

Lemma 1. Let $\rho \in]0, 1[$. Let $A \subseteq]a, b[$ such that for all $]\alpha, \beta[\subseteq]a, b[$,

$$\lambda^*(A \cap]\alpha, \beta[) \leq \rho(\beta - \alpha).$$

Then $\lambda^*(A) = 0$. Consequently A is measurable and $\lambda(A) = 0$.

Proof. We recall that

$$\lambda^*(A) = \inf \left\{ \sum_{n=1}^{+\infty} (\beta_n - \alpha_n) \mid A \subseteq \bigcup_{n=1}^{+\infty}]\alpha_n, \beta_n[\right\}.$$

Since $A \subseteq]a, b[$, in the previous definition it is not restrictive to suppose that, for all n , $]\alpha_n, \beta_n[\subseteq]a, b[$. Let $\varepsilon > 0$. From the definition of outer measure we have that there exists a sequence $(]\alpha_n, \beta_n[)_n$ of subintervals of $]a, b[$, such that

$$A \subseteq \bigcup_{n=1}^{+\infty}]\alpha_n, \beta_n[\quad \text{and} \quad \sum_{n=1}^{+\infty} (\beta_n - \alpha_n) < \lambda^*(A) + \varepsilon.$$

Recalling now that the outer measure is countably subadditive, we have

$$\lambda^*(A) \leq \sum_{n=1}^{+\infty} \lambda^*(A \cap]\alpha_n, \beta_n[) \leq \sum_{n=1}^{+\infty} \rho(\beta_n - \alpha_n) \leq \rho(\lambda^*(A) + \varepsilon).$$

Since this is true for all $\lambda^*(A) + \varepsilon$, we deduce that

$$\lambda^*(A) \leq \rho \lambda^*(A)$$

and the conclusion follows. □