Exercises 5

key words: Distributions. Support of a distribution.

1) (Principal Value of $\frac{1}{x}$) Show that $PV_{\frac{1}{x}} : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$,

$$PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx,$$

is a distribution. Determine the support of $PV_{\frac{1}{x}}$.

2) (Finite Part of $\frac{1}{x^2}$) Show that $FP_{\frac{1}{x^2}} : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$,

$$FP_{\frac{1}{x^2}}(\varphi) = \lim_{\varepsilon \to 0^+} \Big(\int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x^2} \, dx - 2 \, \frac{\varphi(0)}{\varepsilon} \Big),$$

is a distribution. Determine the support of $FP_{\frac{1}{x^2}}$.

3) Show that $T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$,

$$T(\varphi) = \lim_{m \to +\infty} \Big(\sum_{j=1}^{m} \varphi(\frac{1}{j}) - m\varphi(0) - (\log m)\varphi'(0) \Big),$$

is a distribution. Determine the support of T.

4) Let u be a continuous function in $\mathbb{R}^n \setminus \{0\}$. Suppose that

$$u(tx) = t^{-n} u(x), \quad \text{for all } t > 0, \ x \in \mathbb{R}^n \setminus \{0\}.$$

Show that

$$U(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} u(x)\varphi(x) \, dx,$$

exists for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, if and only if

$$\int_{|\omega|=1} u(\omega) \, d\sigma = 0.$$

In such a case prove that U is a distribution.

5) Let f be an integrable function on all the compact sets of $\mathbb{R}^n \setminus \{0\}$. Suppose that there exist $C > 0$, $m \in \mathbb{N} \setminus \{0\}$ such that, for all $|x| \leq 1$,

$$|f(x)| \leq \frac{C}{|x|^m}.$$

Prove that there exists $T \in \mathcal{D}'(\mathbb{R}^n)$ such that, for all $\varphi \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\}$,

$$T(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x) \, dx.$$

6) Let T be a distribution on $\mathbb{R}^n$ and f be a function in $\mathcal{D}(\mathbb{R}^n)$ such that $f = 0$ on the support of T. Can we deduce that $T(f) = 0$?