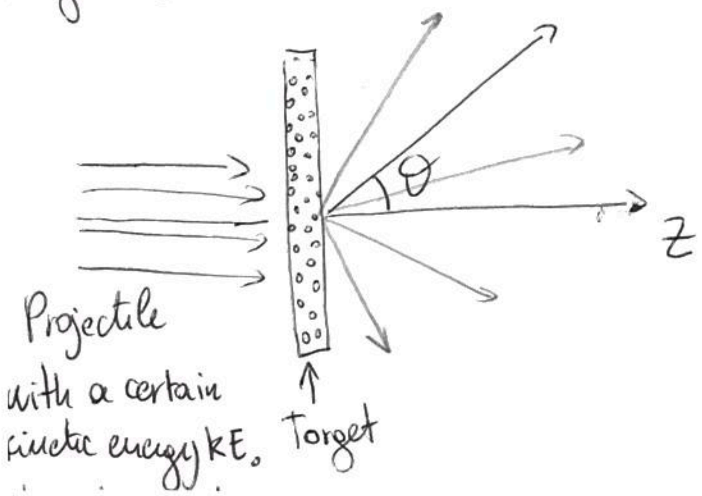
NUCLEON - NUCLEON SCATTERING

The study of the deuteron gives us a number of clues about NN interaction, but the total amount of information is limited. Because there are no exited states, we can study only the $l=0$, parallel spin $d=2$ fm system.

To study N-N interaction, we can perform nucleon-nucleon scattering experiments in which an incident beam of nucleons is scattered from a target of nucleons.

In this case the observed scattering particles will be used to study NN interaction. In order to make the the things as clear as possible, a target of H is used.



n-p scattering

n = projetile

p = H = target

We can measure:

- # of ptc at a particular θ
- # " vs time
- # " vs distance from the scattering point
- # " vs θ
- combination of the measurement

$$dN \propto t,\ d\Omega,\ I,\ N_t$$

dN: # of collected ptc

$\propto$: This gives the characteristics of the interaction

t: time

$d\Omega$: Solid angle

I: Intensity of the beam

N_t: Number of target projetiles

$$\boxed{I = n \cdot A \cdot t}$$

t: Unit time

$$dN = \sigma(\theta,\varphi)\, t\, d\Omega\, I\, N_t = \sigma(\theta,\varphi)\, t\, d\Omega\, I\, n_a A$$

Thikness of target

density of target

Counts how many ptc goes into the target

$$dN = \sigma(\theta,\varphi)\, d\Omega\, n_a N_p$$

$$\boxed{\sigma(\theta,\varphi) = \frac{dN}{d\Omega\, n_a N_p}} \equiv \text{DIFFERENTIAL SCATTERING CROSS-SECTION}$$

We would like to relate the scattering data to NN potential. To do that we will use partial wave expansion rather than relating directly V to data

Our primary interest is in $\frac{dN}{d\Omega}$, which is a function of the bombarding energy as well as the scattering angle.

In the non-relativistic limit, the scattering of one particle off another is described by the Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + (V-E)\psi = 0$$

where μ is the reduced mass of the system.

We will now assume $V=0$: This is true if we study the interaction far away from the region where the scattering takes place.

In the asymptotic region $(r\to\infty)$ the ψ has the form

$$\psi(r,\theta,\phi) \xrightarrow[r\to\infty]{} \underbrace{e^{ikz}}_{\substack{\text{Incident}\\ \text{w.F.}\to\text{unaffected}}} + \underbrace{f(\theta,\phi)\frac{e^{ikr}}{r}}_{\text{Scattered }\psi}$$

where e^{ikz} is the incident plane wave and part of the beam unaffected by the reaction.

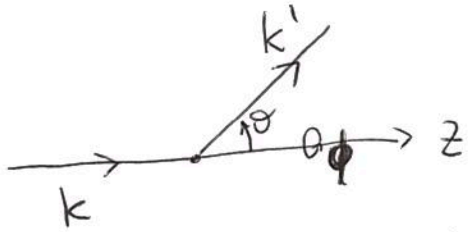
The scattered w.f. is given by a spherical function $\frac{e^{ikr}}{r}$, radiating outward the scattering center. $f(\theta,\phi)$ is the probability of scattering in (θ,ϕ) (scattering amplitude

In the case of elastic scattering, the wave number $k = \sqrt{\frac{2mE}{\hbar^2}}$ has the same value before and after the scatterig. The differential cross-section can be calculated as:

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = |f(\theta,\phi)|^2$$

I want to evaluate ψ

The geometry is such it is convenient to place the origin of the coordinate system at the center of the scattering region



If beam and target are unpolarized the cross section in independent from ϕ.

PARTIAL WAVE-ANALYSIS

For a central potential, the relative angular momentum l beetween the two scattering nucleons is a conserved quantity. Under such conditions, the w.f. can be expanded using partial waves, each with a definite l-value

$$\psi(r,\theta) = \sum_{l=0}^{\infty} a_l \, Y_l^{m}(\theta)\, R_l(k,r)$$

coeff expansion

depends on Energy

Here only spherical harmonics with $m=0$ appears, since the beam and target are unpolarized, and ψ do not depends on ϕ. a_l are the expansion coefficient.

For a free particle, $V=0$, and the radial w.f. become:

$$R_l(k,r) \xrightarrow{\text{FREE}} J_l(kr) \xrightarrow[r\to\infty]{\text{FREE}} \frac{1}{kr}\sin\left(kr - \frac{1}{2}l\pi\right)$$

Where $k = \frac{\sqrt{2\mu E}}{\hbar}$; $J_l(kr)$ are spherical Bessel functions of order l.

Only elastic scattering is allowed by the potential; the probability current density in each partial wave is conserved. =>

The only effect that the potential can have on the w.f. is a change in the phase angle.

$$R_\ell(k,r) \xrightarrow[r\to\infty]{\text{scattering}} \frac{1}{kr}\left(\sin\left(kr - \frac{1}{2}\ell\pi + \delta_\ell\right)\right)$$

δ_ℓ is the PHASE SHIFT in the l-th partial wave analysis.

scattering amplitude

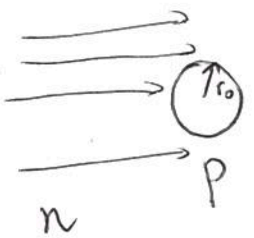
or the $f(\theta)$ can be expressed as: $\boxed{f(\theta) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\, e^{i\delta_\ell}\sin\delta_\ell\, Y_\ell^0(\theta)}$

Since δ_ℓ is related to the "interaction" it must be connected to the scattering cross-section

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{4\pi}{k^2}\left|\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\; e^{i\delta_\ell}\sin\delta_\ell\; Y_\ell^0(\theta)\right|^2$$

$$\boxed{\sigma = \int\frac{d\sigma}{d\Omega}\,d\Omega = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell(k)}$$

Decomposition into partial waves is a useful way to analize scattering results for a given "bombarding" energy. Only few low-order partial waves can contribute to the scattering at low energy.
For realistic nuclear potentials, the orbital angular momentum is not conserved and a PW expansion remains valid only for a limited number of l-values.
In a ideal world, once you have a central potential, you could do the math (in Q.M); get the δ_ℓ; obtain σ and compare it with data.
Let's make some considerations. If energies are small l is small =>
as a first approximation let's consider only $\ell = 0$ → Semi-classical limit calculation.



Scattering occurs ONLY if the neutron goes in the r_0^{Prot} region.

Semi-classical calculation: for $\boxed{E = 1\ \text{MeV}}$ of the angular momentum l can be written as

$l = m v r = \sqrt{2 m E}\, r$ and it's maximum at $r = 0$

$(l\hbar = 2 m v \equiv p$ must be quantized$)$ and E = Kinetic Energy

$$l = m v r_0 = \sqrt{2 m E}\, r_0 = \left(\sqrt{2 \cdot \underset{m_p}{1000} \frac{\text{MeV}}{c^2} \cdot 1\ \text{MeV}}\right) \times 2\,\text{fm} =$$

$l = m v r$

$m v = \sqrt{2 m E}$

$$= \sqrt{2000}\ \frac{\text{MeV}}{c} \cdot 2\ \text{fm} = l$$

Let's take $l = 1$, then $\underbrace{l^2 = l(l+1)\hbar^2}_{\text{Total angular momentum } J^2 = l(l+1)\hbar^2} \Rightarrow l^2 = 1(1+1)\hbar^2 \Rightarrow \frac{l^2}{\hbar^2} = 2$

In our case we have

$$l = 2 \cdot \sqrt{2000}\ \frac{\text{MeV}}{c} \cdot \text{fm} \Rightarrow \frac{l^2}{\hbar^2} = \frac{4 \cdot 2000}{\hbar^2} \cdot \frac{\text{MeV}^2}{c^2} \cdot \text{fm}^2 = 8000 \cdot \frac{1}{\hbar^2 c^2}\ \text{MeV}^2 \cdot \text{fm}$$

$$\hbar c \simeq 200\ \text{MeV} \cdot \text{fm} \Rightarrow (\hbar c)^2 = 4 \cdot 10^4\ \text{MeV} \cdot \text{fm}$$

$$\frac{l^2}{\hbar^2} = \frac{8 \cdot 10^3}{4 \cdot 10^4} = 0{,}2$$ instead of 2! (which is the expected value)

$\Rightarrow$ With $E_K = 1\ \text{MeV}$ only $l = 0$ is allowed!!

On the other hand, if $E_K \simeq 10\ \text{MeV} \rightarrow$ ~~also~~ $l = 1$ should be included even if ~~the~~ use this semi-classical trick.

For LOW-ENERGY scattering $(E < 1\ \text{MeV})$

$$\boxed{\sigma(\theta) = \frac{4\pi}{k^2} \sin^2 \delta_0}$$

$\Rightarrow \sigma(\theta) \propto \frac{1}{E_K}$!(only)

And this can be checked with experiment data.

To solve NN scattering problem using QM we will assume that the interaction can be represented using a square well potential, as for deuteron. The difference is that here $E>0$.

What we assume is:

$V_0 = 36$ MeV

$r_0 = 2,1$ fm

$$\left.\begin{array}{ll} V=-V_0 & r<r_0 \\ V=0 & r>r_0 \end{array}\right\}$$

Let's use $l=0 \Rightarrow \quad H\psi = E\psi \qquad \psi = \phi(r)\, Y_l^m(\theta,\varphi) = \frac{u(r)}{r} Y_l$

FOR A CENTRAL POTENTIAL

$l=0$
$\Rightarrow$ cons

The Shrodinger equation become:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr}\right]u = Eu \qquad l=0$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)\,u = Eu$$

For $r<r_0$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0 u = Eu$$

$$\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_0 u = -Eu \Rightarrow \frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = -\underbrace{(V_0+E)}_{>0 \Rightarrow \text{Scatteri}}u$$

$$\frac{d^2u}{dr^2} = -k^2u \qquad k = \sqrt{\frac{2m(V_0+E)}{\hbar^2}} \qquad \Rightarrow \text{Harmatic oscillator}$$

$u = A \sin kr + B \cos kr$

Since at $r=0$ $u(0)=0$
$\rightarrow$ from the fact that $\frac{u(r)}{r}$ must be finite
$\Rightarrow B=0$

$\boxed{u_I = A \sin kr}$

For $r > r_0$ $V=0$

$$\frac{d^2 u}{dr^2} = -k_2^2 u \qquad k_2 = \sqrt{\frac{2ME}{\hbar^2}}$$

$u = C' \sin kr + D' \cos kr$ which can be rewritten as

$\boxed{u_{II} = C \sin(k_2 r + \delta)}$ where $C' = C\cos\delta$ and $D' = C \sin\delta$

The boundary condition $u_I(r_0) = u_{II}(r_0)$ + $\frac{du}{dr}$ continuos

$$A \sin kr_0 = C \sin(k_2 r_0 + \delta)$$
$$A k \cos kr_0 = C k_2 \cos(k_2 r_0 + \delta)$$

$u_I(r_0) = u_2(r_0)$

$\left.\frac{du_I}{dr_0}\right|_{r_0} = \left.\frac{du_{II}}{dr}\right|_{r_0}$

Diviving the two exprension one gets

$$\boxed{k \cot kr_0 = k_2 \cot(k_2 r_0 + \delta)}$$

Here, everything is known a part from δ.

In case of NO POTENTIAL $\delta = \delta_e$ and δ is the PHASESHIFT!

FOR $\boxed{\text{ALL}}$ radii

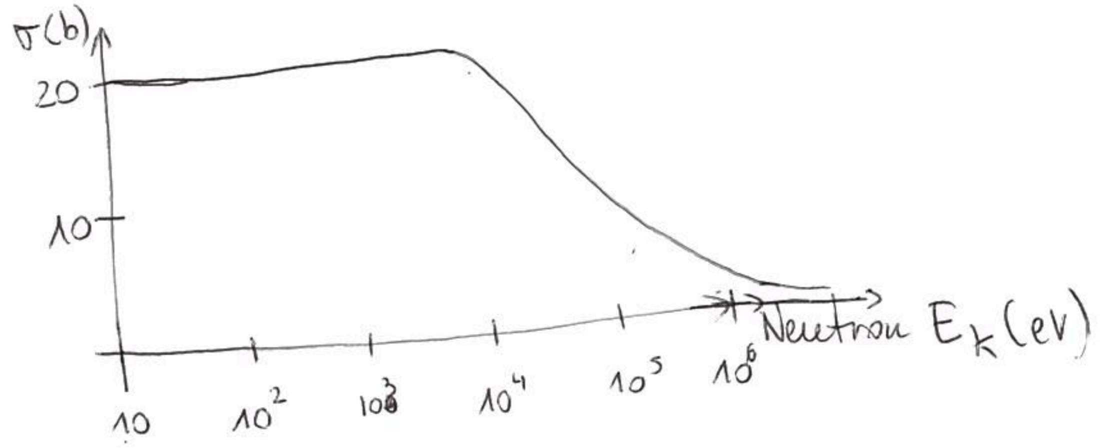
Now we need δ to get σ.

Since $$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0$$

If we do the calculation for square well potential we can caulculate the

$\sigma_{calc} = 5\,b$ Even if the potential is made more realistic, the σ_{calc} is always $\simeq$ few barn

~~Whereas~~ In measurements



Why this happens? Because we did a gross approximation (mistake)
Only in the bound state the spin of deuteron is deffined and $S=1$
But in scattering experiment there is no bound state hence we have to consider also $S=0$ state. In fact neither the p and the n are polarized in initial state

$S=1$ has 3 state ($\Rightarrow$ triplet) $M_S = -1, 0, 1$ $\Rightarrow |\uparrow\uparrow\rangle$; $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$; $|\downarrow\downarrow\rangle$

$S=0$ " 1 state ($\Rightarrow$ Singlet) $M_S = 0$ $\Rightarrow \frac{1}{\sqrt{2}}\left| |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right|$

$$\sigma_{OBS} = \frac{3}{4}\sigma_{S=1} + \frac{1}{4}\sigma_{S=0}$$

Since $\sigma_{obs} = 20\,b$ $\quad \sigma_{S=1} \simeq 5\,b$ $\quad 20\,b = \frac{3}{4}\,5\,b + \frac{1}{4}\sigma_{S=0}$

$\Rightarrow \sigma_{S=0} \simeq 65\,b$ $\quad \Rightarrow$ NUCLEAR FORCE IS SPIN DEPENDENT!

What happens for large Energy? I.e. why the σ decreases?

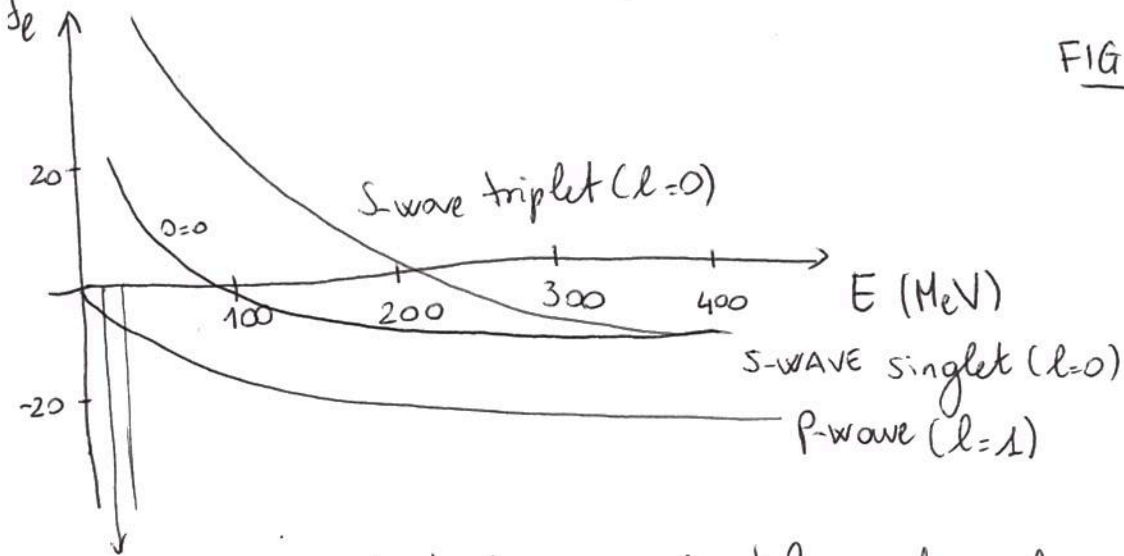


FIG 4.12. KRANE

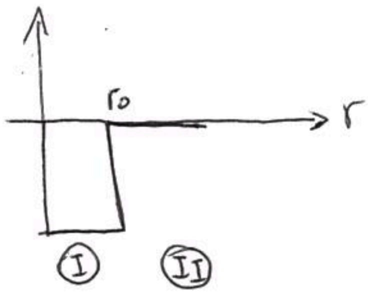
To separate ≠ components one have to measure $\frac{dN}{dt}$ in ≠ θ ranges: $\ell=0$ is isotropic while ≠ ℓ have preferred directions!

$E \lesssim 1\,MeV$ p contribute is negligible and only s-wave matters.

Up to ≃ 300 MeV S is positive then it changes ~~and~~ sign. The effect of S-wave gets negative! ⇒ The nuclear charge become repulsive at short distance

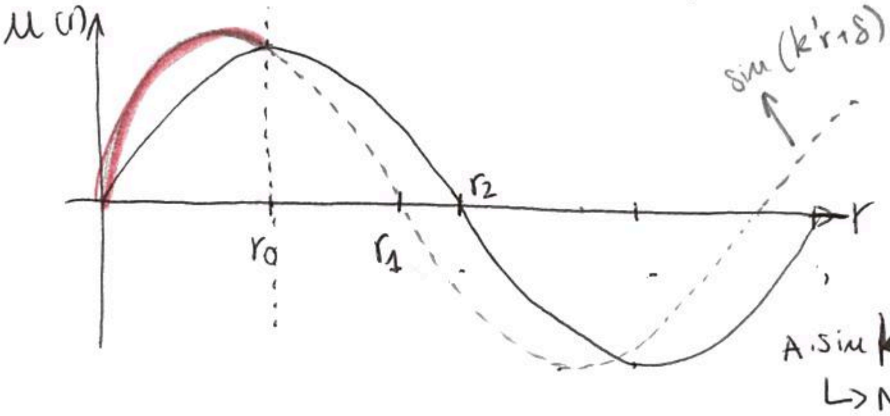
What is the meaning of the fact that δ in ~~[illegible]~~ changes sign? → I.e. why it becomes repulsive at short range.

Let's consider deuteron again (analogy). The solution for square well potential gives us:



$$u_I = A\sin kr \qquad k = \sqrt{\frac{2m(V_0+E)}{\hbar^2}} \quad \leftarrow \text{SHRINKING}$$

$$u_{II} = C'\sin(k'r+\delta) \qquad k' = \sqrt{\frac{2mE}{\hbar^2}} \quad \leftarrow \text{SHIFTING}$$



A sin k'r
↳ NO POTENTIAL

$k > k'$ ⇒ periodicity become smaller and changes of $\frac{2\pi}{k}$ → k large ⇒ smaller period

outside the periodicity ~~[illegible]~~ have to be the same, $\boxed{\text{BUT}}$ the w.f. has to move! To maintain continuity.

When the function is 0? $k'r_2 = \pi$ $(\sin k'r)$
$k'r_1 + \delta = \pi$ $(\sin(k'r+\delta))$ } We subtract and we get