

$$Q = (m_x + m_a - m_y - m_b) c^2$$

Q is equal to the ~~loss~~ excess kinetic energy of final products

$$Q = T_{\text{final}} - T_{\text{initial}} = T_y + T_b - T_x - T_a$$

Q can be positive $\Rightarrow m_i > m_f$ or $T_f > T_i$ = Exoergic or exothermic

Bound-up energy is released by the transformation

or negative $\Rightarrow m_i < m_f$ or $T_i > T_f$ = ENDOERGIC or endothermic

Energy is taken from the kinetic energies of initial reactions

or zero.

The expression $Q + T_a + T_x = T_b + T_y$ is a relativistic expression, so in principle relativistic expressions for kinetic energies have to be used.

$$T_{\text{rel}} = mc^2(\gamma - 1)$$

Since the maximum energy projectile we are using is $\approx 10 \text{ MeV}$, we are in the case $T \ll mc^2$, so it is possible to find a relationship that relates T_{rel} to the non-relativistic expression T_{NR}

$$T_{\text{rel}} = T_{\text{NR}} \left(1 + \frac{3}{4} \beta^2 + O(\beta^4) \right)$$

In the worst case scenario $T_a = 10 \text{ MeV}$ $m_a = m_p$ the relativistic correction amounts to:

$$\frac{3}{4} \beta^2 = \frac{3 T_a^2}{2 M_p c^2} = \frac{3 (10 \text{ MeV})^2}{2 \cdot 1 \text{ GeV}}$$

$$\frac{3}{4} \beta^2 \approx \frac{3 T_a^2}{2 M_p c^2} \approx 0.015 \Rightarrow \text{For } Q=0 \text{ a } 1.5\% \text{ correction is present.}$$

Depending on the accuracy of the measurement, it can be large. This correction represents a systematic uncertainty.

Anyway from now on we will ignore it. Photons are always relativistic, so $T_y = E_y$ and $p = \frac{E_y}{c}$.

LABORATORY FRAME N-R ANALYSIS

The approximations for non-relativistic analysis are:

$$T_a < 10 \text{ MeV}$$

\equiv kinetic energy of projectile

$$\vec{p}_a = p_a \hat{z}$$

\equiv Projectile along z-axis

$$T_x = 0$$

\equiv Target at rest

$$m_a \leq m_b, m_b \leq 4m$$

\equiv Small projectile and observed ptc

$$m_x, m_y \geq 4m$$

\equiv Large target, unobserved particle

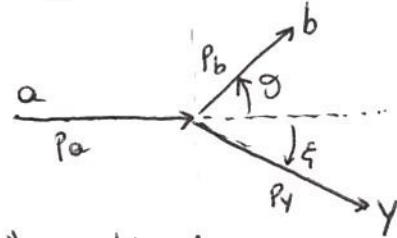
$$\chi_i = 1$$

\equiv No relativistic corrections

With these approximations, conservation of energy and of linear momentum these expressions can be written:

$$Q + T_a = T_b + T_y$$

$$\vec{p}_a = \vec{p}_b + \vec{p}_y$$



Since y is unobserved it can be "eliminated" so that:

$$T_b (m_y + m_b) - 2 \sqrt{m_a m_b T_a} \cos \theta_b \sqrt{T_b} - [m_y (Q + T_a) - m_a T_a] = 0$$

(*)

$$\begin{cases} Q + T_a = T_b + T_y \\ \vec{P}_a = \vec{P}_y + \vec{P}_b \end{cases}$$

Energy conservation
 \vec{P} conservation

1

NON RELATIVISTIC

$$\Rightarrow P = \sqrt{2mE_k}$$

\hookrightarrow x axis:

$$P_a = P_b \cos \theta + P_y \cos \xi$$

y axis

$$0 = P_b \sin \theta - P_y \sin \xi$$

$$\left\{ \begin{array}{l} Q = T_b - T_a + T_y \end{array} \right.$$

3 equations

$$\sqrt{2m_a T_a} = \sqrt{2M_b T_b} \cos \theta + \sqrt{2m_y T_y} \cos \xi$$

3 unknowns

$$\sqrt{2m_b T_b} \sin \theta = \sqrt{2m_y T_y} \sin \xi$$

$$\left\{ \begin{array}{l} Q = T_b - T_a + T_y \end{array} \right.$$

$M_b T_b \cos \xi$

$$\sqrt{m_a T_a} = \sqrt{M_b T_b} \cos \theta + \sqrt{m_y T_y} \cos \xi$$

$$\sqrt{m_b T_b} \sin \theta = \sqrt{M_y T_y} \sin \xi$$

$$\left\{ \begin{array}{l} Q = T_b - T_a + T_y \end{array} \right.$$

$$m_a T_a + m_b T_b \cos^2 \theta - 2 \sqrt{m_a T_a m_b T_b} \cos \theta = M_y T_y \cos^2 \xi = M_y T_y (1 - \sin^2 \xi)$$

$$m_b T_b \sin^2 \theta = M_y T_y \sin^2 \xi$$

$$\sin^2 \xi = \frac{m_b T_b}{M_y T_y} \sin^2 \theta$$

$$m_a T_a + M_b T_b \cos^2 \theta - 2 \sqrt{m_a T_a M_b T_b} \cos \theta = M_y T_y - M_b T_b \sin^2 \theta =$$

$$= M_y T_y - M_b T_b + M_b T_b \cos^2 \theta$$

$$m_a T_a - 2 \sqrt{m_a T_a M_b T_b} \cos \theta + M_b T_b = M_y T_y \Rightarrow \text{To get } T_y$$

\Rightarrow

$$Q = T_b - T_a + \frac{m_a}{M_y} T_a - 2 \frac{\sqrt{m_a T_a M_b T_b}}{M_y} \cos \theta + \frac{M_b}{M_y} T_b$$

$$Q = T_b \left(1 + \frac{M_b}{M_y} \right) - T_a \left(1 - \frac{m_a}{M_y} \right) - 2 \frac{\sqrt{2 m_a T_a M_b T_b}}{M_y} \cos \theta$$

Measurement at fixed angle $\theta \rightarrow$ quadratic relationship between $\sqrt{T_b}$ and T_b

$$T_b \left(\frac{M_y + M_b}{M_y} \right) - 2 \frac{\sqrt{M_a T_a M_b T_b}}{M_y} \cos\theta - \frac{T_a(M_y - M_a) + M_y Q}{M_y} = 0 \quad \textcircled{*}_2$$

$$T_b - 2 \sqrt{T_b} \frac{\sqrt{M_a T_a M_b}}{M_y + M_b} \cos\theta - \frac{T_a(M_y - M_a) + M_y Q}{M_y + M_b} = 0$$

$$\sqrt{T_b} = B \pm \sqrt{B^2 - C}$$

$$B = \frac{\sqrt{M_a T_a M_b}}{M_b + M_y} \cos\theta \quad C = \frac{T_a(M_y - M_a) + M_y Q}{M_b + M_y}$$

Threshold energy

$$B^2 - C = 0$$

$$\frac{M_a T_a M_b \cos^2\theta}{(M_b + M_y)^2} + \frac{T_a(M_y - M_a) + M_y Q}{M_b + M_y} = 0$$

$$T_{a\text{th}} = - \frac{Q M_y (M_b + M_y)}{M_a M_b \cos^2\theta + (M_y - M_a)(M_3 + M_4)}$$

$$= - \frac{Q (M_b + M_y)}{M_b + M_y - M_a - \frac{M_a M_b}{M_y} \sin^2\theta}$$

For $\theta = 0$

$$T_{a\text{th}} = -Q \frac{M_b + M_y}{M_b + M_y - M_a}$$

FOR ENDOTHERMIC reactions
 $T_1 > T_{\text{th}}$ to allow the reaction

FOR EXOTHERMIC reactions
 $T_1 > 0$ to allow for reaction

When $B \geq \sqrt{B^2 + C}$ Two possible values are allowed for T_3

*
3

$$\Rightarrow C < 0 \Rightarrow C = \frac{T_a(M_y - M_b) + M_y Q}{M_3 + M_4} \leq 0$$

$$\Rightarrow T_a < T_a|_{\max, \text{double}} = -Q \frac{M_y}{M_b + M_y}$$

Double T_b values is happening only for $Q < 0$ reactions and for energies between T_a and $T_a|_{\max, \text{double}}$

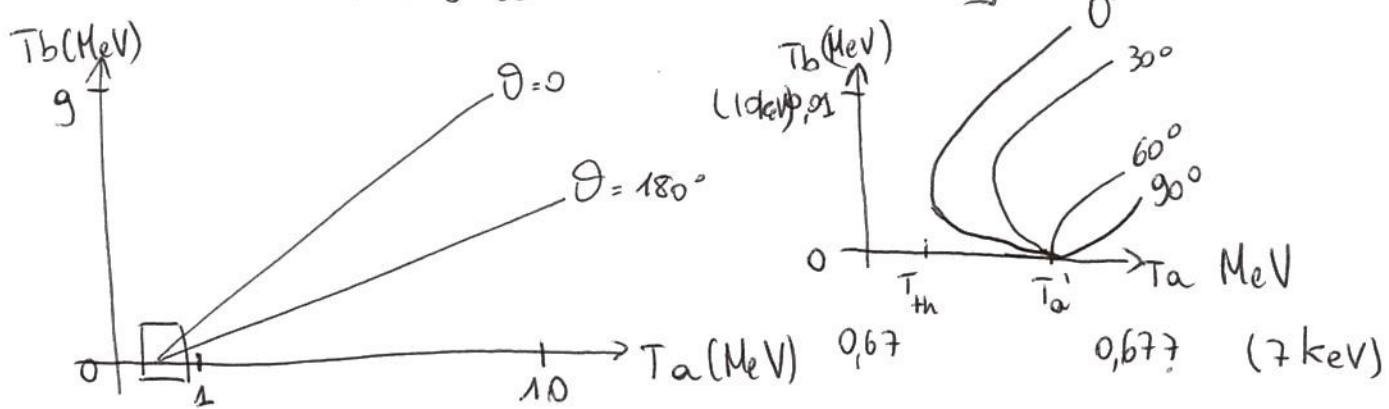
The range of $T_a' - T_{th}$ can be extracted and is equal to:

$$T_a' - T_{th} = T_{th} \frac{M_a M_b}{M_y (M_y - M_a)} \left(1 - \frac{M_b}{M_y} + \dots \right)$$

If a and b have mass $\leq 4u$ and Y is a medium or heavy nucleus, then the range $(T_a' - T_{th})$ become $\ll 1\text{ fm}$.

It is possible, also, to define a maximum scattering angle. The maximum is given by

$$\cos^2 \theta_{\max} = - \frac{(M_y - M_b)}{M_a M_b T_a} [M_y Q + (M_y - M_b) T_a]$$



If we measure T_b for a given θ and T_a we can determine the Q value of the reaction and deduce the mass relationships among the constituent. If we know M_a , M_b and M_x , we can determine M_y .

$$Q = T_b \left(1 + \frac{M_b}{M_y} \right) - T_a \left(1 - \frac{M_a}{M_y} \right) - 2 \cos \theta \left(\frac{M_a}{M_y} \frac{M_b}{M_y} T_a T_b \right)^{1/2}$$

$$Q = (M_a + M_x - (M_b + M_y)) c^2 \Rightarrow \text{Not knowing } Q \text{ is equivalent to not know the masses of the nuclei involved in the reaction.}$$

In experiments where this method is used M_y is the unknown factor. This is the case especially when an excited Y^+ nucleus is created from the reaction. \rightarrow To recover the situation, one solves for M_y and substitutes. Also taking $\theta_b = \pi/2$ to simplify the arithmetic the results is:

$$Q = T_b - T_a + \frac{(T_a M_a C^2 + T_b M_b C^2)}{[M_x C^2 + M_a C^2 - M_b C^2] - Q}$$

It can be solved using a common denominator and solve the quadratic equation in Q .

Alternatively, the equation can be treated as an iterative or recursive equation:

1) Define a time-saving shorthand:

$$Q = S + \frac{()}{[] - Q}, \text{ where } S = T_b - T_a$$

2) From the lowest order solution:

$$Q_0 = S + \frac{()}{[]}$$

3) The n^{th} correction to Q is found from

$$\sum_{i=0}^n Q_i = S + \frac{()}{[] + \sum_{i=0}^n Q_i}$$

4) $Q = \sum_{i=0}^n Q_i \rightarrow$ The iteration is stopped when the answer is "good enough"

$$\Rightarrow Q_0 = S + \frac{()}{[]}$$

$$Q_0 + Q_1 = S + \frac{()}{[] - Q_0} = S + \frac{()}{[]} \left(\frac{1}{1 - Q_0 / []} \right) = S + \frac{()}{[]} \left(1 + \frac{Q_0}{[]} \right) = Q_0 + \frac{Q_0 ()}{[]^2}$$

$$Q_1 = \frac{Q_0 ()}{[]^2}$$

$$\text{So } \frac{Q_1}{Q_0} = \frac{(T_a M_a C^2 + T_b M_b C^2)}{[M_x C^2 + M_a C^2 - M_b C^2]^2}$$

In the worst possible case $m_x = M_a = M_b \approx 4\mu$ and $T_a \approx T_b \approx 10\text{MeV}$, giving $\frac{Q_1}{Q_0} \approx 5 \cdot 10^{-3}$. Since Q -values are typically $\approx 10^{-4}$ this can be ignored especially for target with small A

(10)

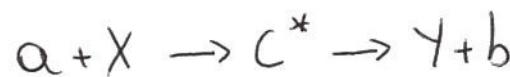
COMPOUND-NUCLEUS REACTIONS

Let's suppose a incident ptc which enters a target nucleus with an impact parameter small compared to the nuclear radius.

It will have a high probability of interacting with one of the nucleons of the target, possibly through a simple scattering.

The recoil nucleus is therefore ^{with some energy} and the incident ptc has now a lower energy. After several interactions, the incident energy is shared among many of the nucleons of the combined system of projectile + target. The averaged increased energy is not enough to free it from the nucleus, but since the number of interaction occurs, there is an increase of probability that the nucleus gains enough energy to "escape".

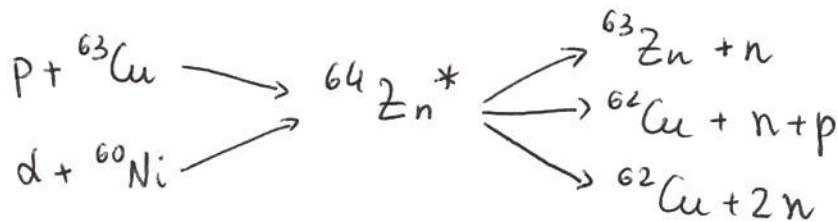
Such reactions have a definite intermediate state after the absorption of the incident ptc but before the emission of the outgoing ptc. This intermediate state is called "compound nucleus".



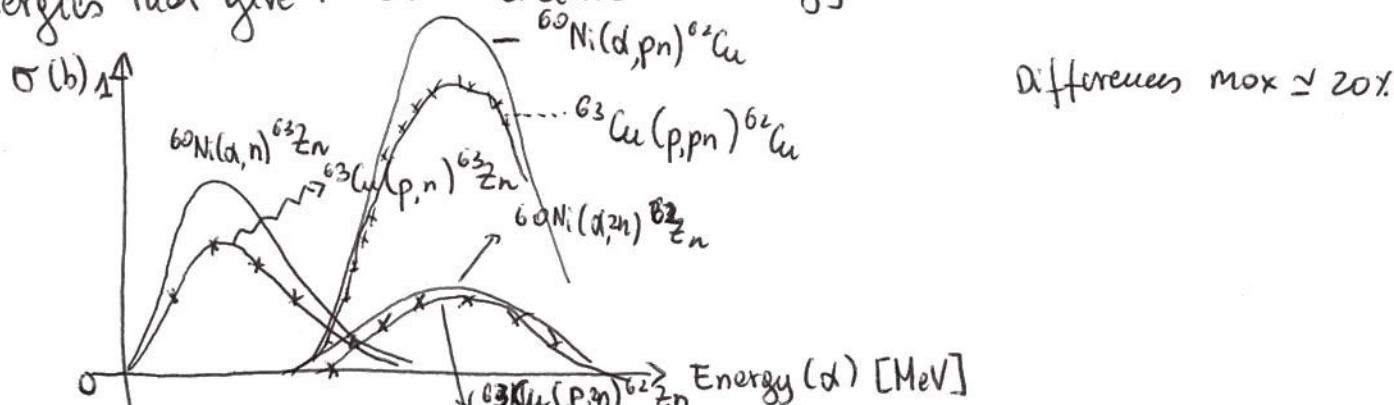
where C^* indicates the compound nucleus.

A given compound nucleus may decay in a variety of different ways and the C^* does not have any memory of how it was formed.

E.g.



If the hp "no memory" is correct, then the relative cross-sections for ${}^{63}\text{Cu}(p, n) {}^{63}\text{Zn}$ and ${}^{60}\text{Ni}(d, n) {}^{63}\text{Zn}$ would be the same at incident energies that give the same excitation energy to ${}^{64}\text{Zn}^*$

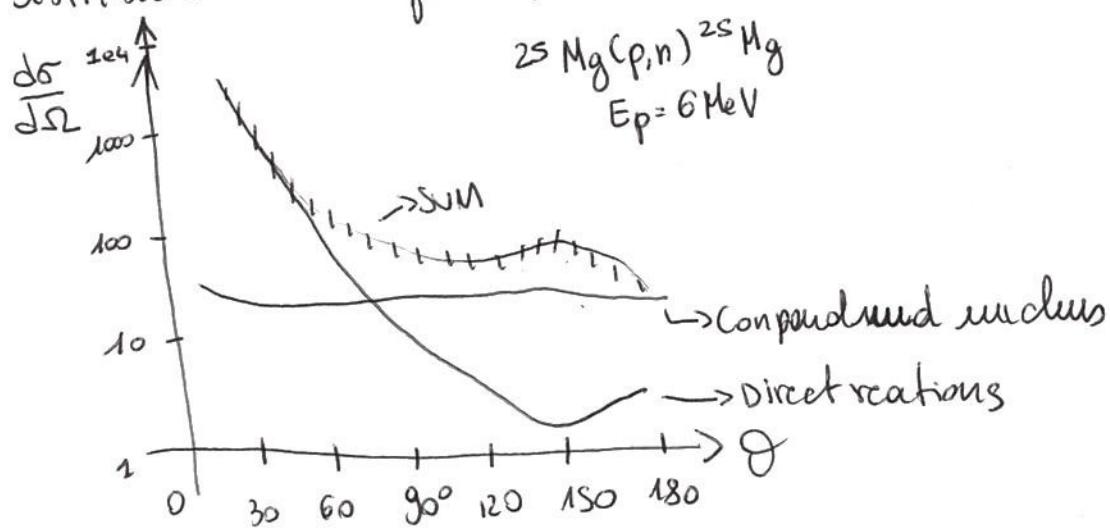


The agreement is remarkably good, showing that the decay of $^{64}\text{Zn}^*$ into a final state is \approx independent of how it was formed.

The compound-nucleus model works best for low incident energies (10-20 MeV) (\Rightarrow the incident nucleus has a low probability to escape) and $\sqrt{\text{A}}$ medium to heavy nuclei.

Another characteristic of compound-nuclear reactions is the angular distribution of products.

Because of the random interactions among nucleons, a isotropic angular distribution is expected.



DIRECT REACTION

At the opposite extreme from compound-nucleus reactions are direct reactions in which the incident particle interacts primarily off the surface of the target nucleus.

As the energy of the incident particle is increased, its de Broglie wavelength decreases, until it becomes more likely to interact with nucleon-sized object than with a nucleus-sized object.

A 1 MeV nucleon has a de Broglie length λ of

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{\sqrt{2m_p c^2 K}} \quad m_p = 10^3 \text{ MeV}, K = 1 \text{ MeV}$$

$$\lambda = \frac{2\pi\hbar c}{\sqrt{2m_p c^2 K}} = \frac{2\pi \cdot 200 \text{ MeV fm}}{\sqrt{2 \cdot 1000 \cdot 1 \text{ MeV}}} = \frac{2\pi \cdot 200}{\sqrt{2 \cdot \sqrt{5} \sqrt{2} \sqrt{5} \sqrt{2} \sqrt{5}}} = \frac{20\pi}{8\sqrt{5}} = \frac{50}{\sqrt{5}} \approx 20 \text{ fm}$$

While a 10 MeV nucleon has a $\lambda \approx 8 \text{ fm}$, still not enough to "see" the structure of nucleons. Increasing the energy the λ decreases and direct

102

processes might occur. Of course it may be possible that direct and compound-nuclear processes both contribute to a given reaction.

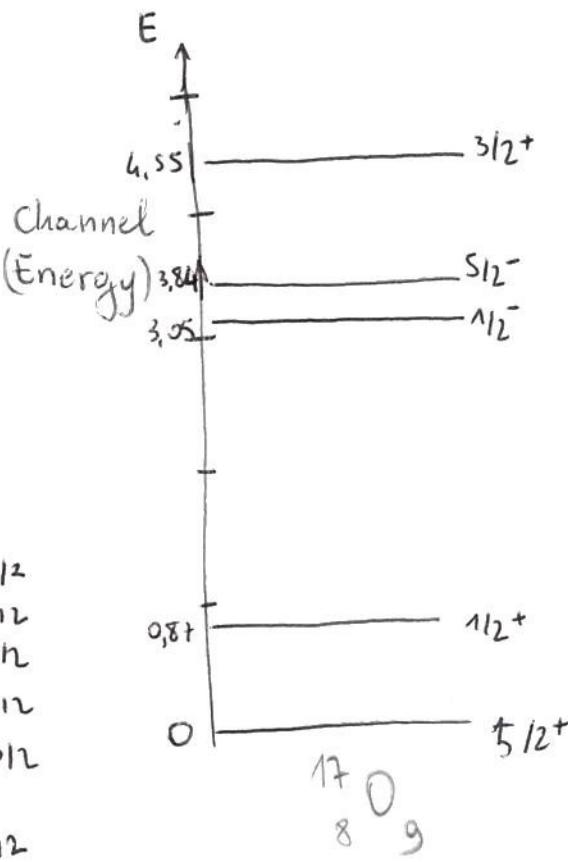
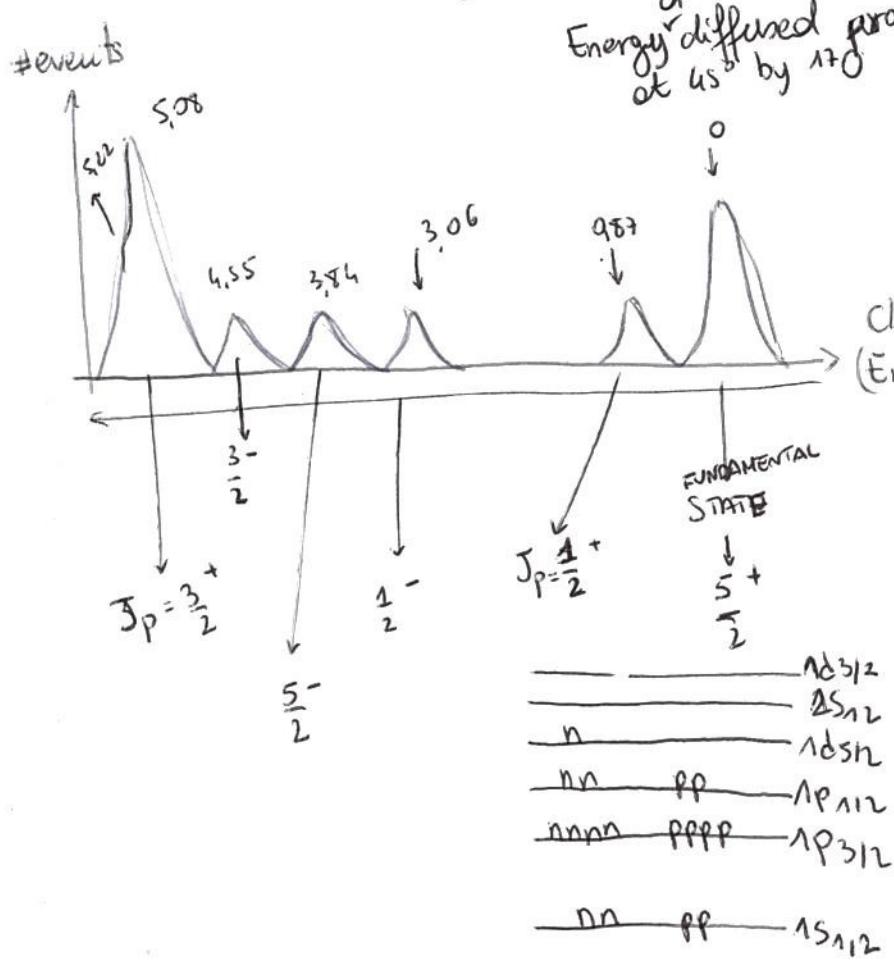
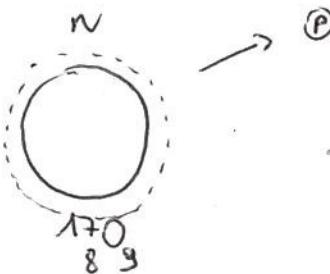
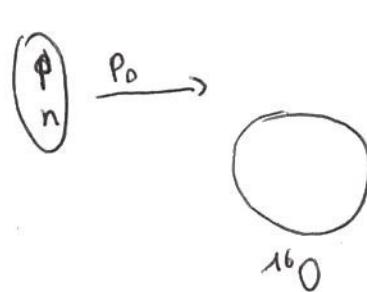
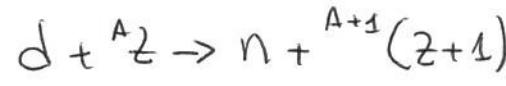
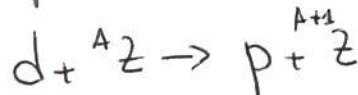
How to distinguish?

- 1) Direct processes occur very rapidly ($\approx 10^{-22}$ s) while compound-nuclear processes has a time $\approx 10^{-10} \div 10^{-18}$ s.
 - 2) The outgoing ptc in direct reactions tend to be sharply peaked at forward angles.

EXAMPLE OF DIRECT REACTIONS

- STRIPANG

One or more nucleons are transferred from projectile to target
- The simplest reactions are those induced by the d

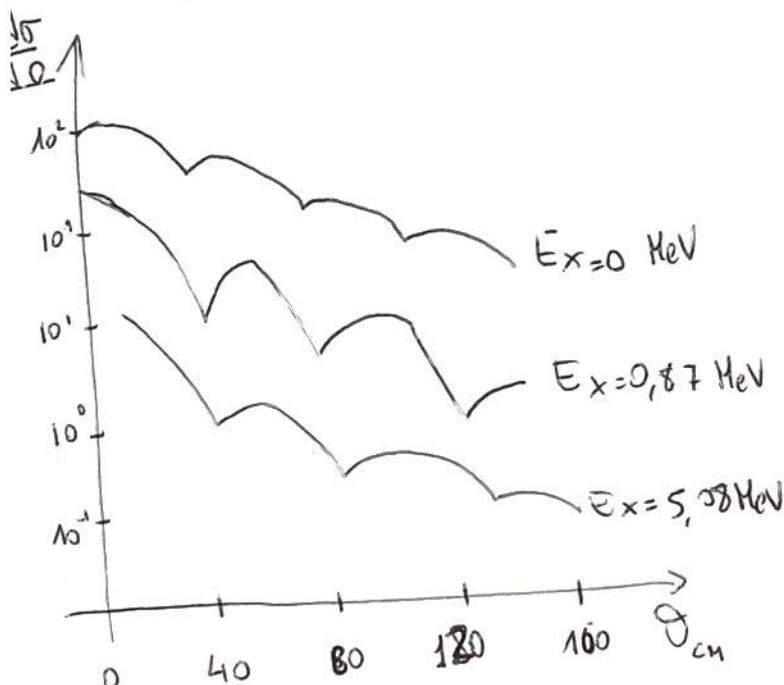


The angular momentum which is transferred in the reaction corresponds to the orbital angular momentum of the single neutron (in this specific case of $d + {}^{16}\text{O} \rightarrow p + {}^{17}\text{O}$) and it can be evaluated as:

$$|q| \approx \frac{\sqrt{l(l+1)\hbar^2}}{R} \approx \frac{l\hbar}{R}$$

$$\begin{aligned}|q|^2 &= P_d^2 + P_p^2 - 2 P_d \cdot P_p \cos\theta = (P_d - P_p)^2 + 2 P_p P_d (1 - \cos\theta) = \\ &= (P_d - P_p)^2 + 4 P_p P_d \sin^2 \frac{\theta}{2}\end{aligned}$$

Again, as we observed for the $e^- + A$ beam, the angular distribution of the diffracted p presents minima and maxima typical of the interference phenomenon. The first maximum tells us how many states are populated, i.e. the orbital from which the neutron has been transferred. It is then possible to determine the angular momentum of nuclear states.



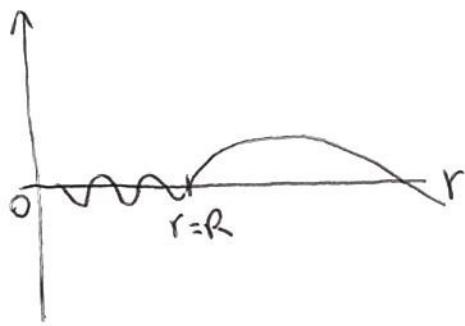
The maximum of the state with $E = 0.87 \text{ MeV}$ is at $\theta = 0^\circ \Rightarrow l = 0$
 \Rightarrow Shell S

Other states (~~(Ex = 5.08 MeV)~~) have a shifted maximum.
 E.g. $E = 5.08 \text{ MeV}$ has a maximum at $\theta = 15^\circ \Rightarrow 1$ unit of l is transferred \Rightarrow neutron in D shell

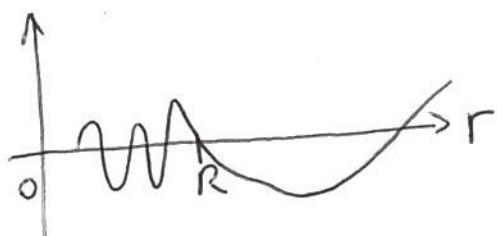
RESONANCE REACTIONS

Between the 2-extreme that we have described before there is the intermediate case of "resonance production": discrete levels in the compound-nucleus reaction. These levels have a high probability to be formed (i.e. large cross-section) and their width are very small because of low incident energy.

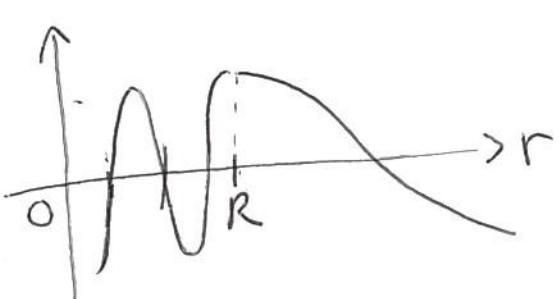
To obtain a qualitative understanding of the formation of resonances we can see the nuclear potential seen by the captured ptc as a square well potential. The w.f. outside and inside the well must be matched smoothly. The match can occur in several ways:



- a) In this case the incident ptc has a little probability to penetrate the nucleus and form the resonance



- b) In this case a higher probability is reached but the relative shift and the amplitudes have to be matched



- c) only certain incident energies give the right conditions to obtain a "resonance".

In a single isolated resonance of energy E_R and width Γ , the energy profile of the cross-section in the vicinity of the resonance will have the character of the energy distribution of any decaying state of lifetime $\gamma = \hbar/\Gamma$. The resonance will occur when the cross-section has a maximum.

The total cross-section, which include all the scattering and reaction processes can be written as:

$$\sigma_T = \sigma_{Sc} + \sigma_r = \sum_{l=0}^{\infty} 2\pi \lambda^2 (2l+1) (1 - R_e M_e)$$

Where $M_e = e^{2iS_e}$ and S_e = phase shift of l -th partial wave

Assuming only one partial wave l (the one that gives resonance), the resonance will occurs if $M_e = -1 \Rightarrow S_e = \frac{\pi}{2}$

The shape of the resonance can be obtained by expanding the phase shift around $S_e = \frac{\pi}{2}$. Better convergence is obtained if the expansion is done using the $\cot(S_e)$

$$\cot S_e(E) = \cot S_e(E_R) + (E - E_R) \left(\frac{\partial \cot S_e}{\partial E} \right)_{E=E_R} + \frac{1}{2} (E - E_R)^2 \left(\frac{\partial^2 \cot S_e}{\partial E^2} \right)_{E=E_R} + \dots$$

where $\left(\frac{\partial \cot S_e}{\partial E} \right)_{E=E_R} = - \left(\frac{\partial S_e}{\partial E} \right)_{E=E_R}$

Defining the width Γ as

$$\Gamma = 2 \left(\frac{\partial S_e}{\partial E} \right)_{E=E_R}^{-1}$$

Then the 2nd-orders terms vanishes and

$$\cot S_e = - \frac{E - E_R}{\Gamma/2}$$

Since Γ is the full width of the resonance the cross-section should fall to half of its central value at $E - E_R = \Gamma/2$.

This occurs when $S_e = \pm 1, \frac{\pi}{4}, \frac{3}{4}\pi$. It is then possible to define

$S_e = \frac{\Gamma}{2}$ is the center of resonance

$$\sin \delta_r = \frac{1/2}{[(E - E_R)^2 + \frac{\Gamma^2}{4}]^{1/2}}$$

And the scattering cross-section becomes:

$$\sigma_{sc} = \frac{\pi}{k^2} (2l+1) \frac{\Gamma^2}{(E - E_R)^2 + \frac{\Gamma^2}{4}}$$

This expression can be generalized in two ways. 1) The spin of reacting ptc have to be taken into account. If S_a and S_x are the spin of incident ptc and target, and I is the total angular momentum of the resonance, $I = S_a + S_x + l$

and $2l+1$ must be replaced with a general statistical factor

$$g = \frac{2I+1}{(2S_a+1)(2S_x+1)}$$

2) Partial entrance and exit width must be considered. (If resonance can decay in \neq ways)

$$\Gamma = \sum_i \Gamma_i$$

$\Gamma = \tau/\gamma$ is the same for all channels, because the resonance has a lifetime γ always.

For elastic reactions $a + X \rightarrow a + X$

$$\sigma = \frac{\pi}{k^2} g \frac{(\Gamma_{ax})^2}{(E - E_R)^2 + \frac{\Gamma^2}{4}}$$

For reactions like $a + X \rightarrow b + x$

$$\sigma = \frac{\pi}{k^2} g \frac{\Gamma_{ax} \Gamma_{bx}}{(E - E_b)^2 + \frac{\Gamma^2}{4}}$$

This happens because the Γ_{ax} denominator is for the "outgoing" ptc while the Γ_{ax} at numerator is for the formation process.

FIG. 11.28: Comparison DATA/MODEL fully fulfilled.