Robust optimisation

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Optimisation under uncertainty

Stochastic programming

Extensions of deterministic methodologies (be they linear or nonlinear), in which a probability distribution is associated with each unexpected event and/or with each uncertain parameter.

Robust optimisation

A set of methodologies whose goal is to determine a solution that is able to cope with a suitably specified set of unexpected events and/or a given variability of certain parameters without becoming unfeasible and too expensive, without any hypothesis on the probabilistic nature about the elements of the system subject to uncertainty.

The shortest path problem

Given a direct graph $G = (N, A)$ where N is the set of nodes, of cardinality n, while A is the set of arcs of cardinality m, and given a cost c_{ij} associated with the arc $(i, j), \forall (i, j) \in A$, the problem of the shortest path consists in identifying a path oriented from a source node $o \in N$ to a destination node $t \in N$, which has minimum cost among all the paths oriented from o to t in G. The cost of a path is defined as the sum of the costs of the arcs that belong to this path.

SPP- formulation

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\sum_{(j,v)\in BS(v)} x_{jv} + \sum_{(v,j)\in FS(v)} x_{vj} = b_v, \quad \forall v \in N$$

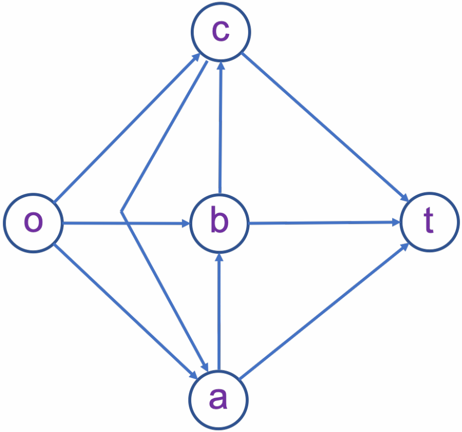
$$x_{ij} \in \{0, 1\}, (i, j) \in A,$$

where $BS(v)$ is the set of arcs entering node v and $FS(v)$ is the set of arcs exiting node v. In addition, $b_v = -1$ if $v = o$, $b_v = 1$ if $v = t$, and $b_v = 0$ otherwise.

SPP - an example

- Consider a telecommunications network described in terms of an oriented graph $\boldsymbol{G}$ in which packets must be sent from a source node $\boldsymbol{o}$ to another destination $\boldsymbol{t}$.
- Let $\boldsymbol{c_{ij}}$ there be the estimated time for sending a packet along the link $(\boldsymbol{i}, \boldsymbol{j}), \forall(\boldsymbol{i}, \boldsymbol{j}) \in \boldsymbol{A}$.
- We want to find a path in $\boldsymbol{G}$ from $\boldsymbol{o}$ to $\boldsymbol{t}$ that minimises the total routing time of a packet.
- The routing problem in question could be solved by means of a shortest path algorithm by formulating the routing problem in terms of minimum path from $\boldsymbol{o}$ to $\boldsymbol{t}$ in G.

Telecommunication network



1. (o,a,t)
2. (o,a,b,t)
3. (o,a,b,c,t)
4. (o,b,t)
5. (o,b,c,t)
6. (o,b,c,a,t)
7. (o,c,t)
8. (o,c,a,t)
9. (o,c,a,b,t)

SPP - Uncertainty in some parameters

- However, due to the variability of the traffic along the network and therefore due to the congestion of the network, which varies over time, the routing time is not usually fully known, but is a parameter subject to uncertainty.
- The network operator would therefore prefer to determine a robust minimum path from o to t, i.e. a path which is associated with an acceptable routing time regardless of what will be the actual congestion in the network when the packet will be sent from o to t. In this regard,
- Note that the routing problem in question must be solved in real time, and therefore it is not conceivable the use of an algorithmic approach based on the resolution of a problem of minimum path from scratch every time a variation occurs at network congestion level.

Scenarios

- Now we describe the representation through scenarios.
- According to this representation, the uncertainty is structured through a set $\boldsymbol{S}$ of possible scenarios, that is, of possible realisations of the uncertain parameter.
- In the event that the number of possible realisations is finite, we can explicitly define this set by means of the notation $\boldsymbol{S} = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_k\}$ where each scenario $\boldsymbol{s}_i, \boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{k}$ denotes a possible realisation of the parameter affected by uncertainty.

Scenarios - example

Considering our example problem and denoting with $\boldsymbol{c} = [\boldsymbol{c_{ij}}]$ the vector of the costs associated with the arcs of the network, each scenario $\boldsymbol{s_i}$ therefore denotes a possible realisation of the uncertain parameter $\boldsymbol{c}$, i.e. a possible realisation of the costs (i.e. routing times) associated with the network connections.

Two scenarios

- Suppose that the routing time c_{ij} along arc (i, j) depends on the traffic conditions which vary according to the time considered.
- For the sake of simplicity, suppose that this routing time is a function of day and night.
- In this case, to represent the uncertainty of the routing time parameter we can consider two scenarios $S = \{day, night\}$, which describe the routing times along the network connections in the daytime (*day* scenario) and in the night time (*night* scenario).
- The two scenarios are shown in the next table.

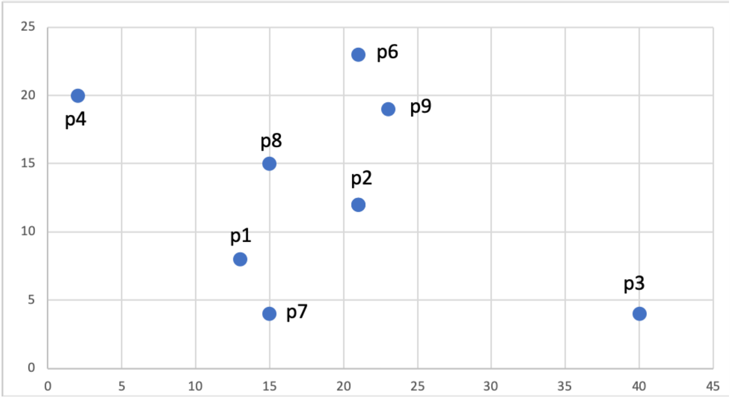
Arc costs per scenario

arc	day	night
$(\boldsymbol{o}, \boldsymbol{c})$	5	3
$(\boldsymbol{o}, \boldsymbol{b})$	1	10
$(\boldsymbol{o}, \boldsymbol{a})$	10	1
$(\boldsymbol{c}, \boldsymbol{t})$	10	1
$(\boldsymbol{b}, \boldsymbol{t})$	1	10
$(\boldsymbol{a}, \boldsymbol{t})$	3	7
$(\boldsymbol{a}, \boldsymbol{b})$	10	1
$(\boldsymbol{b}, \boldsymbol{c})$	10	1
$(\boldsymbol{c}, \boldsymbol{a})$	7	5

Finding an acceptable path

- Suppose we want to find a path oriented from the node $\boldsymbol{o}$ to the node $\boldsymbol{t}$ in which the routing time is "acceptable" both day and night.
- The uncertainty of the parameter leads to the fact that there is not necessarily a solution (in our case, a path oriented from $\boldsymbol{o}$ to $\boldsymbol{t}$) that is preferable to the others regardless of the scenario considered.
- For example, consider paths $\boldsymbol{p_4} = (\boldsymbol{o}, \boldsymbol{b}, \boldsymbol{t})$, $\boldsymbol{p_3} = (\boldsymbol{o}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{t})$, and $\boldsymbol{p_8} = (\boldsymbol{o}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{t})$. During the day $\boldsymbol{p_4}$ involves a routing cost equal to $\mathbf{2}$ and is therefore preferable to $\boldsymbol{p_3}$, which costs $\mathbf{40}$. On the contrary, at night it is preferable to use $\boldsymbol{p_3}$ which has a routing cost of $\mathbf{4}$, while $\boldsymbol{p_4}$ costs $\mathbf{20}$.
- $\boldsymbol{p_8}$, on the other hand, shows an intermediate behaviour in both scenarios, always requiring a cost of $\mathbf{15}$.

Day and night scenarios



Interval representation

- Uncertainty is structured assuming that the uncertain parameter varies in given ranges
- E.g., each arc cost $c_{i,j}, \forall (i,j) \in A$ can assume any value belonging to a given range $[c_{ij}^{-}, c_{ij}^{+}]$
- We see that a realisation of all the arc costs in the respective uncertainty intervals is in fact a scenario.
- Hence the representation of uncertainty by intervals can be seen as a special case of the representation by scenarios, characterised by an infinite set of possible scenarios.

Interval representation – example

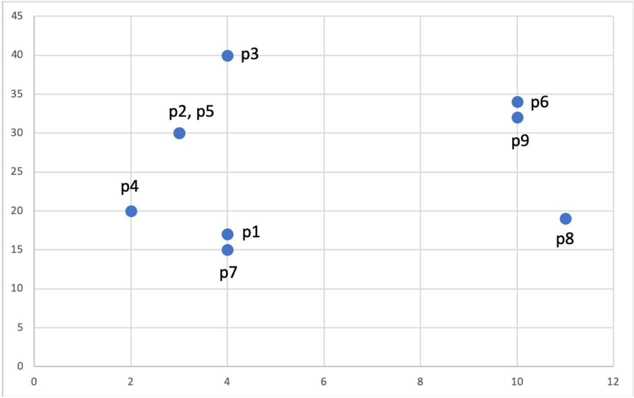
- Suppose that the uncertain parameters of the problem, i.e., routing times, have a variability that does not allow to clearly distinguish the case of daytime and nighttime.
- Instead, we are able to provide an estimate of the best and worst time along each arc of the network.
- It can then be assumed that the actual routing time at the time of the use of the arc can be any value belonging to this interval.
- Therefore *day* and *night* scenarios constitute a subset of the scenarios implicitly described by the uncertainty intervals.

Interval representation – example

arc	c_{ij}^-	c_{ij}^+
(o, c)	3	5
(o, b)	1	10
(o, a)	1	10
(c, t)	1	10
(b, t)	1	10
(a, t)	3	7
(a, b)	1	10
(b, c)	1	10
(c, a)	5	7

Thanks to a greater number of scenarios, path p_8 has a cost of **19** in the most unfavourable case (whereas considering only the day and night scenarios, the cost is **15** in the most unfavourable case). Considering instead the most favourable scenario the routing time is reduced from **15** to **11**.

Interval scenarios



Polyhedron representation

- In this representation we assume that the uncertain parameter, let call it α, can vary in a polyhedron $\boldsymbol{Q} = \{\alpha : \boldsymbol{H}\alpha \leq \boldsymbol{g}\}$ where $\boldsymbol{H}$ generically indicates a matrix of constraints, while $\boldsymbol{g}$ is the corresponding right hand side vector.
- The interval representation is a special case of the polyhedron representation.
- Considering the SPP, where $\boldsymbol{c} = [\boldsymbol{c_{ij}}]$ is the vector of the costs associated with the arcs (in this case $\boldsymbol{c}$ is therefore the parameter subject to an uncertainty), we can in fact define the set of uncertainty by intervals in polyhedral terms:

$$\boldsymbol{Q} = \{\boldsymbol{c} : \boldsymbol{c}_{\boldsymbol{ij}}^{-} \leq \boldsymbol{c_{ij}} \leq \boldsymbol{c}_{\boldsymbol{ij}}^{+}, \forall(\boldsymbol{i},\boldsymbol{j}) \in \boldsymbol{A}\}$$

Polyhedron representation - example

- Suppose we have certain information derived from technological constraints.
- For example, the sum of the routing times associated with the connections exiting node $\boldsymbol{o}$ does not exceed the value **15**.
- Similarly, the sum of the routing times associated with the connections entering the node $\boldsymbol{t}$ does not exceed the value **15**.
- Such information relating to the uncertain parameter can therefore be expressed in terms of linear constraints, and therefore by means of a polyhedral structure.

$$
\begin{aligned}
c_{oa} + c_{ob} + c_{oc} &\leq 15 \\
c_{at} + c_{bt} + c_{ct} &\leq 15 \\
c_{ab} + c_{bc} + c_{ca} &\leq 20 \\
c_{oc} + c_{bc} + c_{ct} &\geq 5 \\
c_{ob} + c_{cbc} + c_{bt} + c_{ab} &= 15
\end{aligned}
$$

Finding a robust solution

- The generic optimisation problem affected by uncertainty is defined as

$$\min\{\boldsymbol{c}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{F}\} \tag{1}$$

  where $\boldsymbol{F}$ represents the feasible region while $\boldsymbol{c}(\boldsymbol{x})$ represents the objective function or cost to be minimised.
- This problem is also called the nominal problem, whereas the optimisation problem whose resolution allows the determination of a robust solution is indicated as robust version (or counterpart).
- The uncertain α parameter that causes uncertainty in the optimisation problem can appear at level of the objective function or in the constraints that define its feasible region.

Absolute robustness

- Let $\boldsymbol{S}$ be the set of possible scenarios, or realisations, of the uncertain parameter α.
- Let us assume that α appears in the definition of the objective function $\boldsymbol{c}(\boldsymbol{x})$.
- It follows that at each solution $\boldsymbol{x} \in \boldsymbol{F}$ there are associated several realisations of the objective function, depending on the realisation of the parameter $\alpha \in \boldsymbol{S}$.
- In the following we will denote by $\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x})$ the cost of the solution $\boldsymbol{x}$ in case the uncertain α parameter is realised as indicated by the scenario $\boldsymbol{s}, \boldsymbol{x} \in \boldsymbol{F}$ and $\boldsymbol{s} \in \boldsymbol{S}$.

Minimax criterion

The robustness criterion in absolute sense, or minimax criterion, consists in determining a solution $\boldsymbol{x}^* \in \boldsymbol{F}$ such that the maximum cost associated with $\boldsymbol{x}$, due to the variation of α in $\boldsymbol{S}$, is the lowest possible. Formally, a robust solution $\boldsymbol{x}^*$ can be determined by solving the following optimisation problem

$$\min\{\max\{\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x}) : \boldsymbol{s} \in \boldsymbol{S}\} : \boldsymbol{x} \in \boldsymbol{F}\}. \tag{2}$$

Problem (2) is therefore the robust version (in absolute sense) of the optimisation problem (1).

Minimax criterion

- The robustness criterion in the absolute sense is a very conservative criterion.
- Its purpose is in fact to protect the system under study from the worst possible realisations of the uncertain parameter α, i.e. with respect to the most unfavourable scenario allowed by the set of scenarios $\boldsymbol{S}$.

Minimax criterion - SPP

- Assume that we want to identify a robust minimum path in absolute sense from node $\boldsymbol{o}$ to node $\boldsymbol{t}$, in the hypothesis that the routing times can assume any value belonging to the intervals specified in slide 15.
- To identify this robust minimum path it is sufficient to solve a classic SPP in which the cost of each arch is the maximum cost associated with the arc itself (refer to column $\boldsymbol{c}_{\boldsymbol{ij}}^{+}$ in the table of slide 15).
- In our example, path $\boldsymbol{p_7} = (\boldsymbol{o}, \boldsymbol{c}, \boldsymbol{t})$, which corresponds to a value of the objective function equal to **15**, is obtained as the minimum robust path in absolute sense.

Feasible region's uncertainty

- Let's now assume that α appears on the feasible region.
- As a consequence, the feasible region can vary according to the α realisation, i.e., according to the scenario $\boldsymbol{s} \in \boldsymbol{S}$ considered.
- Let $\boldsymbol{F}(\boldsymbol{s})$ be the set of feasible solutions relating to the scenario $\boldsymbol{s} \in \boldsymbol{S}$.
- Let $\boldsymbol{F}^*$ be the set of all the solutions that are feasible whatever is the scenario $\boldsymbol{s} \in \boldsymbol{S}$, i.e.,

$$\boldsymbol{F}^* = \bigcap_{\boldsymbol{s} \in \boldsymbol{S}} \boldsymbol{F}(\boldsymbol{s})$$

Feasible region's uncertainty

Definition

In the case of uncertainty at the level of the feasible region, the robustness criterion in the absolute sense, or minimax criterion, allows in determining a solution $\boldsymbol{x}^* \in \boldsymbol{F}^*$ that is of minimal cost. Formally, a robust solution $\boldsymbol{x}^*$ can be determined by solving the following optimisation problem

$$\min\{\boldsymbol{c}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{F}^*\} \tag{3}$$

Problem (3) is the robust version (in absolute sense) of problem (1).

The minimax criterion is always very conservative: it determines a minimum cost solution by only considering those solutions whose feasibility is granted, regardless of the realisation of $\boldsymbol{\alpha}$.

Robustness with control parameter

- We consider only the case in which α appears in the objective function.
- We assume that α is a vector of n components, and that for each component α_i there exists a reference value μ_i, called nominal value, which represents the average realisation of α_i.
- Each solution $x \in F$ is actually associated with multiple values of the objective function, depending on the realisation of α in S
- $c^s(x)$ is the cost of solution x in the event the uncertain parameter α is realised as indicated in the scenario $s, x \in F$ and $s \in S$.

Robustness with control parameter

- In addition, an integer parameter Γ, $\mathbf{0} \leq \Gamma \leq \boldsymbol{n}$, called the control parameter, is given.
- The role of this parameter is to control the relationship between the level of robustness and the cost of the solution, thus acting on the desired level of conservatism.
- Γ states that only a few scenarios are worthy of consideration.
- Precisely, only those scenarios in $\boldsymbol{S}$ in which at most Γ components of α deviate from the respective nominal value are worthy of consideration.
- Let $\boldsymbol{S}(\Gamma) \subseteq \boldsymbol{S}$ be the subset of scenarios identified by Γ.

Robustness with control parameter

Definition

The criterion of robustness in the absolute sense with the Γ control parameter consists in determining a solution $\boldsymbol{x}^*(\Gamma) \in \boldsymbol{F}$ such that the maximum cost of $\boldsymbol{x}^*$, assessed as the uncertain parameter $\boldsymbol{\alpha}$ varies in $\boldsymbol{S}(\Gamma)$, is the lowest possible one. Formally, a robust solution $\boldsymbol{x}^*(\Gamma)$ can be determined by solving the following optimisation problem:

$$\min\{\max\{\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x}) : \boldsymbol{s} \in \boldsymbol{S}(\Gamma)\} : \boldsymbol{x} \in \boldsymbol{F}\} \tag{4}$$

Problem (4) is called the robust version with control parameter of the nominal problem (1).

Robustness with control parameter

- The role of Γ is to control the relationship between the level of robustness and the cost of the solution.
- If $\Gamma = \mathbf{0}$, we assume that each component is realised by its nominal value, thus ignoring any form of parameter variability.
- $\Gamma = \boldsymbol{n}$ is equivalent to consider all possible variations of α in $\boldsymbol{S}$ and therefore $\boldsymbol{S}(\Gamma) = \boldsymbol{S}$.
- It follows that, for $\Gamma = \boldsymbol{n}$, the criterion with control parameter is reduced to the absolute robustness.
- In general, the higher the Γ value and the higher the robustness level of the solution: on the other hand, the higher the cost of the robust solution.
- As Γ increases, therefore, the conservatism level of the method increases.

Binary robust optimisation

- Assume the nominal problem is a binary linear optimisation problem
$$\min\{cx : x \in F\} \tag{5}$$
where x is a vector of n binary variables, and therefore $F \subseteq \{0, 1\}^n$.
- Let c be the uncertain parameter (i.e., $\alpha = c$)
- Each component c_i can vary within the interval $[\mu_i, \mu_i + \rho_i]$ where
  - μ_i is the nominal value of component i
  - ρ_i is the deviation from the nominal value of component i

Binary robust optimisation

- Given the integer control parameter $\Gamma, 0 \leq \Gamma, \leq n$, we want to determine a robust solution $x^*(\Gamma)$ such that at the most Γ components of c deviate from their respective nominal value.
- Hence, identifying a robust solution is therefore equivalent to solving the following robust counterpart

$$\min\{\mu x + \max\{\sum_{i \in S'} \rho_i x_i : S' \subseteq \{1, \ldots, n\}, |S'| \leq \Gamma\} : x \in F\}$$

- Z^* represents the optimal value.

SPP with $\Gamma = 1$

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arc max	(o,c) 5	(o,b) 10	(o,a) 10	(c,t) 10	(b,t) 10	(a,t) 7	(a,b) 10	(b,c) 10	(c,a) 7	path max
p1	4	4	13	4	4	8	4	4	4	13
p2	3	3	12	3	12	3	12	3	3	12
p3	4	4	13	13	4	4	13	13	4	13
p4	2	11	2	2	11	2	2	2	2	11
p5	3	12	3	12	3	3	3	12	3	12
p6	10	19	10	10	10	14	10	19	12	19
p7	6	4	4	13	4	4	4	4	4	13
p8	13	11	11	11	11	15	11	11	13	15
p9	12	10	10	10	19	10	19	10	12	19