Robust optimisation

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Binary robust optimisation

Without loss of generality, we assume that the indices of the n variables of problem $\min\{cx : x \in F\}$ are ordered such that $\rho_1 \geq \rho_2 \geq \ldots, \rho_n$. For convenience, we also set $\rho_{n+1} = 0$.

Theorem

$$Z^* = \min\{Z_h : h = 1, \ldots, n+1\}$$

where

$$Z_h = \Gamma\rho_h + \min\{\mu x + \sum_{i=1}^{\min\{h,n\}} (\rho_i - \rho_h)x_i : x \in F\}, h = 1, \ldots, n+1$$

Binary robust optimisation - Algorithm

Step 1 Solve $n + 1$ binary optimisation problems

$$\min\{\mu x + \sum_{i=1}^{\min\{h,n\}} (\rho_i - \rho_h)x_i : x \in F\}, h = 1, \dots, n + 1$$

Let x_h^* be the optimal solution of the h-th problem, and Z_h^* the corresponding value of the objective function, $h = 1, \dots, n + 1$. Determine the values

$$Z_h = \Gamma\rho_h + Z_h^*, h = 1, \dots, n + 1$$

Step 2 Let $h^* = \arg\min\{Z_h : h = 1, \dots, n + 1\}$

Step 3 $x^* = x_{h^*}$ is a robust solution, with cost $Z^* = Z_{h^*}$

Nominal values and deviations

arc	h	μ_{ij}	ρ_{ij}
(o, b)	1	1	9
(o, a)	2	1	9
(c, t)	3	1	9
(b, t)	4	1	9
(a, b)	5	1	9
(b, c)	6	1	9
(a, t)	7	3	4
(o, c)	8	3	2
(c, a)	9	5	2
-	10	-	0

Robust binary optimisation - Arc costs

Since $\rho_1 = \rho_2 = \cdots = \rho_6 = 9$ and $\rho_8 = \rho_9 = 2$ there are only four significant indices h, where to compute $Z_h : h \in \{1, 7, 8, 10\}$.

h	Arcs (o,b)	(o,a)	(c,t)	(b,t)	(a,b)	(b,c)	(a,t)	(o,c)	(c,a)
1	1	1	1	1	1	1	3	3	5
7	6	6	6	6	6	6	3	3	5
8	8	8	8	8	8	8	5	3	5
10	10	10	10	10	10	10	7	5	7

The modified arc costs for each arc (i, j) are calculated as

$$c_{ij} = \begin{cases} \mu_{ij} + (\rho_{ij} - \rho_h) & \text{if the index of } (i,j) \text{ is } \leq h \\ \mu_{ij} & \text{otherwise} \end{cases}$$

Robust binary optimisation - Path costs

If we assume $\Gamma = \mathbf{3}$ the optimal solution is $\boldsymbol{p_7} = (\boldsymbol{o}, \boldsymbol{c}, \boldsymbol{t})$, with a cost $\boldsymbol{Z}^* = \mathbf{17}$, as shown below

h	Z^*_h	P^*_h	ρ_h	$\Gamma\rho_h$	$Z^*_h + \Gamma\rho_h$
1	2	$p_4 = (o, b, t)$	9	27	29
7	9	$p_7 = (o, c, t)$	4	12	21
8	11	$p_7 = (o, c, t)$	2	6	17
10	19	$p_8 = (o, c, a, t)$	0	0	19

Robust binary optimisation

Γ	h: 1	7	8	10	P_h^*
1	11	13	13	19	$p_4 = (o, b, t)$
3	29	21	17	19	$p_7 = (o, c, t)$
5	47	29	21	19	$p_8 = (o, c, a, t)$

Here we show how the robust minimum path varies as the control parameter Γ changes.

Relative robustness

- Let $\boldsymbol{S}$ be the set of possible scenarios, or realisations, of the uncertain parameter α.
- Let us assume that α appears in the definition of the objective function $\boldsymbol{c}(\boldsymbol{x})$.
- At each solution $\boldsymbol{x} \in \boldsymbol{F}$ there are associated several realisations of the objective function, depending on the realisation of the parameter $\alpha \in \boldsymbol{S}$.
- We denote by $\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x})$ the cost of the solution $\boldsymbol{x}$ in case the uncertain α parameter is realised as indicated by the scenario $\boldsymbol{s}, \boldsymbol{x} \in \boldsymbol{F}$ and $\boldsymbol{s} \in \boldsymbol{S}$.

Relative robustness

- Absolute robustness: associate to each solution $\boldsymbol{x} \in \boldsymbol{F}$ the maximum possible cost depending on the realisation α and minimise this maximum value. An excessively conservative approach.
- It may be more reasonable to evaluate, for each scenario $\boldsymbol{s}$, the cost of the optimal solution if $\boldsymbol{s}$ is realised, $\boldsymbol{c}_{\boldsymbol{s}}^{*}$, and to associate to each solution $\boldsymbol{x} \in \boldsymbol{F}$ the maximum difference $(\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x}) - \boldsymbol{c}_{\boldsymbol{s}}^{*})$ as the variation of $\boldsymbol{s}$ in $\boldsymbol{S}$.
- The objective of the criterion of robustness in a relative sense is to determine a feasible solution which exhibits the least deviation from the optimality when all the possible realisations of the uncertain parameter α in $\boldsymbol{S}$ vary.

Minimax criterion with regret

Definition

The robustness criterion in relative sense, or minimax criterion with regret, consists in determining a solution $\boldsymbol{x}^* \in \boldsymbol{F}$ such that the maximum deviation of the cost associated with $\boldsymbol{x}^*$ from the optimal cost, due to the variation of α in $\boldsymbol{S}$, is the lowest possible one. Formally, a robust solution in relative terms $\boldsymbol{x}^*$ can be determined by solving the following optimisation problem

$$\min\{\max\{(\boldsymbol{c}^s(\boldsymbol{x}) - \boldsymbol{c}_s^*) : \boldsymbol{s} \in \boldsymbol{S}\} : \boldsymbol{x} \in \boldsymbol{F}\}. \quad (1)$$

where $\boldsymbol{c}_s^*$ denotes the cost of the optimal solution associated with scenario $\boldsymbol{s}$, assuming such optimal solution exists.

Problem (1) is therefore the robust version (in relative sense) of the *nominal* optimisation problem.

Absolute and relative robustness

- minimax criterion (absolute)

$$\min\{\max\{\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x}) : \boldsymbol{s} \in \boldsymbol{S}\} : \boldsymbol{x} \in \boldsymbol{F}\}.$$

- minimax criterion with regret (relative)

$$\min\{\max\{(\boldsymbol{c}^{\boldsymbol{s}}(\boldsymbol{x}) - \boldsymbol{c}^*_{\boldsymbol{s}}) : \boldsymbol{s} \in \boldsymbol{S}\} : \boldsymbol{x} \in \boldsymbol{F}\}.$$

Robust SPP with regret

$$\min \max\{\sum_{(i,j)\in A} (c_{ij}^s x_{ij} - c_s^*) : s \in S\}$$

$$\sum_{(j,v)\in BS(v)} x_{jv} + \sum_{(v,j)\in FS(v)} x_{vj} = b_v, \quad \forall v \in N$$

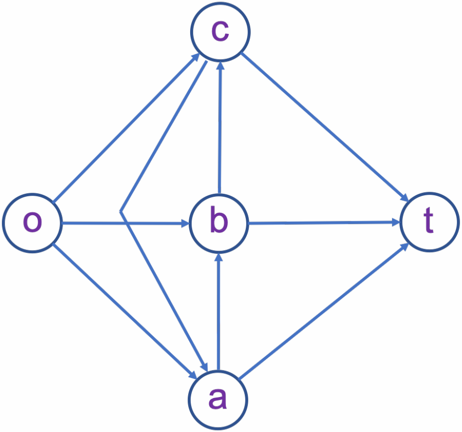
$$x_{ij} \in \{0,1\}, (i,j) \in A,$$

where $BS(v)$ is the set of arcs entering node v and $FS(v)$ is the set of arcs exiting node v. In addition, $b_v = -1$ if $v = o$, $b_v = 1$ if $v = t$, and $b_v = 0$ otherwise. c_{ij}^s is the cost of arc (i,j) according to scenario s.

Robust SPP with regret

- If interval representation of uncertainty, i.e., $c_{ij} \in [c_{ij}^{-}, c_{ij}^{+}]$,
- for each feasible solution, i.e., for each oriented path p from o to t, it is simple to characterise the scenario that maximises the deviation of the cost of path p from the cost of the shortest path. i.e., $\arg\max\{\sum_{(i,j)\in A}(c_{ij}^{s}x_{ij} - c_{s}^{*}) : s \in S\}$
- set the cost of each arc $(i,j) \in p$ to its upper bound, i.e., c_{ij}^{+}
- set the cost of each arc $(i,j) \notin p$ to its lower bound, i.e., c_{ij}^{-}

Telecommunication network



1. (o,a,t)
2. (o,a,b,t)
3. (o,a,b,c,t)
4. (o,b,t)
5. (o,b,c,t)
6. (o,b,c,a,t)
7. (o,c,t)
8. (o,c,a,t)
9. (o,c,a,b,t)

Interval representation – example

arc	c_{ij}^{-}	c_{ij}^{+}
(o, c)	3	5
(o, b)	1	10
(o, a)	1	10
(c, t)	1	10
(b, t)	1	10
(a, t)	3	7
(a, b)	1	10
(b, c)	1	10
(c, a)	5	7

Robust SPP with regret

path max	p1	p2	p3	p4	p5	p6	p7	p8	p9	c_s^*	max cost
p1	17	13	13	4	4	4	4	8	4	2	15
p2	13	30	22	11	12	19	4	11	28	3	27
p3	13	21	40	2	21	19	13	11	19	2	38
p4	4	12	4	20	12	19	4	11	19	4	16
p5	4	3	22	11	30	28	13	11	10	3	27
p6	8	3	13	11	21	34	4	17	12	3	31
p7	4	3	13	2	12	12	15	13	12	2	13
p8	8	3	4	2	3	16	15	19	14	2	17
p9	4	21	13	11	13	10	6	15	32	3	29