It means that s, p, d, ... states are DEFINED, but the m_ℓ states are mixed up $\Rightarrow L^2 = \Phi$ (45)

$m_\ell = 0, 1, +1$ (AND)

NOT (0, or 1, or -1)

and L^2 commutes with J^2, J_z where $\vec{J} = \vec{L} + \vec{S}$ is the TOTAL ANGULAR MOMENTUM OF each NUCLEONS (not the nucleus)

$J^2 = J(J+1)\hbar^2$

$J_z = M_j \hbar$ $\quad m_j = J, J-1, \dots, -J \quad m_J = (m_\ell + m_s)$

$\Rightarrow$ How the energy changes due to this.

ℓ, s, J, m_J are defined

$$\langle \vec{\ell} \cdot \vec{s} \rangle = \left\langle \frac{J^2 - \ell^2 - s^2}{2} \right\rangle$$

$J^2 = (L+S)^2 = L^2 + 2L\cdot S + S^2$

$\ell \cdot S = \frac{J^2 - L^2 - S^2}{2}$

$\langle \ell \cdot s \rangle = \left\langle \frac{J^2 - L^2 - S^2}{2} \right\rangle$

$\vec{J} = \vec{\ell} + \vec{S} \quad S = \frac{1}{2}$

$\ell = 0 \quad J = \frac{1}{2}$

$\ell^2 + \frac{3}{2}\ell + \frac{1}{2}\ell + \frac{3}{4} = \ell^2 + 2\ell + \frac{3}{4}$

$\ell \neq 0 \quad J = \ell + \frac{1}{2}, \ \ell - \frac{1}{2}$

$$\text{if } J = \ell + \frac{1}{2} \quad \langle \vec{\ell} \cdot \vec{s} \rangle = \frac{\overset{J}{(\ell + \frac{1}{2})}\overset{J+1}{(\ell + \frac{3}{2})}\hbar^2 - \ell(\ell+1)\hbar^2 - \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2}{2}$$

$$= \frac{\hbar^2}{2}\left[\cancel{\ell^2} + 2\ell + \cancel{\frac{3}{4}} - \cancel{\ell^2} - \ell - \cancel{\frac{3}{4}}\right] = \frac{\hbar^2}{2}\ell$$

$$J = \ell - \frac{1}{2} \quad \langle \vec{\ell} \cdot \vec{s} \rangle = \frac{(\ell - \frac{1}{2})(\ell + \frac{1}{2})\hbar^2 - \ell(\ell+1)\hbar^2 - \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2}{2} =$$

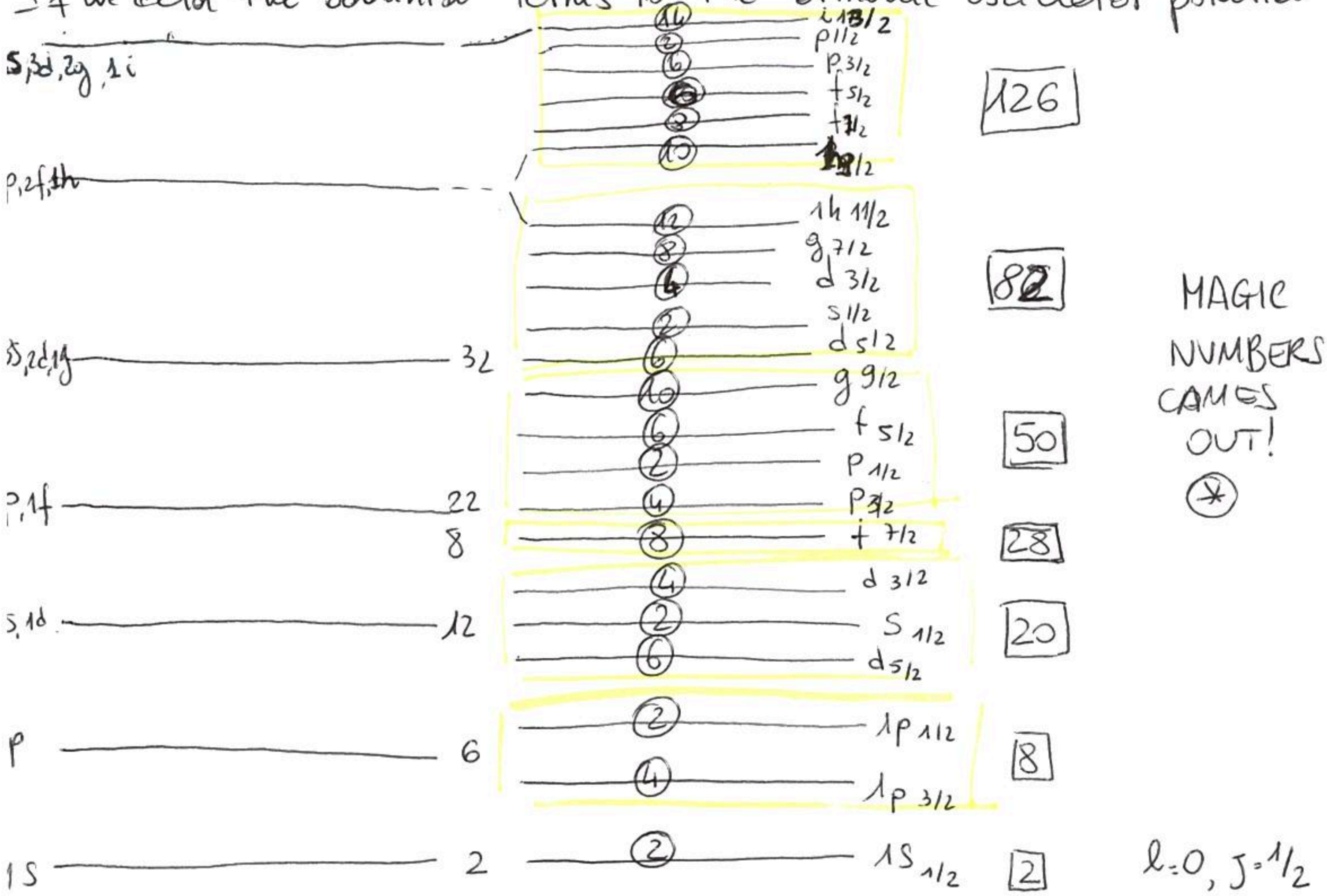
$$= \frac{\hbar^2}{2}\left[\ell^2 - \frac{1}{4} - \ell^2 - \ell - \frac{3}{4}\right] = -\frac{\hbar^2}{2}(\ell + 1)$$

⇒ The energy is splitted into 2 parts and to match the data $f(r)$ have to be negative.

For $J = l - \frac{1}{2}$ energy will decrease by an amount larger than the value of which it increases ⇒ We introduce an asimmetry

The splitting is also $\propto$ to l $\Delta E = \frac{\hbar^2}{2}(2l+1)$ so, the larger is l the larger is the split. Moreover we learn that S ($l=0$) do not split

If we add the addinial terms to the armonic oscillator potential



ow many quantum states there are for each level? Orbital state + 1

luick summary on l, J states

$l = 1$ $J = \frac{1}{2}$ (↑up) $J = \frac{3}{2}$ (↓down) p

$l = 2$ $J = \frac{3}{2}$ $J = \frac{5}{2}$ d

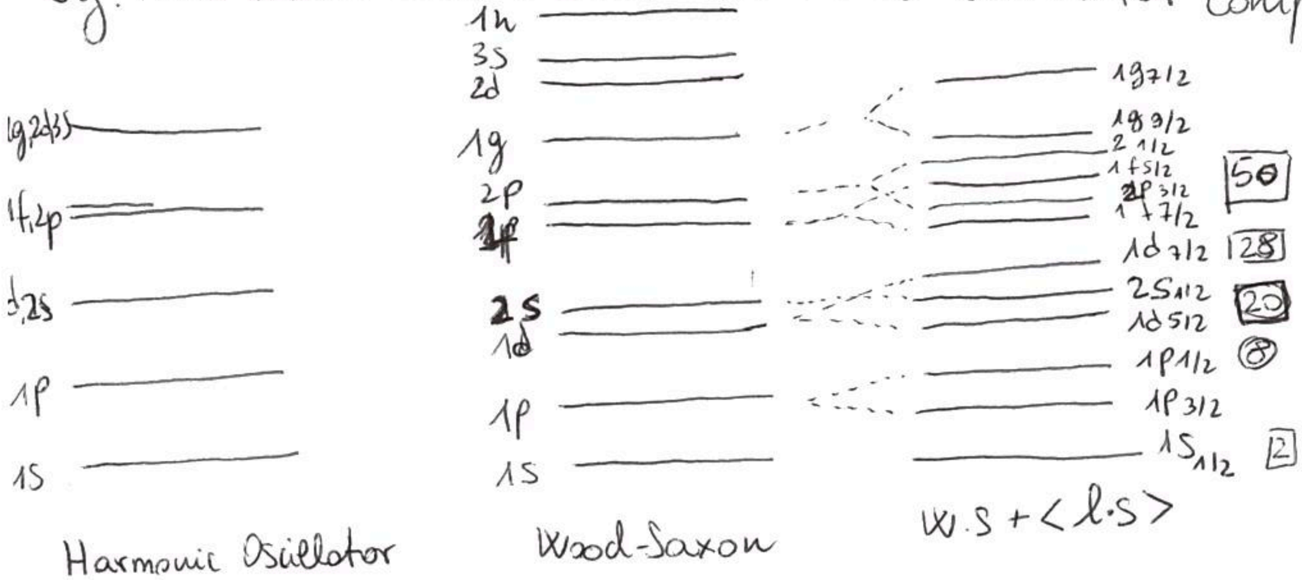
$l = 3$ $J = \frac{5}{2}$, $J = \frac{7}{2}$ f

$l = 4$ $J = \frac{7}{2}, \frac{9}{2}$ g

$l = 5$ $J = \frac{9}{2}, \frac{11}{2}$ h

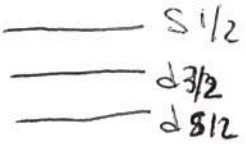
(*) Magic numbers comes out also for harmonic potential, even if we saw that the most realistic one is the Wood-Saxon one. This shown here is not "the true" because other effects might play a role

Eg. How Wood Saxon and harmonic oscillator compare?



SOME CAVEATS

FOR $A \simeq 50$ the energy levels are



i.e. the energy level are "exchanged" w.r.t. the usual

Similarly, for $A \simeq 65$ the 2p lines exchange order.

Shell model works very well for "light" nuclei. For heavy nuclei energy levels ~~can~~ can change order (cross-over)

Average separation decreases as A increases. Since the well of the potential does not vary much with A, it follows that the average separation between levels decreases.

Protons suffers from Coulomb interactions ⇒ n e p have different potentials. For light nuclei the Coulomb effect is not as relevant as for heavy nuclei.

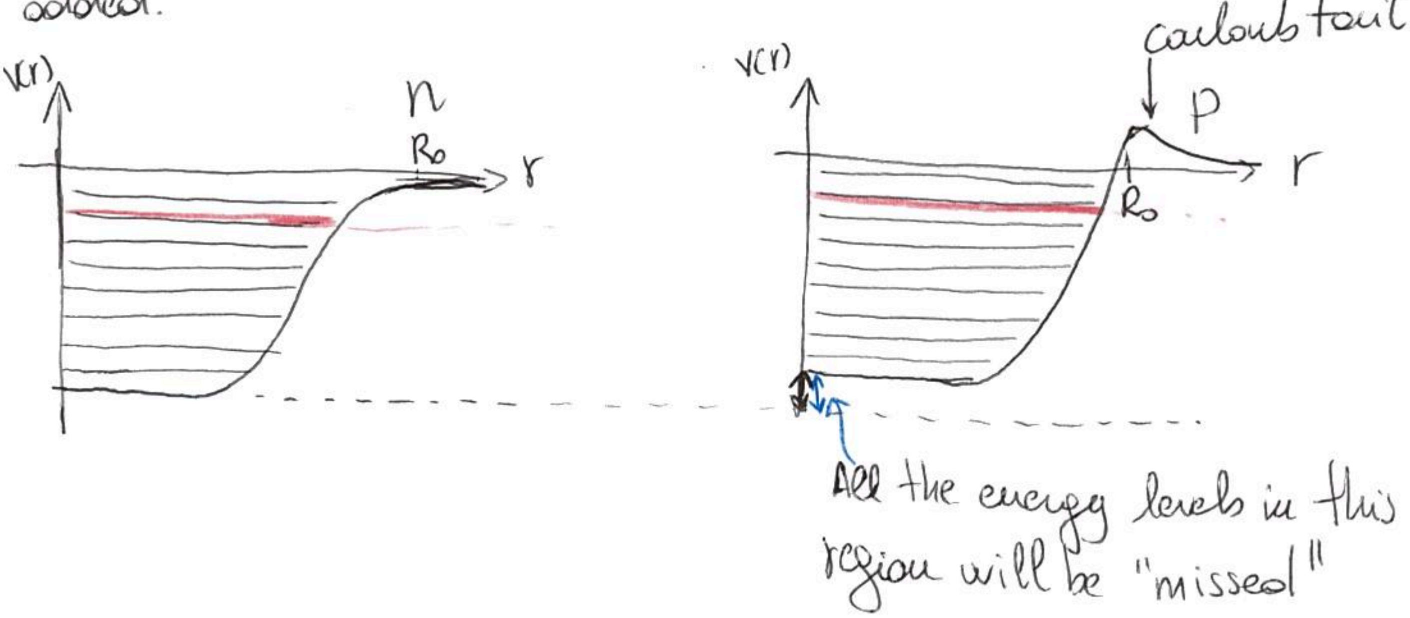
$$V_{coul}(r) = \frac{Z \cdot e^2}{4\pi\varepsilon_0 R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)$$

Inside the nucleus +

the effect of Coulomb is ...

w Z → Small contribution

$V_c(r)$ is Positive $\Rightarrow$ the potential and the well are shifted up. For $r > R_0$ a Coulomb tail have to be added.



And since p and n can exchange $\Rightarrow$ We can fill n only up to a certain "level" $\Rightarrow$ If we want to have a "minimal energy", the number of neutron must be larger than protons! ($\rightarrow$ i.e. Shell model is able to reproduce a feature that we already observed!)

SHELL MODEL & NUCLEUS SPIN PARITY IN GROUND STATE

The total angular momentum J^π of a nucleus can be understood using the shell model.

We just recall that parity π refers to the space part of the w.f. and tell us if it is symmetric or not.

π depends on l : l even $\Rightarrow$ positive parity $\pi = +$

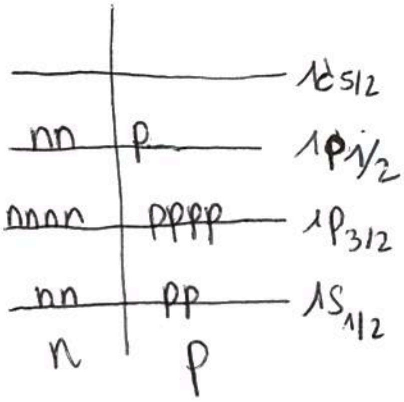
l odd $\Rightarrow$ negative " $\pi = -$

Even-Even nuclei are <u>all</u> $J = 0^+$ $\Rightarrow$ All nucleons are paired $\Rightarrow \sum_{ij} J_i J_j = \sum 0 = 0$

For odd-A nuclei, of course, this is not true and the J^π can be "any". But with shell-model we can see which is the J^π of almost any odd-A nucleus.

EXAMPLE

${}^{15}_{7}N_8$



Which is the last nucleous not paired? The p in $1p_{1/2}$

So the parity of

$${}^{15}_{7}N_8 = \frac{1}{2}^{-}$$

P (l=1)

last orbital

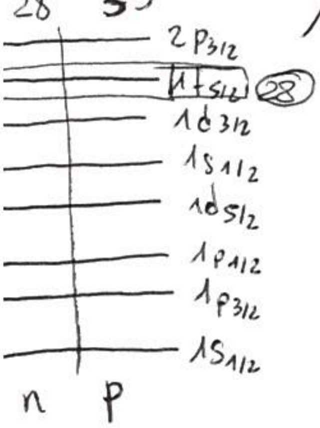
Similarly we can guess the J^+ of other nuclei

${}^{15}_{8}O_7 : J^\pi = \frac{1}{2}^-$ ${}^{17}_{8}O_9 : J^\pi = \frac{5}{2}^+$ ${}^{13}_{6}C_7 : J^\pi = \frac{1}{2}^-$

And this is correct for most of the element. This model is called EXTREME SINGLE PARTICLE MODEL and is working rather well. Of course, as for many of the "fenomenological" and a posteriori model there are some cases where the single particle model do not work.

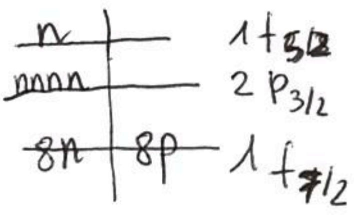
As an example let's take

${}^{61}_{28}Ni_{33}$



If you recall, the magic numer 28 closes in $1f_{7/2}$.

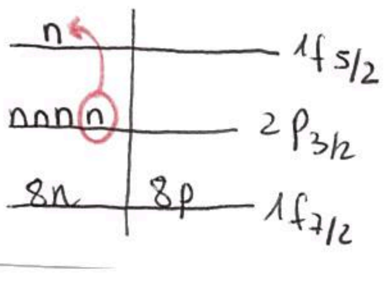
If we take ${}^{61}_{28}Ni_{33}$ we would expect:



$$J^\pi = \frac{5}{2}^-$$

But the measured value is $\frac{3}{2}^-$!

...ll if the following "operation" happens:

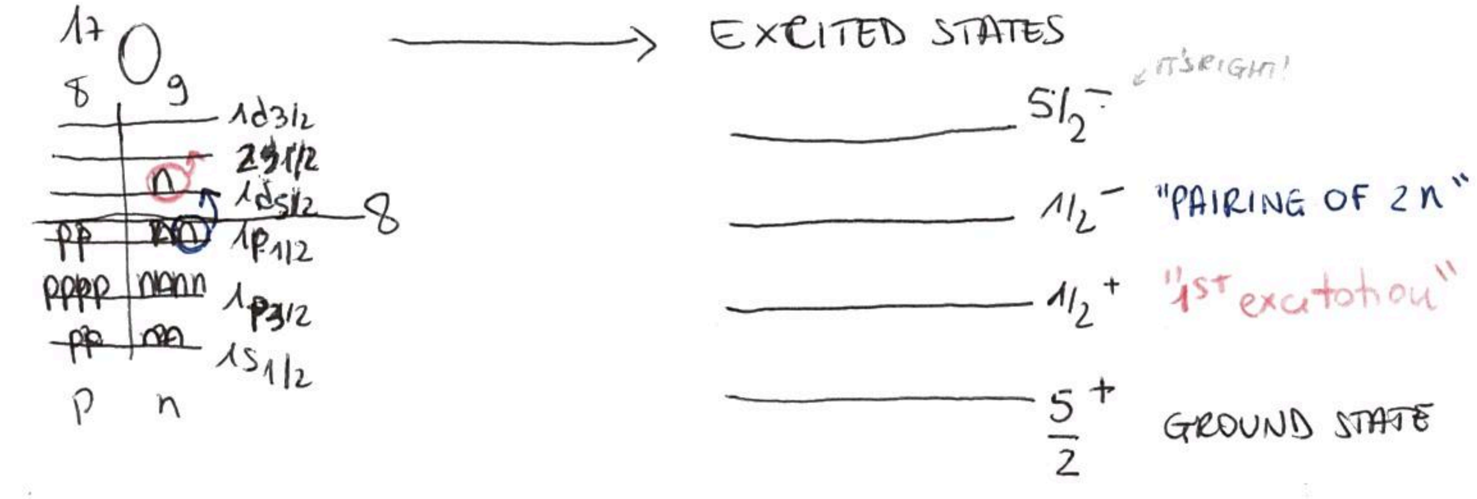


=> Then the observed J^π would match with observed measurent.

Why this should happen? Because of the PAIRING. As we already observed, if 2 nucleons pairs, the energy of the nucleus decreases $\Rightarrow$ it's more stable $\Rightarrow$ it will prefer to stay in such state!

BUT! pairing in p level is $\neq$ than pairing in f level because the pairing in large l values implies a larger energy $\Rightarrow$ more stability. In the specific case the pairing energy is larger than the energy gap $\Rightarrow$ the preferred state would be the one with 2n in $1f_{5/2}$. This implies that with the shell-model we cannot "guess" but we can understand the measurements. The model is "STILL OK" also for J^{π} measurements!

The shell model can also be used to understand exited states of specific nuclei:



$\Rightarrow$ Single particle shell model can be used also to understand the spin parity of excited states!

SHELL MODEL & MAGNETIC MOMENTS OF A NUCLEUS

$\boxed{\text{MAGNETIC DIPOLE MOMENTS}}$

An other case in which the shell model gives a reasonable agreement with data is the magnetic dipole moments.

It is computed from the expectation value of the magnetic moment operator in the state with maximum z projection of angular momentum. Including both l and s terms:

$$\mu = \mu_N \left(g_\ell l_z + g_s s_z\right)$$

Where μ_N = Nuclear magneton $= \frac{e\hbar}{2m_p}$

FOR p: $\begin{cases} g_\ell = 1 \\ g_s = 5{,}5857 \end{cases}$ For n: $\begin{cases} g_\ell = 0 \\ g_s = -3{,}8260 \end{cases}$

For each nucleon $\vec{l} + \vec{s} = \vec{J}$ (l·s is strong)

If $\vec{J}$ is precisely defined, l_z and s_z have not a precise definition
=> only J_z is important. =>

$J_z = l_z + S_z$

$m_J = m_\ell + m_s$

Each state is characterize by:
$l, \underset{1/2}{s}, J, m_j, |m_\ell, m_s|$
$-\ell < x < \ell$ step of 1

Each state is defined by m_ℓ; m_J is fixed and m_ℓ and m_s can assume any values under this assumption.

Secondly, One measure the averaged value of $\langle \mu_z \rangle$ => We have to evaluate it, also using the averaged $m_{j,\ell s}$

$$\langle \mu_z \rangle = \langle J, m_J = J \mid g_\ell l_z + g_s \hat{s}_z \mid J, m_J = J \rangle \mu_N$$

Why? Because everything should be aligned to the max

We can have 2 cases: a) $J = \ell + \frac{1}{2}$ (spin)

b) $J = \ell - \frac{1}{2}$

Let's start from a) $J = l + \frac{1}{2}$

$$m_J = m_e + m_s$$

$$J \quad \| \quad l + \frac{1}{2} = m_e + m_s$$

$$m_e = l, l-1, l-2, \dots, -l$$
$$m_s = \frac{1}{2}, -\frac{1}{2}$$

which values of $m_e + m_s$ give $l + \frac{1}{2}$?

$\Downarrow$

ONLY $\boxed{m_e = l \text{ and } m_s =}$

$$\langle l_z \rangle = l$$
$$\langle S_z \rangle = \frac{1}{2}$$

$$\Rightarrow \langle \mu_z \rangle = \left(g_e l + g_s \frac{1}{2} \right) \mu_N$$

$$= \left(g_e l + \frac{g_s}{2} \right) \mu_N$$

If the last nucleus is a p: $\langle \mu_z \rangle = \left(l + \frac{5{,}5877}{2} \right) \mu_N = \left(l + 2{,}7929 \right) \mu_N =$

$$= \left(J - 2{,}2929 \right) \mu_N$$

If the last nucleus is a n: $\langle \mu_N \rangle = \frac{g_s}{2} \mu_N = -\frac{3{,}8260}{2} \mu_N = -1{,}9130 \mu$

Case b) $J = l - \frac{1}{2}$

$$J = l - \frac{1}{2}$$
$$m_J = l - \frac{1}{2}$$
$$m_e + m_s = l - \frac{1}{2}$$

m_e	m_s
l	$\frac{1}{2}$
$l-1$	$-\frac{1}{2}$
$l-2$	
$\vdots$	
l	

$$\Rightarrow \left. \begin{array}{l} m_e = l \quad m_s = -\frac{1}{2} \\ m_e = l-1 \quad m_s + \frac{1}{2} \end{array} \right\} \text{MIX OF 2 STATES!}$$

$$| J, m_e = J \rangle = a \left| m_e = l, m_s = -\frac{1}{2} \right\rangle + b \left| m_e = l-1, m_s = \frac{1}{2} \right\rangle$$

$$= a \psi_1 + b \psi_2$$

$$\langle \mu_z \rangle = \langle a\psi_1 + b\psi_2 | g_e l_z + g_z S_z | a\psi_1 + b\psi_2 \rangle \mu_z$$

Lets evaluate the <II> part.

$$\langle a\psi_1 + b\psi_2 | g_e l_z + g_s s_z | a\psi_1 + b\psi_2 \rangle =$$

$$= \langle a\psi_1 | \mu_z | a\psi_1 \rangle + \langle b\psi_2 | \mu_z | b\psi_2 \rangle + \underbrace{\langle a\psi_1 | \mu_z | b\psi_2 \rangle + \langle b\psi_2 | \mu_z | a\psi_1 \rangle}$$

In ψ_1 e ψ_2 the m_l and m_s are defined.
The ψ_1 and ψ_2 are also hortogonal
$\psi_1 \psi_2 = 0$

$$= \langle a\psi_1 | \mu_z | a\psi_1 \rangle + \langle b\psi_2 | \mu_z | b\psi_2 \rangle + 0 + 0$$

$$= |a|^2 \langle \psi_1 | \mu_z | \psi_1 \rangle + |b|^2 \langle \psi_2 | \mu_z | \psi_2 \rangle$$

To solve the eq. we have 2 tasks: 1) evaluate $\langle \psi_n | \mu_z | \mu_n \rangle$; 2) Evaluate a and b

Let's start from task ①

$$\langle \psi_1 | \mu_z | \psi_1 \rangle = \langle m_l = l, m_s = -\tfrac{1}{2} | g_e l_z + g_s s_z | m_l = l, m_s = -\tfrac{1}{2} \rangle$$

ARE 2 OPERATORS

$$\langle m_l = l, m_s = -\tfrac{1}{2} | g_e l | m_l = l, m_s = -\tfrac{1}{2} \rangle + \langle m_l = l, m_s = -\tfrac{1}{2} | g_s \left(\tfrac{\hbar}{2}\right) | m_l = l, m_s = -\tfrac{1}{2} \rangle$$

are the same

*

$$g_e l \hbar + g_s \left(-\frac{\hbar}{2}\right)$$

*

$$\langle \psi_2 | \mu_z | \psi_2 \rangle = \langle m_l = l-1, m_s = \tfrac{1}{2} | g_e l_z + g_s s_z | m_l = l-1, m_s = \tfrac{1}{2} \rangle$$

$$= g_e (l-1)\hbar + g_s \left(\frac{\hbar}{2}\right)$$

To solve task ② we have to use a trick. The trick uses the Raising and lowering operators. (J_+, J_-)

$J_- = J_x - i J_y$ $\quad$ J_- = Lowering operator

$J_+ = J_x + i J_z$ $\quad$ J_+ = Raising operator

$$\langle J_+ | J, m_J \rangle = \sqrt{(J-m_J)(J+m_J+1)}\ |J, m_j+1\rangle$$

$$\psi = a\,|m_e = l, m_s = -\tfrac{1}{2}\rangle + b\,|m_e = l+1, m_s = \tfrac{1}{2}\rangle$$

$\Uparrow$

$|J, m_J = J\rangle$

Let's apply J_+ $\Rightarrow$ m_e cannot be increased because it is already maximum $\Rightarrow$ J_+ applied to our ket must return 0.

$\boxed{J_+ |J, m_J = J\rangle = 0}$ We will use this operator to get a and b

$\vec{J} = \vec{l} + \vec{s}$

Ⓐ

$$J_+ |m_e = l, m_s = -\tfrac{1}{2}\rangle = (L_+ + S_+)\,|m_e = l, m_s = -\tfrac{1}{2}\rangle =$$

$$= \underbrace{L_+ |m_e = l, m_s = -\tfrac{1}{2}\rangle}_{=0 \Rightarrow \text{Already max } l} + S_+ |m_e = l; m_s = -\tfrac{1}{2}\rangle$$

$$= 0 + \sqrt{\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)\left(\frac{1}{2} - \frac{1}{2} + 1\right)}\ |m_e = l; m_s = -\tfrac{1}{2}+1\rangle$$

$$J_+ |m_e = l, m_s = -\tfrac{1}{2}\rangle = |m_e = l, m_s = +\tfrac{1}{2}\rangle$$

(b)

$$J_+ \left| m_\ell = \ell - 1, m_s = \frac{1}{2} \right\rangle = (L_+ + S_+) \left| m_\ell = \ell - 1, m_s = \frac{1}{2} \right\rangle$$

$$= L_+ \left| m_\ell = \ell - 1, m_s = \frac{1}{2} \right\rangle + S_+ \left| m_\ell = \ell - 1, m_s = \frac{1}{2} \right\rangle$$

$$= \sqrt{(\ell - (\ell - 1))(\ell + (\ell - 1) + 1)} \left| m_\ell = 1, m_s = \frac{1}{2} \right\rangle$$

$\Rightarrow$

$$0 = a \left| m_\ell = \ell, m_s = +\frac{1}{2} \right\rangle + b\sqrt{2\ell} \left| m_\ell = \ell, m_s = \frac{1}{2} \right\rangle$$

$$\Rightarrow \quad a + b\sqrt{2\ell} = 0 \quad \underset{\text{2 variables}}{\Rightarrow} \text{2 equation} : \begin{cases} |a|^2 + |b|^2 = 1 \\ a = -b\sqrt{2\ell} \end{cases}$$

$$\begin{cases} |b|^2 = \dfrac{1}{2\ell + 1} \\ |a|^2 = |b|^2 \cdot 2\ell = \dfrac{2\ell}{2\ell + 1} \end{cases}$$