Optimisation Models: exercises

Dipartimento di Ingegneria e Architettura
Università degli studi di Trieste

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Exercise 1: Manpower planning

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1	3	3,4,2	10.2
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Personnel	5	6	7	7	6	6

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- for each month, it is not possible to use more manpower than is available

- maximise the total benefit obtained through the projects **once they are finished**.

Notation

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$$
\begin{aligned}
\mathcal{P} &: \text{set of projects} \\
\mathcal{M} &: \text{set of months } = 1, \ldots, nM \\
D_p &: \text{duration of project } p \\
G_p &: \text{monthly benefit once project } p \text{ is finished} \\
CAP_m &: \text{available manpower for month } m
\end{aligned}
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$\mathcal{M}$: set of months $= 1, \ldots, nM$

D_p : duration of project p

G_p : monthly benefit once project p is finished

CAP_m : available manpower for month m

$RU_{p,k}$: resource usage of project p in **it's** k-th month

Decision variables

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$$x_{p,m} = \begin{cases} 1 & \text{if project } p \text{ starts in month } m \\ 0 & \text{otherwise} \end{cases} \qquad \forall p \in \mathcal{P}, m \in \mathcal{M}$$

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$start_p$: starting month of project p

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$$start_p \leq nM - D_p + 1 \qquad \forall p \in \mathcal{P}$$

Decision variables constraints

```
! Each project starts once and only once
 forall(p in P) sum(m in M) x(p,m) = 1

! Connect variables x(p,t) and start(p)
 forall(p in P) sum(m in M) m*x(p,m) = start(p)

! Finish everything by the end of the planning period
 forall(p in P) start(p) <= nM-D(p)+1
```

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```
! Resource availability
 forall(m in M)
  sum(p in P,s in 1..m) RU(p,m-s+1)*x(p,s) <= CAP(m)
```

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$$max \sum_{p \in \mathcal{P}} G_p \cdot n_p$$

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$$max \sum_{p \in \mathcal{P}} G_p \cdot n_p$$

```
! Objective: Maximize Benefit
MaxBen:=
 sum(p in P) (G(p)*(nM-start(p)-D(p)+1))

! Solve the problem
maximize(MaxBen)
```

SOS1

Special Ordered Sets of type 1: an ordered set of non-negative variables at most **one** of which can take a non-zero value.

- the **order** is specified by assigning weights (*reference row values*) to each variable
- used for branching on **sets** of variables, rather than individual variables
- the search procedure will generally be noticeably faster

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```
if not USESOS then  ! Turn variables x into binaries
 forall(p in P,m in M) x(p,m) is_binary
else                ! Define SOS-1 sets
 forall(p in P) sum(m in M) m*x(p,m) is_sos1
end-if
```

SOS1

Standard branching

- "1-branch" (**strong decision**) eliminates a very large number of possible solutions
- "0-branch" (**weak decision**) eliminates only a few
- **unbalanced** tree

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SOS1 branching

- all the variables in one subset are set to 0
- any combination of the variables in the other subset sums to 1
- **strong decision deferred**
- unpromising variables placed in the new "set-to-0" subset
- **better balanced** and **smaller** tree

Exercise 2: Purchasing with price breaks

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Supplier	Max%	Unit Price	Breakpoint	UP	BP	UP	BP
1	40%	9.2	100	9	200	7	1000
2	35%	9	50	8.5	250	8.3	2000
3	40%	11	100	8.5	300	7.5	4000

- if you buy 220 items from supplier 1, you pay $9.2 \cdot \mathbf{100} + 9 \cdot \mathbf{100} + 7 \cdot \mathbf{20}$. We can buy at most 40% of the requirement from supplier (240 items)

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$\mathcal{S}$: set of suppliers

$\mathcal{R}$: set of price ranges

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$B_{s,r}$: breakpoint where the unit cost change from $COST_{s,r}$ to $COST_{s,r+1}$ for supplier s

T : total amount we wish to buy

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Decision variables

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Objective function

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$$min \sum_{s \in \mathcal{S}, r \in \mathcal{R}} x_{s,r} \cdot COST_{s,r}$$

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$$\sum_{s \in \mathcal{S}, r \in \mathcal{R}} x_{s,r} \geq T$$

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$$\sum_{r \in \mathcal{R}} x_{s,r} \leq Q_s \cdot T \qquad \forall s \in \mathcal{S}$$

Decision variables constraints

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$$(B_{s,0} := 0 \quad \forall s \in \mathcal{S})$$

SOS2

Special Ordered Sets of type 2: an ordered set of non-negative variables, of which at most two can be non-zero, and if two are non-zero these must be consecutive in their ordering

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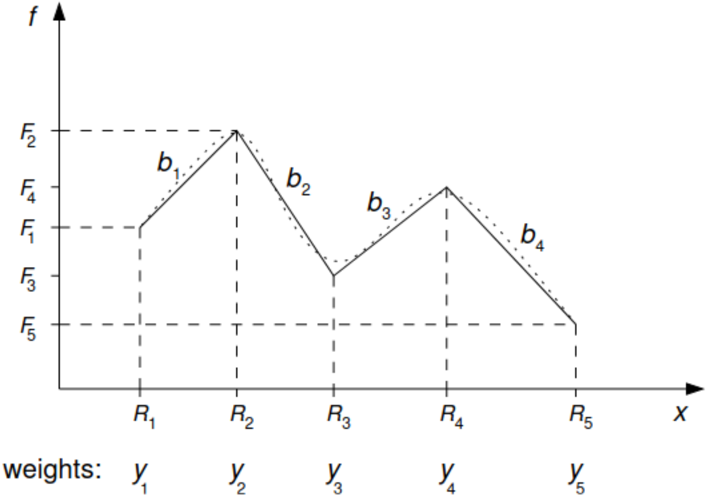
- used for branching on **sets** of variables, rather than individual variables
- the search procedure will generally be noticeably faster
- generally used for modeling **piecewise approximations of functions** of a single variable

SOS2: piecewise functions



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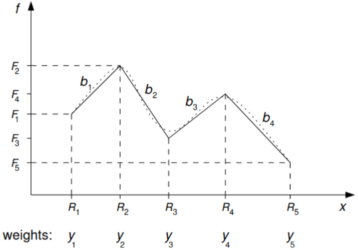
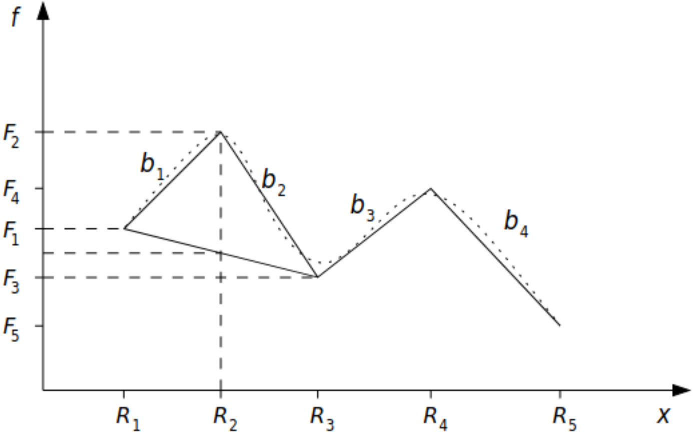


$$x = \sum_{i=1}^{5} R_i \cdot y_i \qquad \textbf{(reference row)}$$

$$f = \sum_{i=1}^{5} F_i \cdot y_i$$

$$\sum_{i=1}^{5} y_i = 1 \qquad \textbf{(convexity row)}$$

SOS2: adjacency condition



$y_1 = 0.5$

$y_2 = 0$

$y_3 = 0.5$

$y_4 = 0$

$y_5 = 0$

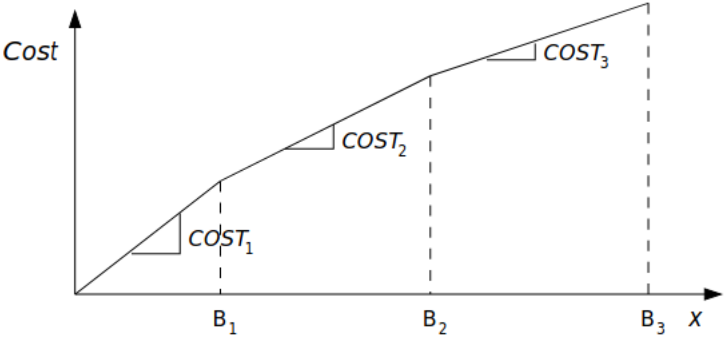
SOS2: alternative formulation

$$b_i = \begin{cases} 1 & \text{if the value of } x \\ & \quad \text{lies between } R_i \text{ and } R_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in 1, \dots, 4$$

$$\begin{aligned} y_1 &\leq b_1 \\ y_2 &\leq b_1 + b_2 \\ y_3 &\leq b_2 + b_3 \\ y_4 &\leq b_3 + b_4 \\ y_5 &\leq b_4 \end{aligned}$$

$$b_1 + b_2 + b_3 + b_4 = 1$$

SOS2: Purchasing with price breaks



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$\mathcal{P}_0$: set of breakpoints including $0 \ = 0, 1, 2, 3$

$CBP_{s,p}$: total cost at breakpoint p for supplier s

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$$CBP_{s,0} := 0$$

$$CBP_{s,p} := CBP_{s,p-1} + COST_{s,p} \cdot (B_{s,p} - B_{s,p-1}) \quad \forall p \in \mathcal{P}_0, p > 0$$

$$(B_{s,0} := 0 \qquad \forall s \in \mathcal{S})$$

Decision variables

$x_{s,r}$: number of items bought from suppliers s at the price range r

$$y_{s,r} = \begin{cases} 1 & \text{if we have bought any items from} \\ & \quad \text{supplier } s \text{ at the price range } r \\ 0 & \text{otherwise} \end{cases} \qquad \forall s \in \mathcal{S}, r \in \mathcal{R}$$

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n_s : number of items bought from suppliers s

$w_{s,p}$: weight variables associated with breakpoint p

Decision variables constraints

$$
\begin{aligned}
y(s,r) &\geq y(s,r+1) && \forall s \in \mathcal{S}, r : r, r+1 \in \mathcal{R} \\
x_{s,r} &\leq (B_{s,r} - B_{s,r-1}) \cdot y(s,r) && \forall s \in \mathcal{S}, r \in \mathcal{R} \\
(B_{s,r} - B_{s,r-1}) \cdot y(s,r+1) &\leq x_{s,r} && \forall s \in \mathcal{S}, r : r, r+1 \in \mathcal{R}
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(**reference row**) $$n_s = \sum_{p \in \mathcal{P}_0} w_{s,p} \cdot B_{s,p} \quad \forall s \in \mathcal{S}$$

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(**convexity row**) $$\sum_{p \in \mathcal{P}_0} w_{s,p} = 1 \quad \forall s \in \mathcal{S}$$

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$$
\forall s \in \mathcal{S} \quad \bigcup_{p \in \mathcal{P}_0} \{w_{s,p}\} \quad \textbf{SOS-2} \text{ with coeff. } B_{s,p}
$$

Decision variables constraints

```
! Define z and also order the weight variables by breakpoint
    quantities
 forall(s in S) sum(p in P0) B(s,p) * w(s,p) = z(s)

! The convexity row (w sum to 1)
 forall(s in S) sum(p in P0) w(s,p) = 1

! Define the w as SOS-2 as we can linearly interpolate
   between the breakpoints
forall(s in S)
makesos2(union(p in P0){w(s,p)}, sum(p in P0) B(s,b)*w(s,p))

! Alternative formulation:
!The weight coefficients BR are all augmented by EPS
!since Mosel does not accept 0-valued weights with ‘is_sos2’
forall(s in S)
  sum(p in P0) (B(s,p)+10E-20) * w(s,p) is_sos2
```

Constraints

$$\sum_{s \in \mathcal{S}, r \in \mathcal{R}} x_{s,r} \geq T$$

$$\sum_{r \in \mathcal{R}} x_{s,r} \leq Q_s \cdot T \qquad \forall s \in \mathcal{S}$$

Constraints

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$$\sum_{s \in \mathcal{S}} n_s \geq T$$

Constraints

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$$\sum_{r\in\mathcal{R}} x_{s,r} \leq Q_s \cdot T \qquad \forall s \in \mathcal{S}$$

$$\sum_{s\in\mathcal{S}} n_s \geq T$$

$$n_s \leq Q_s \cdot T \qquad \forall s \in \mathcal{S}$$

```
! The minimum quantity that must be bought
 sum(s in S) z(s) >= T

! No more than the maximum percentage from each supplier
 forall(s in S) z(s) <= Q(s)*T
```

Objective function

$$min \sum_{s \in \mathcal{S}, r \in \mathcal{R}} x_{s,r} \cdot COST_{s,r}$$

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$$min \sum_{s \in \mathcal{S}, p \in \mathcal{P}_0} w_{s,p} \cdot CBR_{s,p}$$

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$$min \sum_{s \in \mathcal{S}, p \in \mathcal{P}_0} w_{s,p} \cdot CBR_{s,p}$$

```
! Objective: sum of costs*weights
 MinCost:= sum(s in S, p in P0) CB(s,p) * w(s,p)

 minimize(MinCost)
```