Exercises 6

key words: Derivative in the sense of distributions. Multiplication of a distribution with a C^∞ function.

1) Let $T \in \mathcal{D}(\mathbb{R}^2)$,

$$T(\varphi) = \int_{\mathbb{R}} \varphi(x, -x)\, dx.$$

i) Prove that T is a distribution and find supp (T).

ii) Find $(\partial_{x_1} - \partial_{x_2})T$.

2) Let $\varphi \in \mathcal{D}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$.
Is there any relation between the statements "$T(\varphi) = 0$" and "$\varphi T = 0$"?

3) Consider the differential operator

$$P = \frac{d^2}{d^2x} + a\frac{d}{dx} + b, \qquad a,\ b \in \mathbb{R}.$$

Let $f,\, g \in C^2(\mathbb{R})$ such that $Pf = Pg = 0$, $f(0) = g(0) = 0$ and $f'(0) - g'(0) = 1$. Define $h(x) = (1 - H(x))f(x) + H(x)g(x)$, where H is the Heaviside function, and let $T \in \mathcal{D}'(\mathbb{R})$,

$$T(\varphi) = -\int_{\mathbb{R}} h(x)\varphi(x)\, dx.$$

Find PT.

4) Let H be the Heaviside function and $E:\ \mathbb{R}^2 \to \mathbb{R}$,

$$E(x, t) = \frac{1}{2} H(t - |x|).$$

Prove that E define a distribution and find

$$\Box E = (\partial_t^2 - \partial_x^2)E.$$

5) Let H be the Heaviside function and $E:\ \mathbb{R}^2 \to \mathbb{R}$,

$$E(x, t) = \frac{H(t)}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

Prove that E define a distribution and find

$$(\partial_t - \partial_x^2)E.$$

6) Consider $E_n:\ \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$E_n(x) = \begin{cases} \log|x| & \text{if } n = 2, \\ |x|^{2-n} & \text{if } n \geq 3. \end{cases}$$

Prove that E_n define a distribution and find

$$\Delta E_n = (\sum_{j=1}^{n} \partial_{x_j}^2) E_n.$$