$
$6 + u_6 - u_4 \leq 5$
$6 + u_4 - u_3 \leq 5$
$6 + u_3 - u_2 \leq 5$
$6 + u_2 - u_5 \leq 5$

Since $u_1 = 1$ then

$u_6 = 2$
$u_4 = 3$
$u_3 = 4$
$u_2 = 5$
$u_5 = 6$

is a feasible solution