(B)

$$J_+ \left| m_\ell = \ell-1, m_s = \tfrac{1}{2} \right\rangle = (L_+ + S_+) \left| m_\ell = \ell-1, m_s = \tfrac{1}{2} \right\rangle$$

$$= L_+ \left| m_\ell = \ell-1, m_s = \tfrac{1}{2} \right\rangle + S_+ \left| m_\ell = \ell-1, m_s = \tfrac{1}{2} \right\rangle$$

$$= \sqrt{(\ell-(\ell-1))(\ell+(\ell-1)+1)} \left| m_\ell = 1, m_s = \tfrac{1}{2} \right\rangle$$

$\Rightarrow$

$$0 = a \left| m_\ell = \ell, m_s = +\tfrac{1}{2} \right\rangle + b\sqrt{2\ell} \left| m_\ell = \ell, m_s = \tfrac{1}{2} \right\rangle$$

$$\Rightarrow \quad a + b\sqrt{2\ell} = 0 \quad \xRightarrow{\text{2 variables}} \text{2 equation} \quad \begin{cases} |a|^2 + |b|^2 = 1 \\ a = -b\sqrt{2\ell} \end{cases}$$

$$\begin{cases} |b|^2 = \dfrac{1}{2\ell+1} \\ |a|^2 = |b|^2 \cdot 2\ell = \dfrac{2\ell}{2\ell+1} \end{cases}$$

<u>So summarizing:</u>

In shell model we can use EXTREME SINGLE PARTICLE MODEL to understand MAGNETIC dipole moment of a nucleus of total angular momentum J and parity π.
For even-even nuclei it is always 0, because all n e p are paired up and the total mometum $\vec{J}$ in the last shell is always 0 $\Rightarrow \langle \mu_z \rangle = 0$
For even-odd nuclei we can <u>evaluate</u> how the $\langle \mu_z \rangle$ should behave.
To do so, we start from 2 cases (A) $J = \ell + \frac{1}{2}$ and B) $J = \ell - \frac{1}{2}$)
Case A) is "easy", only one combination of m_ℓ an m_s is "allowed" and the

$$\langle \mu_z \rangle = \left[g_\ell \ell + g_s \tfrac{1}{2} \right] \mu_N = \left[\ell + \frac{5,5857}{2} \right] \mu_N = \left[J - \tfrac{1}{2} + 2,7928 \right]$$

of the last nucleons

a) In case of p
$g_\ell = 1 \quad g_s = 5,5857$

$$= \left[J + 2,2928 \right] \mu_N$$

$$\langle \mu_z \rangle_n = -1{,}9130\ \mu_N \qquad g_\ell = 0$$
$$g_s = -3{,}8260$$

Case b) More complicated: There are 2 states that can lead with $J = \ell - \frac{1}{2}$

$\psi_1 : m_\ell = \ell \quad m_s = -\frac{1}{2}$

$\psi_2 : m_\ell = \ell - 1 \quad m_s = \frac{1}{2}$

$$\psi = a\psi_1 + b\psi_2$$

• NORMALIZATION +
• RAISING OPERATOR
to get a and b

$$|a|^2 = \frac{2\ell}{2\ell+1}, \quad |b|^2 = \frac{1}{2\ell+1}$$

$$\langle \psi_1 | \mu_z | \psi_1 \rangle = \left[g_\ell \ell + g_s\left(-\frac{1}{2}\right) \right] \mu_N$$ pure ψ_1

$$\langle \psi_2 | \mu_z | \psi_2 \rangle = \left[g_\ell(\ell - 1) + g_s\left(\frac{1}{2}\right) \right] \mu_N$$ pure ψ_2

$$\langle \psi | \mu_z | \psi \rangle = |a|^2 \langle \psi_1 | \mu_z | \psi_1 \rangle + |b|^2 \langle \psi_2 | \mu_z | \psi_2 \rangle$$

$$= \frac{2\ell}{2\ell+1} \left[g_\ell \ell + g_s\left(-\frac{1}{2}\right) \right] \mu_N + \frac{1}{2\ell+1} \left[g_\ell (\ell - 1) + g_s\left(\frac{1}{2}\right) \right] \mu_N$$

$J = \ell - \frac{1}{2} \quad \ell = J + \frac{1}{2}$

$$= \frac{\mu_N}{2\ell+1} \left[g_\ell \left(2\ell^2 + \ell - 1 \right) + g_s \left(-\ell + \frac{1}{2} \right) \right]$$

$$= \frac{\mu_N}{2(J+1)} \left[g_\ell \left(2\left(J + \frac{1}{2}\right)^2 + J + \frac{1}{2} - 1 \right) + g_s \left(-J - \frac{1}{2} + \frac{1}{2} \right) \right]$$

$J^2 + J + \frac{1}{4}$

$2J^2 + 2J + \frac{1}{2} + J + \frac{1}{2} - 1$

$2J^2 + 2J$

$$= \frac{\mu_N}{2(J+1)} \left[g_\ell \left(2J^2 + 2J + \cancel{\frac{1}{2}} + J + \cancel{\frac{1}{2}} \cancel{-1} \right) + g_s(-J) \right]$$

$$= \frac{\mu_N}{2} \frac{J}{(J+1)} \left[g_\ell (2J + 3) - g_s \right]$$

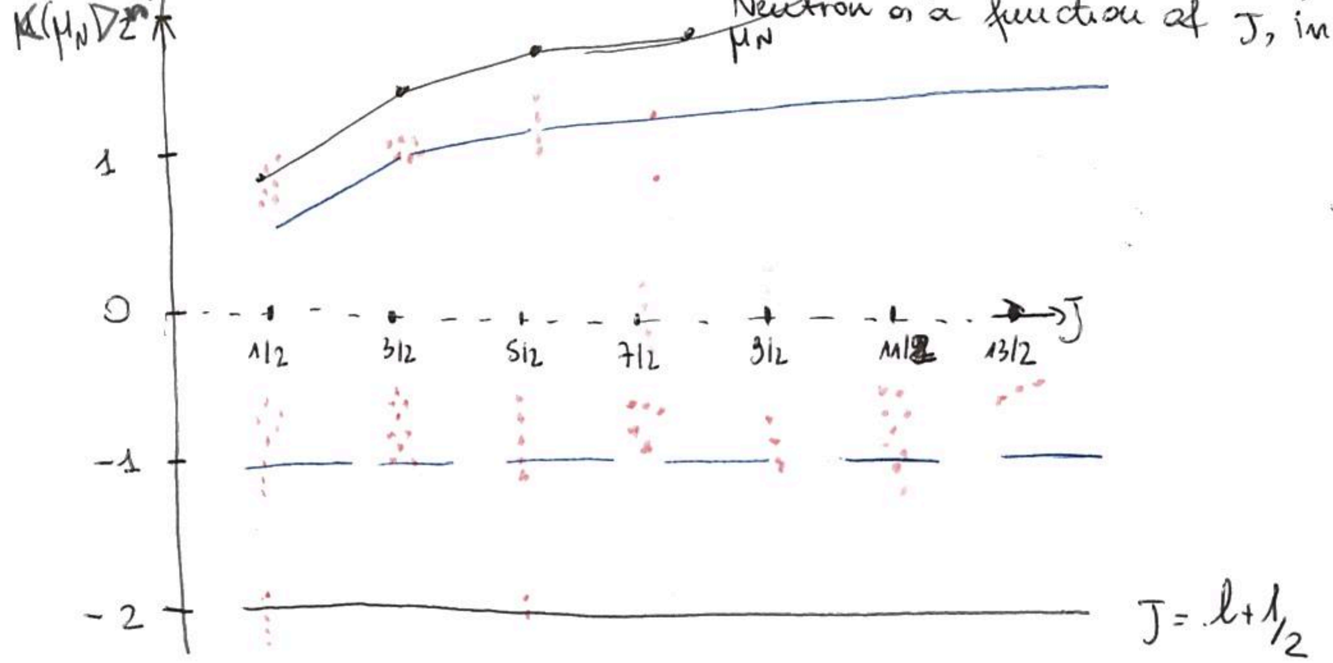
FOR PROTON (LAST NUCLEON): $\langle \mu_z \rangle = \frac{\mu_N J}{2(J+1)}\left[2J+3-5{,}5857\right] = \frac{\mu_N J}{2(J+1)}\left[2J-2{,}5857\right] \cdot$

$$= \frac{J}{(J+1)}\left[J-1{,}2928\right]\mu_N$$

FOR LAST NEUTRON: $\langle \mu_z \rangle = \frac{\mu_N J}{2(J+1)}\left[+3{,}8260\right] = \frac{J}{J+1}\left[1{,}9130\right]\mu_N$

There calculations were done in 1937 by Schmidts

~~Now~~ let's compare with data. Here we have $\langle \mu_z \rangle$ in case the last nucleons ~~of~~ the shell model is a Neutron as a function of J, in terms of μ_N



The black lines are called "Schmidts" lines and are the predictions from extreme single model.

DATA falls within the Schmidt lines but are generally smaller ~~out~~ in magnitude and have a considerable scatter.

One defect of the theory is that g_s for a nuclear in the nucleus is assumed to be ~~the~~ same as a g_s of free nucleon.

The effect of "mesons" surrounding the nucleus can lead without major concerns to a lowering of g_s for nucleons inside the nucleus. If $g_s^{NUCL} = 0{,}6\, g_s^{free}$ the blue lines are obtained.

The overall agreement gets better, but the scatter of the points suggests

that the shell model is oversimplified in the calculation. (i.e. spin parity not considered)
Nevertheless, the success indicates that the shell model works when we try to understand the structure of nuclei.

SHELL MODEL & ELECTRIC QUADRUPOLE MOMENT

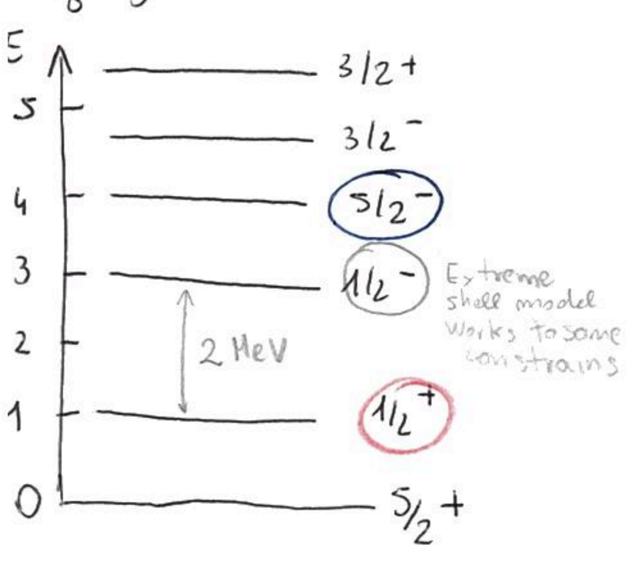
The electric quadrupole moments calculation in shell model is done by evaluating the electric quadrupole operator $(3z^2 - r^2)$ in a state in which the total angular momentum of the odd particle has its maximum projection along the z axis.

In a nucleus, if the charge is distributed spherically there is no electric quadrupole moment. It can be obtained from odd nuclei using the extreme single particle model. The "sign" will be ok but the magnitude would be 10 times smaller than the measurements.
$\Rightarrow$ Electric quadrupole moment is not properly working within the shell model.

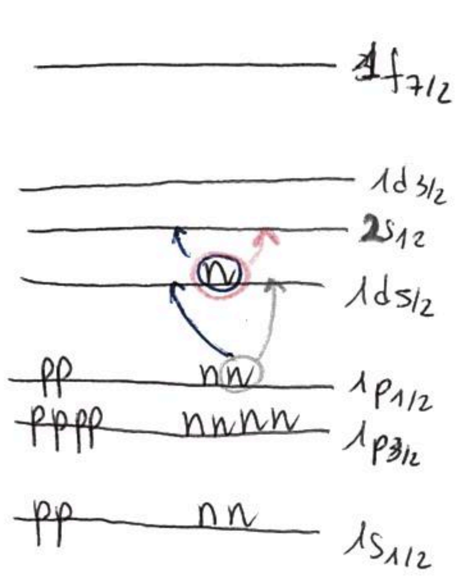
EXCITED NUCLEI, EVEN-Z, EVEN-EVEN NUCLEI AND COLLECTIVE STRUCTURES

Let's go back to oxigen and its excited states that we have already looked at few pages ago more in details

$^{17}_{8}O_9$



In extreme single particle model we can explain the ground state and the first 2 excited levels.



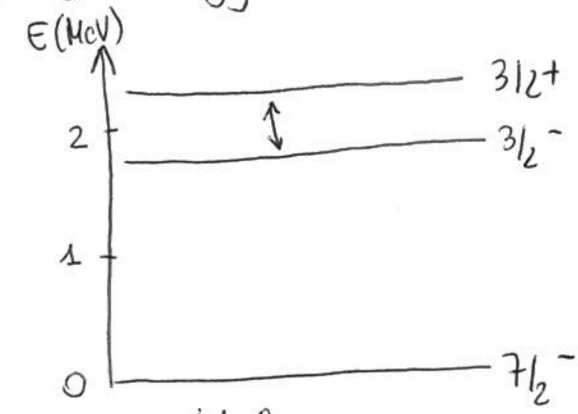
$\frac{5}{2}^-$: either we go "very high" in s (but this is very unlikely) or we look "here". How can we get $\frac{5}{2}^-$? We consider to have 3 unpaired n; one in $p_{1/2}$, 1 in $d_{5/2}$ and 1 in $s_{1/2}$ (52)

$$\Rightarrow \left.\begin{array}{l} J_1^\pi = \frac{1}{2}^- \\ J_2^\pi = \frac{5}{2}^+ \\ J_3^\pi = \frac{1}{2}^+ \end{array}\right\} \Rightarrow J^\pi \to \text{can be } \frac{5}{2}^-$$

$\Rightarrow$ To some more extend the extreme shell model still works

Now let's consider ${}^{41}_{20}Ca_{21}$ Calcium and ${}^{41}_{21}Sc_{20}$ Scandium

Their energy levels of excited states are

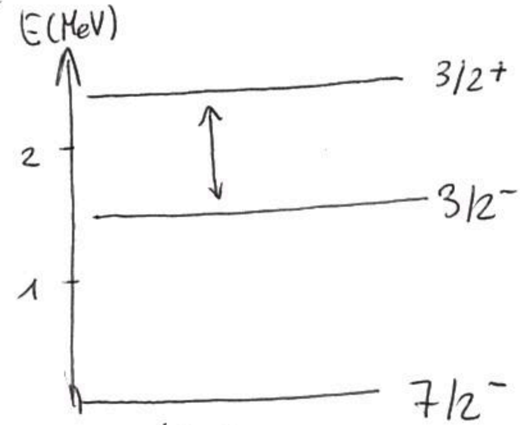


${}^{41}_{20}Ca_{21}$

$Z=20$ $d_{3/2}$ shell closer

$N=21$

last ptc : n



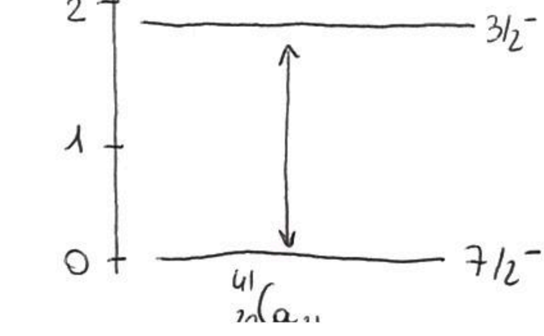
${}^{41}_{21}Sn_{20}$

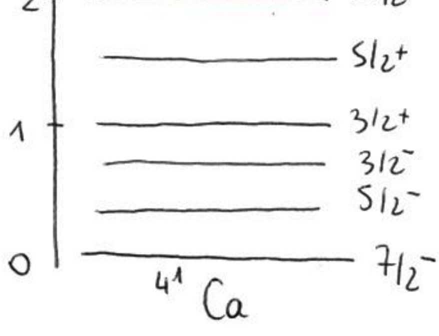
"

last ptc p

The energy "schemes" are very similar but the values of the energies are different because of COULOMB interaction!

Now, let's consider ${}^{41}_{20}Ca_{21}$ ${}^{43}_{20}Ca_{23}$





In extreme single particle model there should be no difference! But there is a huge difference, suggesting that the shell model is not working completly.

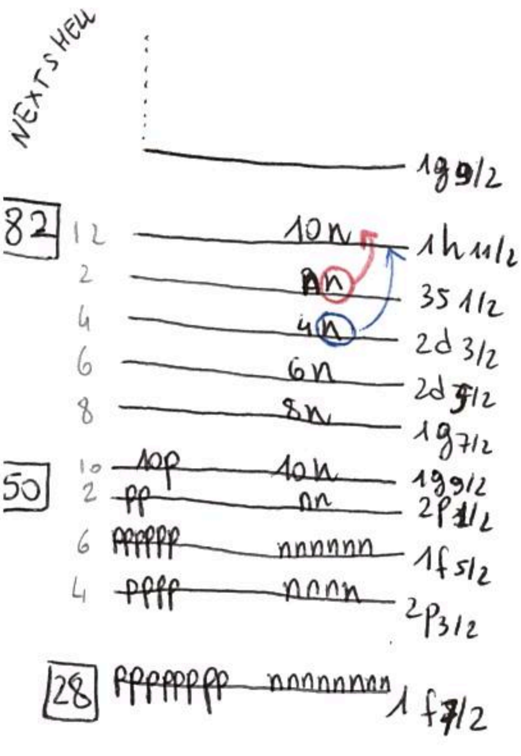
We can predicts some characteristics:

1) J of spin parity
2) Magnetic dipole
3) Justification of pairing

But not all ⇒ good base line, but not complete desciption, maybe we can "just" add something more.

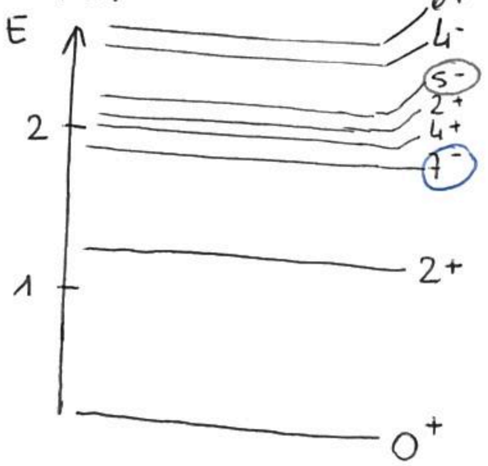
Before move, let's have a look to even-even nucleus. We said several times that even-even nuclei should all have the ground state in 0^+. They actually have it, but we can see what happens for exitided states. As an example let's start from $^{130}_{50}Sn_{80}$ Tin/Stagno

According to shell model p and n should be disposed like:



p Fill the shell up to the shell-closer $1g_{9/2}$, n fill the shell up to $1h_{11/2}$, leaving "2-empty" slots.

The excited state scheme of $^{130}_{50}Sn_{80}$ is:



form an excited state, we can breake one pair and excited next state. E.g. we can break $3s_{1/2}$ and we will ...

$\frac{\vec{1}}{2} + \frac{\vec{11}}{2} \Rightarrow 6^-, 5^-$ $\Rightarrow$ We can explain not the first excited state but the 5^{th}... (52)

We can try to breake $2d_{3/2}$ and we will have

$\frac{\vec{3}}{2} + \frac{\vec{11}}{2} \Rightarrow 7^-, 6^-, 5^-, 4^-$

And we could explain the $\boxed{3^{rd}}$ excitided state.

How would it be possible to get 2^+? ~~A~~ way would be to break one of the $h_{11/2}$ pairs and, keeping both members of the pair in the $h_{11/2}$ shell recouple them to a spin $\neq 0$. In this way, according to spin algebra the possibilities would be any ranging from 11 to 0, the parity of the state would be $\oplus$ since the ~~2~~ neutron would be in the same (odd) shell.

But before considering "all" pairs we have to take into account that the 2 pct would be identical.

To recall the algebra let's start from the core $J_1 = J_2 = 2$.

$\vec{J}_1 + \vec{J}_2 = 4, 3, 2, 1, 0$ in principle, but if we consider $m_J = m_{S_1} + m_{S_2}$

m_{S_2} \ m_{S_1}	2	1	0	-1	-2	
2	•4	×	×	×	×	
1	•3	•2	×	×	×	
0	•2	•1	•0			$\to J=0$
-1	•1	•0	•-1	•-2	×	$\to J=2$
-2	•0	•-1	•-2	•-3	•-4	$\to J=4$

× are repetition of •
($\Rightarrow$ IDENTICAL PARTICLES)
$\hookrightarrow$ It is related the pairing term!

How they sum up?

$J = 1, 3$ are not ALLOWED!

Similarly for $\frac{\vec{11}}{2} + \frac{\vec{11}}{2}$ only even states are allowed $\Rightarrow 10^+, 8^+, 6^+, 4^+, 2^+, 0^+$

$\Rightarrow$ The first excited state would be 2^+. The same consideration ~~wo~~
But the same consideration can be done by all the states occupied by 2 n...
Can we make some more considerations?

A major exception to this succesfull interpretation of 2^+ is that its energy is $\simeq 1{,}2$ MeV, while usually the energy of moving nucleons is $\simeq$ 2MeV. Of course we can say that shell model is just an approximate model that the a more complex schema should exists. For example we could write the ψ of (2^+) state as the linear combination of different states.

In principle this could work, but before going on, one should look also at other even-~~odd~~ nuclei. Of lots of nuclei each one has an "anomalous" 2^+ state at an energy which in $\simeq 0{,}5E$ needed to break a pair.

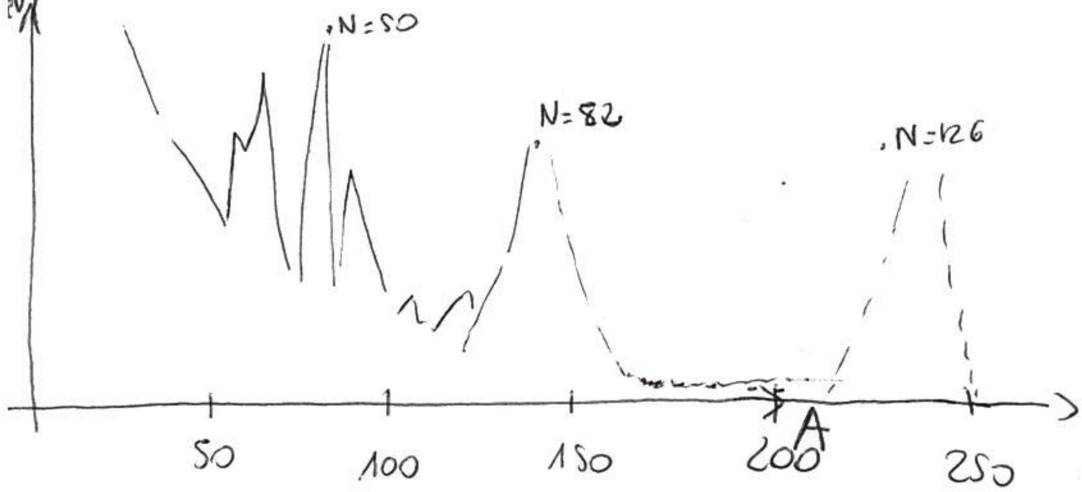
So this is not an exception, but rather a general property of even-even nuclei, and be identified as a collective property of nuclei.

So we can look for other collective properties and see what we can learn from them.

ollective properties ~~and their~~ origin lies in the nuclear collective notion of the entire nucleus.

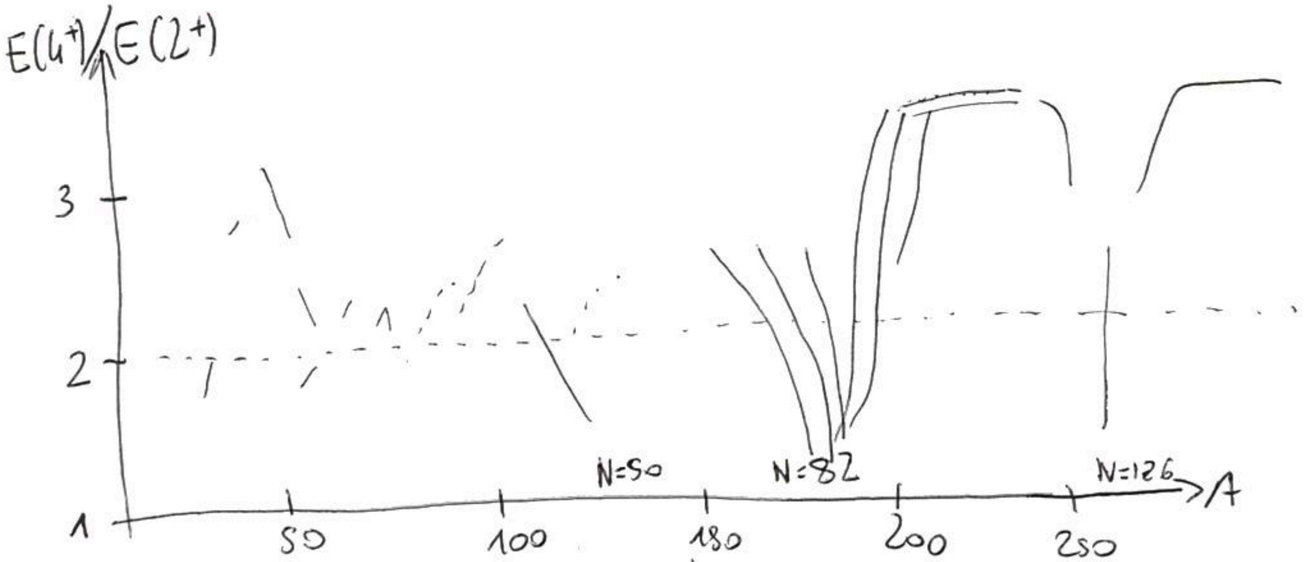
hese properties lies smothly and gradually with mass number and e almost independent from of the number and kind of valence ucleons outside filled subshell.

t's have a look at two other properties of even-even nuclei, amely the energy of lowest 2^+ excited state and the ratio etween the E of 4^+ and that of 2^+



The energy of the first 2^+ excited state decrease smothly as a function of A except where the shells close.

For A betwen 150 and 190 the velus are small and



Except for shell-closer nuclei the $E(4^+)/E(2^+)$ is roughly 2 for nuclei with $A<150$ and very constant at 3.3 for A between 150 and 190 and $A>230$

The nuclei with $A<150$ are generally treated in terms of a model based on vibrations about a spherical equilibrium, while nuclei with A between 150 and 190 show structures most characteristic of rotations of non spherical system.

Let's start from VIBRATIONS

NUCLEAR VIBRATIONS

Moving from spherical symmetry to "something else" means that the energy can be described in terms of distance from the center of the nucleus

- Spherical surface:
  r is the distance to any point on the surface
- Non-spherical surface:
  $r = r(\theta, \varphi) \Rightarrow r$ depends on the θ and φ

Any $R(\theta, \varphi)$ can be expressed using sherical harmonics expansion

$$R(\theta,\varphi) = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} a_{\lambda\mu} Y_{\lambda}^{\mu}(\theta,\varphi)$$

In steps of 1

$a_{\lambda\mu}$ is not a constant, in general, but can vary as a function of time t.

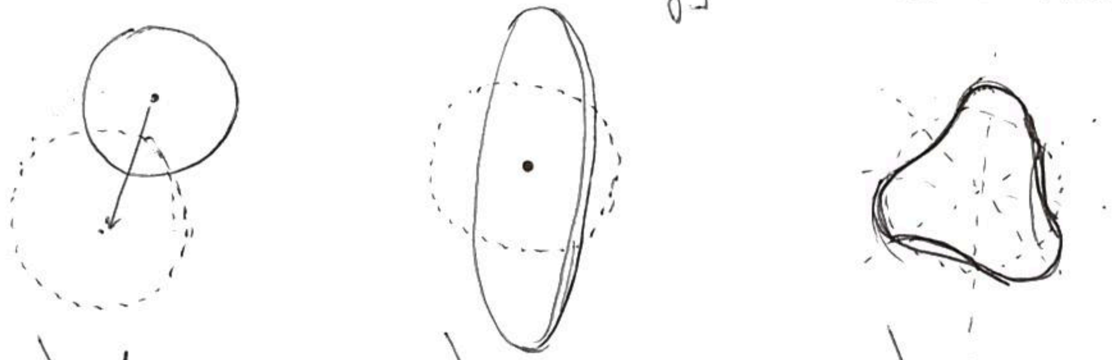
for $\lambda = 0$

$$Y_0^0(\theta, \varphi) = \sqrt{\frac{1}{4\pi}} \equiv R_{AVERAGE} \rightarrow$$ Oscillation around a spherical shape $\Leftrightarrow$ is the same thing that we have analyzed up to now.

$\lambda = 1$

$Y_1^M(\theta, \varphi) \Rightarrow$ Shift toward one direction $\Rightarrow$ Vibration of the whole nucleus $\Rightarrow$ No excitation energy

$\lambda = 2$ might interest us (Quadrupole mode vibration) and also $\lambda = 3$ octupole mode vibrations, because when the nuclei vibrate they produce a phonon. A phonon will have a quantizied energy that depends on the energy associated to $\lambda = 2, 3, \ldots$



$\lambda = 1$ DIPOLE $\qquad$ $\lambda = 2$ QUADRUPOLE $\qquad$ $\lambda = 3$ OCTUPOLE

When a nucleus vibrates in quadrupole state it emits a phonon with energy $E_p = \hbar\omega_2$ $\Rightarrow$ 1 phonon of $\lambda = 2$ type with parity +
he next level would create 2 phonons with the same energy

$\boxed{\lambda = 2}$ Angular momentum $k = 2$ $(k^2 = k(k+1))$ with parity +

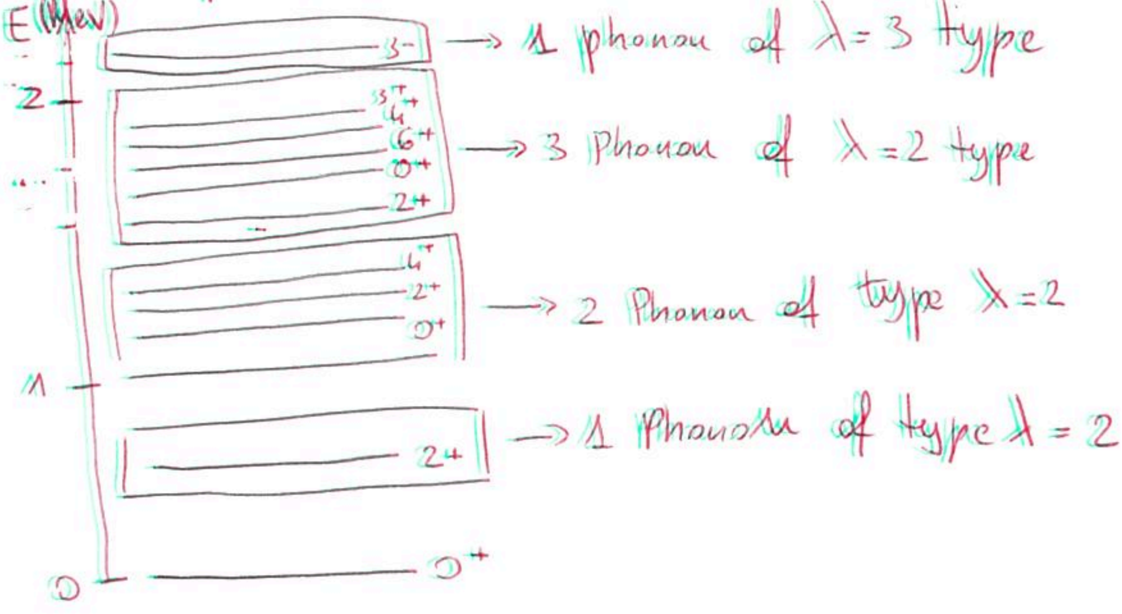
in $\lambda = 2$ we can have also 2 phonon emitted with energy $E = 2\hbar\omega_2$
he total angular momentum would be $\vec{p}_1 + \vec{p}_2 = \vec{2} + \vec{2} = 4, 3, 2, 1, 0$
In principle, but since they are identical ptcs:
$\Rightarrow \vec{2} + \vec{2} = 4^+, 2^+, 0^+$ with π ⊕

milarly if 3 phonon are emitted, $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{2} + \vec{2} + \vec{2} = 6^+, 4^+, 2^+, 0^+$.

For $\lambda = 3$ the emitted energy would be $\hbar\omega_3$, where $\omega_2 \neq \omega_3$
angular momentum would be 3 and parity negative.

The energy of $\lambda = 3 \simeq 2E\ \lambda = 2$ $\Rightarrow$ and the vibration energy must be added to the shell model.

For example for Tellerium 120



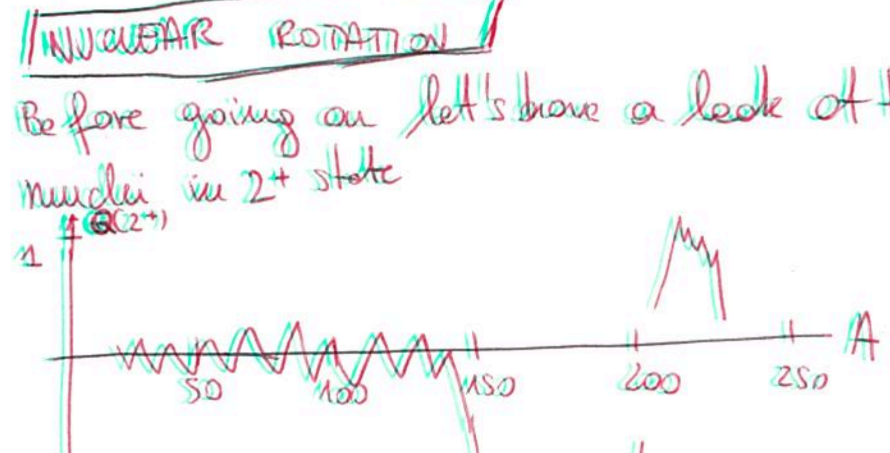
^{120}Te

If we go back to the previous figure we can understand what happens for $A < 150$ nuclei: Smooth decrease of $E(2^+)$ with increasing A and $E(2^+)/E(4^+)$ rather stable around 2.

To understand $A > 150$ we have to use another fenomenon, namely the

NUCLEAR ROTATION

Before going on let's have a look at the electric quadrupole of nuclei in 2^+ state



For $A > 150$ a different motion and/or charge distribution is present.
In fact a quadrupole moment $\neq 0$ indicates a deviation from spherically symmetric charge distribution.

The nuclei in ground state are deformed, so they can be put in rotation and change their energy (spherical nuclei that rotate do not change their charge configuration).

The energy of the rotation in quantum mechanics can be expressed as: $E = \frac{1}{2} I \omega^2 = \frac{L^2}{2I}$ $\quad L^2 = l(l+1)\hbar^2$

$L = I\omega$

I = momento di inerzia

$l = 0 \quad E = 0$

$\cancel{l = 1 \quad E = \frac{2\hbar^2}{2I}}$

$l = 2 \quad E = \frac{\hbar^2}{2I} \cdot 6$

$\cancel{l = 3 \quad E = \frac{\hbar^2}{2I} \cdot 12}$

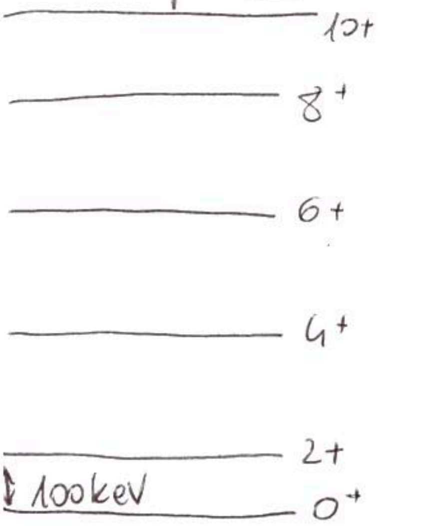
$l = 4 \quad E = \frac{\hbar^2}{2I} \cdot 20$

$l = 6 \quad E = \frac{\hbar^2}{2I} \, 72$

Here we restrict our calculations only to l even, because we would like to deal with some averaged simmetry.

Also in this case we would have $0^+, 2^+, 4^+, 6^+$. but The ratio between $E(2^+)$ and $E(4^+)$ is $\frac{20}{6} = 3{,}3 \simeq$ exactly what was observed. more over the energy levels related to rotation won't be separated but a "constant". This is observed.

For example in ^{174}W:



Where 2+ has an energy of only ≃ 100 keV while in phonon approach the energy that are using are in the MeV sector.

To conclud this section, up to now we have discumed evidence for types of nuclear structure based on the static properties of nuclei (energy levels, spin parity onigments, magnetic dipole and electric quadrupole moments).

The wave functions that result from solving Schrodinger equation perm to calculate many features of nuclear structure.

In same cases (e.g. exited states) observations can be interpred using two alternative interpretations.

The discrimitation between the models can be done by studying the transitio probabilities, i.e. by studing radioactive decays and nuclear reactions.

In both cases we can compare calculations and experimtal values to better understand nuclar states.

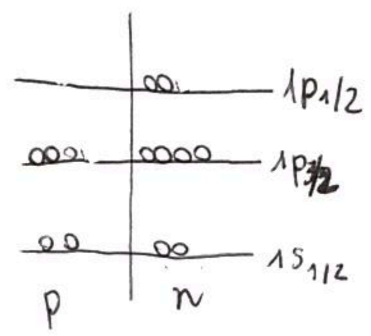
EXERCISE

a) What spin parity would the shell model predict for the ground state of ${}^{13}_{5}B$, ${}^{13}_{6}C$ and ${}^{13}_{7}N$?

b) Order the isobaric triad according to the mass (from lowest to highest). Justify the order.

c) How could you estimate rather closely the energy difference between the two lowest-mass member of the triad?

A)



${}^{13}_{5}B_{8} \Rightarrow J^{\pi} = \frac{3}{2}^{-}$

Proton in $1p_{3/2}$ is not paired

${}^{13}_{6}C_{7} \Rightarrow J^{\pi} = \frac{1}{2}^{-}$

neutron unpaired in $1p_{1/2}$

${}^{13}_{7}N_{6}$

proton unpaired in $1p_{1/2}$

b) From lower to higher mass: ${}^{13}_{6}C$, ${}^{13}_{7}N$, ${}^{13}_{5}B$

${}^{13}_{6}C$ and ${}^{13}_{7}N$ are in the same isospin dublet (same J).

The mass difference arises from different Coulomb energy.

${}^{13}_{7}N$ has a proton more than ${}^{13}_{6}C$, so greater Coulomb energy $\Rightarrow$ larger mass (or in B.E. Coulomb Term is $-a_C \frac{Z(Z-1)}{A^{1/3}}$)

${}^{13}_{5}B$ has fewer protons and more neutrons $\Rightarrow$ less bound $\Rightarrow$ larger mass

c) We have to evaluate the ${}^{13}_{6}C$ and ${}^{13}_{7}N$ mass difference.

Assuming nuclei to be hard sphere with unifor charge the electrostatic energy will be: $W = \frac{3}{5}\frac{Q^2}{R}$ with $R = R_0 A^{1/3} \simeq 1{,}44 \cdot A^{1/3}$

The mass difference will be:

$$M({}^{13}_{7}N) - M({}^{13}_{6}C) = \frac{3}{5R}\left(Q_N^2 - Q_C^2\right) - [m_n - m_p]c^2$$

$$= \frac{3}{5R} e^2 \left(7^2 - 6^2\right) - 0{,}78 \text{ MeV}$$

$$= \frac{3\hbar c}{5R}\Big(\underbrace{\frac{e^2}{\hbar c}}_{\alpha}\Big)(49-36) - 0{,}78 \text{ MeV}$$

$$= \frac{3}{5} \cdot \frac{200 \text{ MeV}\cdot \cancel{\text{fm}}}{137} \times \frac{\overset{13}{49-36}}{1{,}44 \cdot 13^{1/3}\,\cancel{\text{fm}}} - 0{,}78 \text{ MeV}$$

$$= 2{,}62 \text{ MeV}$$

C 13,003355 u

N 13,005799 u

$\Delta m = 0{,}002444\ u = (931{,}5 \cdot 0{,}002444)\ \text{MeV}/c^2 = 2{,}277$ MeV