Elasticity

Mechanical properties used in describing linear elastic, isotropic deformation are **elastic modulus, *E*, and Poisson's ratio, ν**.

Other parameters, such as shear modulus *G* and bulk modulus *K*, can be calculated from the values of *E* and ν.

$K=E/3(1-2\nu)$ $G=E/2(1+\nu)$

These mechanical properties can be determined using standardized tests and basic testing equipment.

Other common metrics that are derived from a tensile test include

- Ultimate Tensile Strength (UTS),

- Yield Strength (σ_y),

- Failure Strength (σ_y),

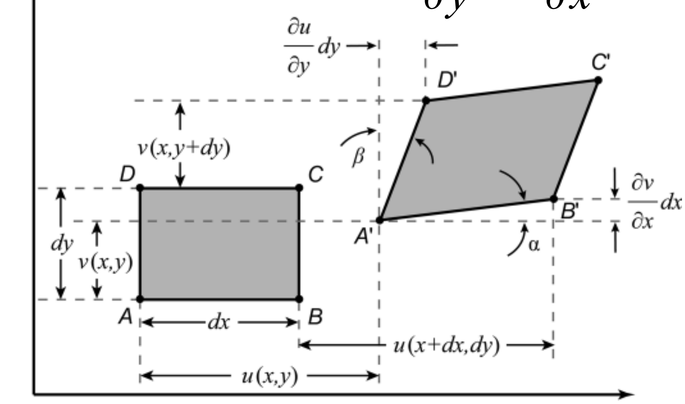
- Strain to failure (ε_f).

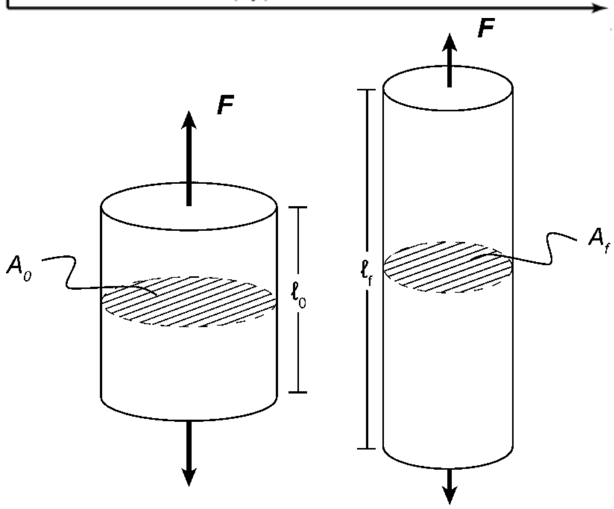
Stress and strain

When a component is loaded, there is normally some deformation in response to this loading. This deformation is captured in the concept of **strain**, a **non-dimensional quantity that describes the change in a component's physical configuration during loading.**

$$\text{Normal strain: } \varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\text{Shear strain: } \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$





$$\varepsilon = \frac{l_0 - l_f}{l_0} = \frac{\Delta l}{l_0}$$

STRESS

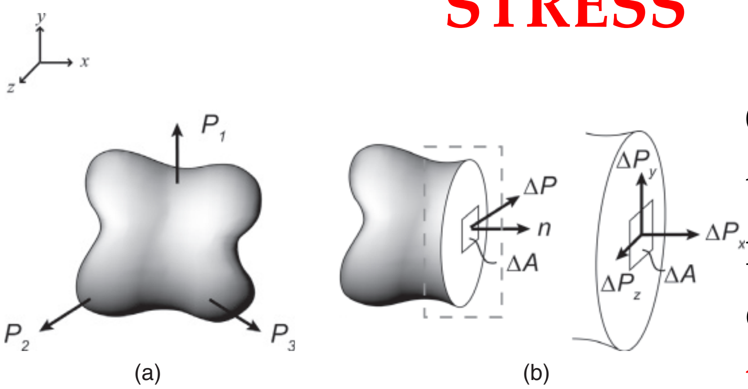


Figure 6.4

(a) An object under loading, showing (b) the definition of the traction vector.

Given the object in Figure, the **force vector** ΔP can be resolved into three coordinate directions, **the traction vector** is then defined as follow:

$$\tau_{xx} = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A}, \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A}, \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A}.$$

the first subscript of the traction vector refers to the plane on which the force is acting, and the second subscript refers to the direction of the force.

By convention, $\tau_{xx} = \sigma_x$, the **normal stress**, while τ_{xy} and τ_{xz} are **shear stresses**.

It is important to note here that the orientation of the surface ΔA will have an effect on the magnitude and direction of the stresses, even if the load ΔP remains the same.

In a simple uniaxial case stress is defined simply as: $\sigma = \dfrac{F}{A_0}$.

Stress tensor

imagine that there is an infinitesimal cube inside of an object under loading, whose sides are parallel with the three coordinate planes. In this configuration, **there are three normal stresses** (σ_x, σ_y, and σ_z) and **six shear stresses** ($\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{zx} = \tau_{xz}$).

The stress state of this object demonstrates observer invariance – that is, it does not change when viewed by different observers.

However, the components of stress, the normal and shear stresses that act on the orthogonal sides of the infinitesimal element, can be different in different configurations.

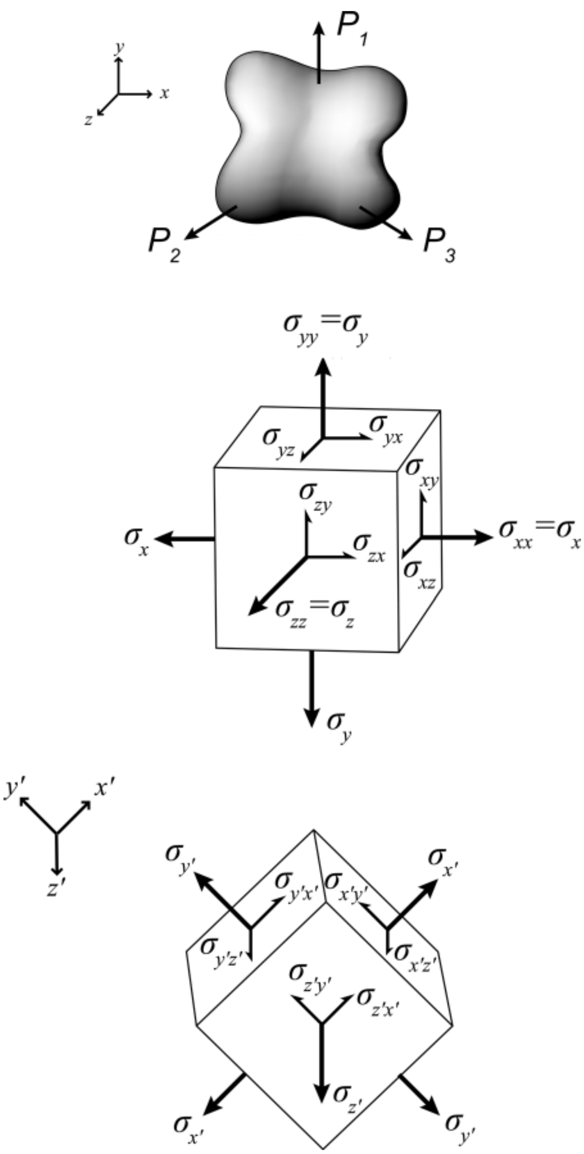


Figure 6.5

An infinitesimal cube inside an object under loading (top), before (middle), and after (bottom) rotation.

If this infinitesimal cube is rotated, the stresses in the new configuration will be related to the stresses in the original configuration, but they will generally not be of the same magnitude. **This change in stresses made by changing from one set of coordinate axes to another is known as a stress transformation.**

It is common to group the values for stress together in the **stress tensor**, defined as σ_{ij}, or:

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

This is sometimes written as

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}.$$

It can be shown that this tensor, order for a stress not to move the material, must be symmetric or that $\tau_{ij} = \tau_{ji}$

When an object is under stress, **this stress state can be broken into dilatational and deviatoric components.** The **dilatational** component is responsible for volume change and is sometimes referred to as the **hydrostatic component**, while the **deviatoric component** is responsible for shape change, or distortion

The **dilatational** stress tensor, $p\,\delta ij$, is written as:

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$p = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

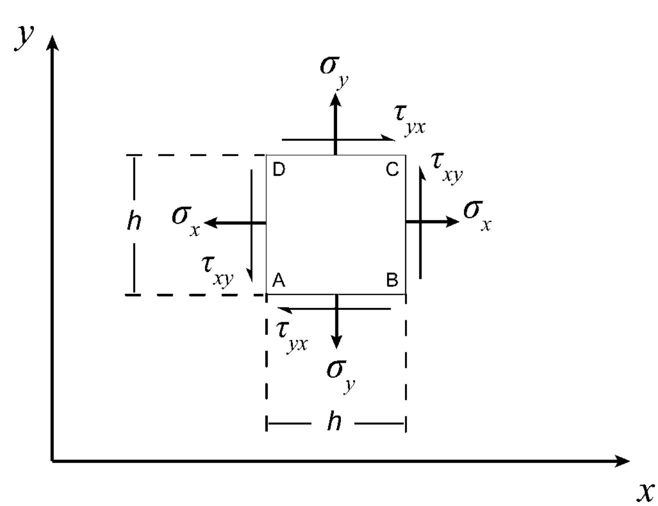


Figure 6.6

Summation of moments for the *x*-*y* plane of an infinitesimal element.

In **crystalline metals** plastic deformation occurs by **slip**, **a volume-conserving process that changes the shape of a material through the action of shear stresses**. On this basis, it might therefore be expected that **the yield stress of a crystalline metal does not depend on the magnitude of the hydrostatic stress**; this is in fact exactly what is observed experimentally.

In **amorphous metals**, a very slight dependence of the yield stress on the hydrostatic stress is found experimentally.

The **deviatoric stress tensor**, σ_{ij}, is defined as the difference between the stress tensor and the dilatational stress tensor as follows:

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}.$$

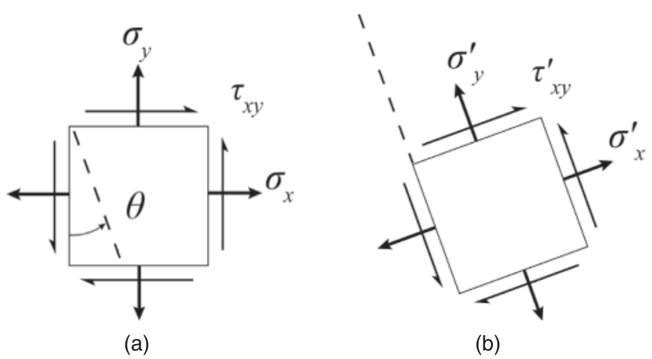


Figure 6.7

(a) An infinitesimal element, (b) undergoing rotation by an angle θ.

Sometimes it is useful to look at the stresses in several different sets of coordinate axes. It is common to apply coordinate transformations in order to find **the set of coordinate axes that have the highest stress values (either normal or shear) for yield or failure predictions**.

It can also be useful to know in which coordinate systems normal or shear stresses are minimized.

Imagine an infinitesimal element in plane stress as given in Figure a. If it is examined in a different coordinate system rotated by θ as shown in Figure b, the stresses σ_x, σ_y, and τ_{xy} are transformed to σ'_x, σ'_y, and τ'_{xy} respectively. It is important to remember that these quantities are merely a new (and equivalent) representation of the initial stress state.

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta.$$

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta.$$

The general stress tensor has six independent components and could require us to do a lot of calculations. To make things easier it can be rotated into the **principal stress tensor** by a suitable change of axes.

For every stress state, we can rotate the axes, so that the only non-zero components of the stress tensor are the ones along the diagonal:

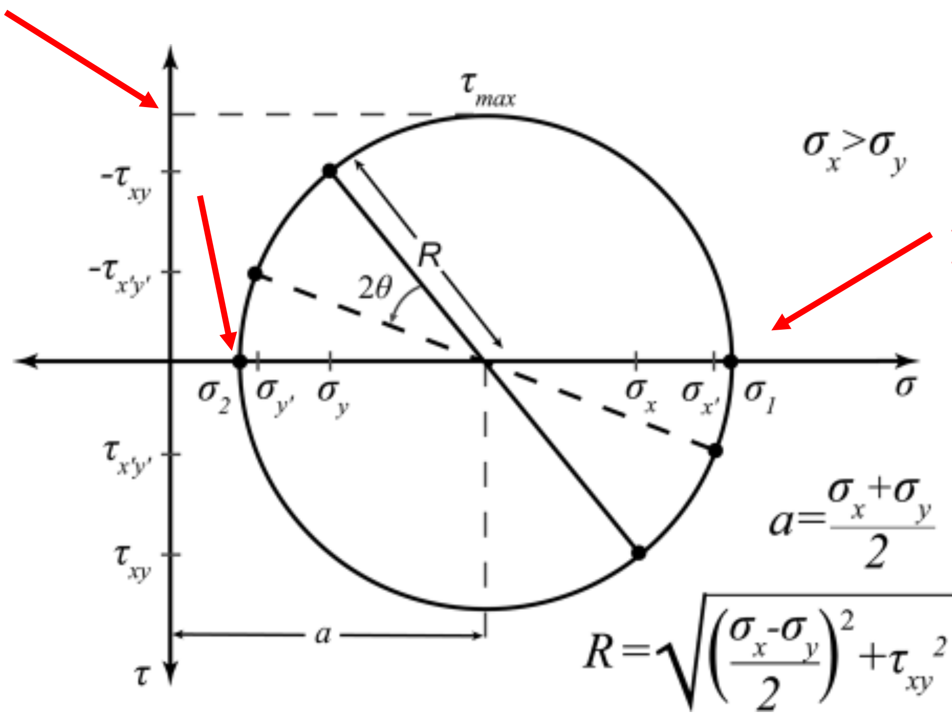
$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

that is, there are no shear stress components, only normal stress components. the elements σ_1, σ_2, σ_3 are the **principal stresses**. The positions of the axes now are the **principal axes.**

The largest principal stress is bigger than any of the components found from any other orientation of the axes. **Therefore, if we need to find the largest stress component that the body is under, we simply need to diagonalise the stress tensor**.

Remember – we have not changed the stress state, and we have not moved or changed the material – we have simply rotated the axes we are using and are looking at the stress state seen with respect to these new axes.

It is not necessary to memorize these derivations, as a convenient graphical method also exists for determining the stresses in a system after a change of coordinate axes. ***Mohr's circle***, introduced by Otto Mohr in the 1880s, can be used in any loading situation, although each plane rotation must be addressed separately.



plot the normal and shear stresses on the *x* and *y* faces as shown in Figure.

Figure 6.8

Mohr's circle representation of a two-dimensional problem.

Use these two points to draw a circle.

Rotation of an angle 2θ in Mohr's circle space represents a rotation of θ in actual space. For example, a rotation of 30◦ in actual space to reach maximum shear stress is represented by a rotation of 60◦ in Mohr's circle space. Again, geometry can be used to find the values for the normal and shear stresses after a coordinate transformation

Forces in an intramedullary rod

A tibial fracture occurs at 80° to the longitudinal axis of a tibia and is repaired using an intramedullary rod as shown in Figure.

When the person is standing on both feet, assuming her body weight, *BW*, is split evenly between her right/left tibias.

What are the normal and shear forces in the intramedullary rod in the plane of the fracture?

Also assume for this example that the rod has a diameter, *d*, and that it is carrying all of the load in the tibia.

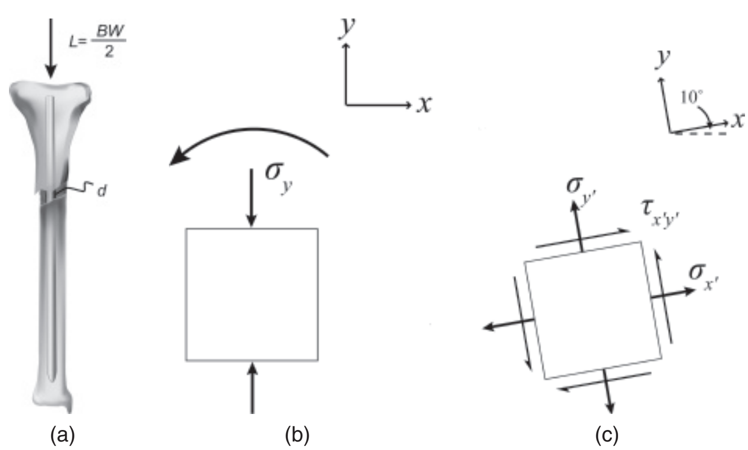


Figure 6.9

(a) A bone with fracture and intramedullary rod, (b) an infinitesimal element in this loading scheme, and (c) the infinitesimal element rotated to match the plane of the crack.

$$\sigma_y = \frac{-\frac{BW}{2}}{\frac{1}{4}\pi d^2} = -\frac{2BW}{\pi d^2}, \ \sigma_x = 0, \ \tau_{xy} = 0$$

$$\sigma_{x,80^\circ} = \frac{\sigma_y}{2} + \frac{-\sigma_y}{2}\cos 2(80^\circ) = -\frac{BW}{\pi d^2}(1 - \cos(160^\circ))$$

$$\sigma_{y,80^\circ} = \frac{\sigma_y}{2} + \frac{\sigma_y}{2}\cos 2(80^\circ) = -\frac{BW}{\pi d^2}(1 + \cos(160^\circ))$$

$$\tau_{xy,80^\circ} = \frac{\sigma_y}{2}\sin 2(80^\circ) = -\frac{BW}{\pi d^2}\sin(160^\circ).$$

Using Mohr's circle to determine maximum shear stress

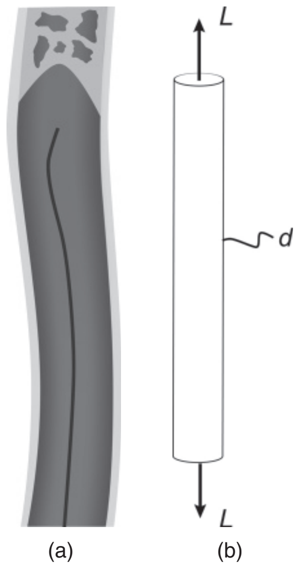
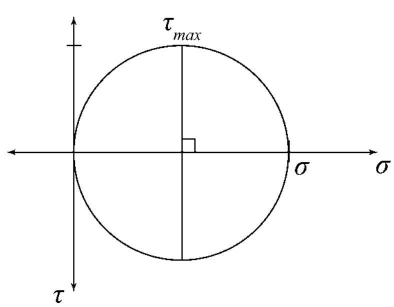


Figure 6.10

(a) A microguidewire in an artery, and (b) the same guidewire under tensile loading during retraction.

A micro-guide-wire used in catheterization is made from a ductile metal (Figure). Knowing that if it fails, it will fail in shear

Use Mohr's circle to draw the coordinate system that maximizes shear stress in order to find the weakest plane and the value for maximum shear stress.



$$\tau_{\max} = \frac{\sigma}{2} = \frac{2L}{\pi d^2} \text{ and occurs at } 45^\circ.$$

Figure 6.11

Mohr's circle representation of the stress state described in Example 6.2.

As described earlier, **finding the highest stresses is necessary when exploring failure or yield situations**. **Another way to represent this is to rotate the stress tensor such that all of the shear stresses are eliminated and the stress tensor can be written as**

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

with σ_1, σ_2, and σ_3 referred to as the **principal stresses** corresponding to **principal directions** p_1, p_2, and p_3.

One method for finding the principal stresses is to take the derivative of Equations

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{xy}' = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta.$$

with respect to θ and set equal to zero. The results will give the principal stresses in two dimensions:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

In the coordinate transformation that results in the principal stresses, **the shear stresses are found to be zero**.

Conversely, if the shear stresses in a representation are zero, then the normal stresses are the principal stresses.

It may also be useful to know the direction and value for maximum shear stress.
Using a similar technique, the maximum shear stress is found to be

$$\tau_{\max} = \left|\frac{\sigma_x - \sigma_x}{2}\right|$$

and the rotation necessary to achieve this is θ = 45°

Principal stresses in an artificial spinal disk

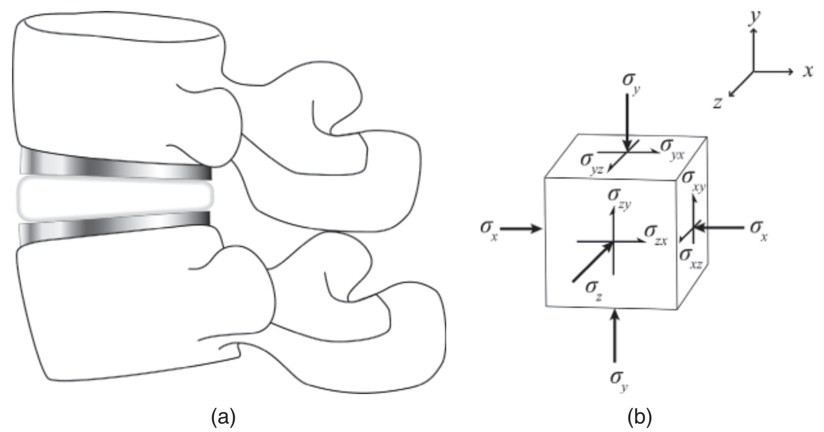
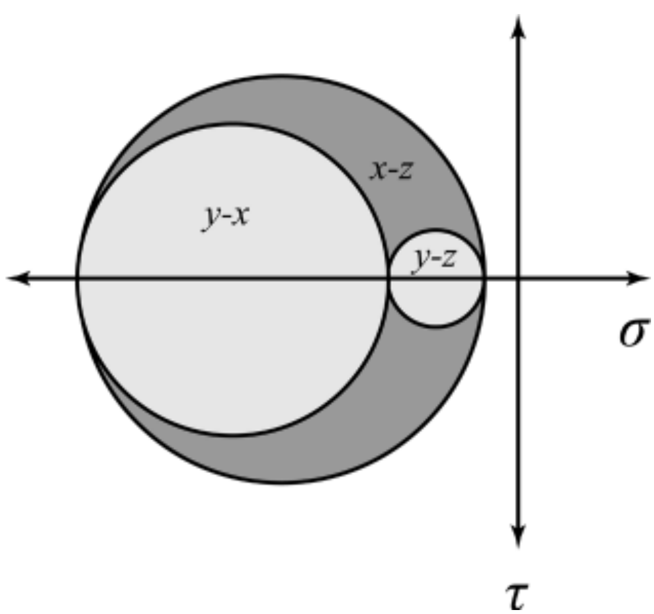


Figure 6.12

(a) An artificial intervertebral disk, and (b) the stress tensor associated with its loading.

An infinitesimal element in an artificial spinal disk is loaded as shown in Figure with $\sigma_x = -2.2$ MPa, $\sigma_y = -1.1$ MPa, $\sigma_z = -0.58$ MPa, $\tau_{xy} = -0.57$ MPa, $\tau_{yz} = -0.33$ MPa, $\tau_{zx} = -0.79$ MPa.

Use Mohr's circle to find the principal stresses in this situation.



From Mohr's circle the principal stresses are $\sigma_1 = -0.25$ MPa, $\sigma_2 = -0.86$ MPa, $\sigma_3 = -2.77$ MPa.

Figure 6.13

Mohr's circle representation of the stress state described in Example 6.3.

The **eigenvalues** of the stress tensor are the principal stresses and the **eigenvectors** are the principal directions.

Given the stress tensor in Equation

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}.$$

the principal stresses can be found by solving

$$\det[\sigma - \lambda I] = 0.$$

The principal directions are then easily found by solving

$$[\sigma - \lambda I][p] = 0.$$

AGAIN lets calculate the Principal stresses in an artificial spinal disk

$$\det \begin{bmatrix} -2.2-\lambda & -0.57 & -0.79 \\ -0.57 & -1.1-\lambda & -0.33 \\ -0.79 & -0.33 & -0.58-\lambda \end{bmatrix} = 0$$

$$(-2.2-\lambda)[(-1.1-\lambda)(-0.58-\lambda)-0.33^2]$$
$$+0.57[-0.57(-0.58-\lambda)+0.33(-0.79)]$$
$$-0.79[-0.57(-0.33)+0.79(-1.1-\lambda)]=0$$

$$\lambda_1 = -0.25\,\text{MPa}$$
$$\lambda_2 = -0.86\,\text{MPa (Principal stresses)}$$
$$\lambda_3 = -2.77\,\text{MPa}$$

$$\begin{bmatrix} 2.2-\lambda_i & 0.57 & 0.79 \\ 0.57 & 1.1-\lambda_i & 0.33 \\ 0.79 & 0.33 & 0.58-\lambda_i \end{bmatrix} \begin{bmatrix} p_{i,1} \\ p_{i,2} \\ p_{i,3} \end{bmatrix} = 0$$

$$p_1 = \begin{bmatrix} -0.86 \\ -0.36 \\ -0.36 \end{bmatrix},\ p_2 = \begin{bmatrix} 0.39 \\ -0.92 \\ 0.01 \end{bmatrix},\ p_3 = \begin{bmatrix} 0.34 \\ 0.13 \\ -0.93 \end{bmatrix} \text{ (Principal directions)}$$

Check orthogonality by verifying that $p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0$.
Check unit length by verifying that $|p_1| = |p_2| = |p_3| = 1$.

During uniaxial elastic deformation of an isotropic material, the relationship between stress and strain is linear and is known as **Hooke's Law** and is given as:

$$\sigma = E\varepsilon$$

In some cases, such as **metals**, even different microstructures will not affect the elastic modulus. However, for **polymers and soft tissues**, the measured value for E is highly dependent on test conditions such as strain rate

Multiaxial loading

Uniaxial deformation is not a realistic model for many medical device applications. For this reason, it is important to consider how equations such as Hooke's Law can be applied to a multiaxial loading situation.

The principle of **linear superposition**, which states that for a linear system, the overall response to two or more stimuli is equal to the sum of responses to those stimuli individually, will be used in this proof. In this case, this means that if the strain responses to applied loads in the *x*-, *y*-, and *z*-directions are analyzed individually, and then these responses are summed, the result is the strain responses for an object under multiple applied loads.

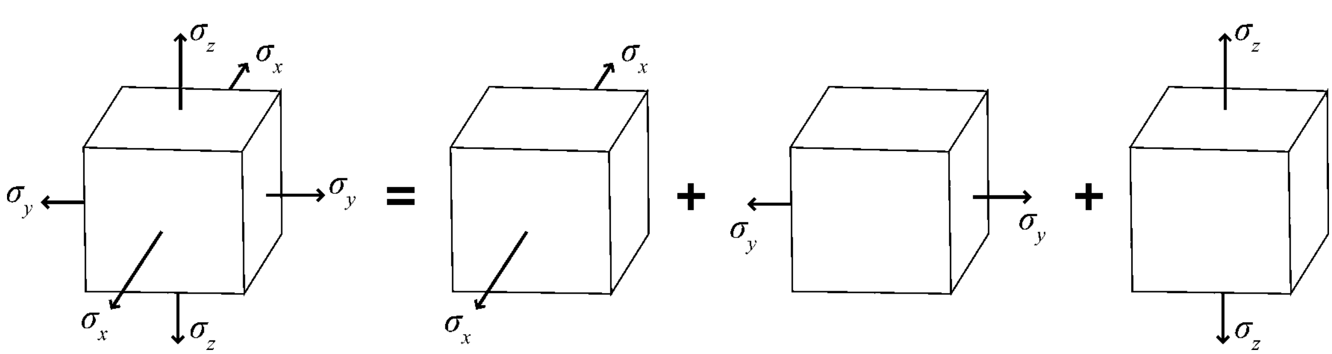


Figure 6.14

The summation of applied stresses leading to the development of three-dimensional Hooke's Law.

Table 6.1 Development of 3D Hooke's Law

	Strain response to:		
	σ_x	σ_y	σ_z
ε_x	$=\dfrac{1}{E}\sigma_x$	$=-\nu\varepsilon_y=-\dfrac{\nu}{E}\sigma_y$	$=-\nu\varepsilon_z=-\dfrac{\nu}{E}\sigma_z$
ε_y	$=-\nu\varepsilon_x=-\dfrac{\nu}{E}\sigma_x$	$=\dfrac{1}{E}\sigma_y$	$=-\nu\varepsilon_z=-\dfrac{\nu}{E}\sigma_z$
ε_z	$=-\nu\varepsilon_x=-\dfrac{\nu}{E}\sigma_x$	$=-\nu\varepsilon_y=-\dfrac{\nu}{E}\sigma_y$	$=\dfrac{1}{E}\sigma_z$

To derive three-dimensional Hooke's Law, sum the strain contributions from each stress:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Constrained loading in a tibial plateau

Imagine a design for a tibial plateau that can be modeled for simplicity as shown in Figure

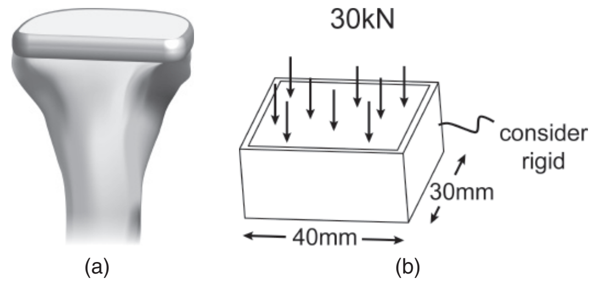


Figure 6.15

(a) A tibial plateau, which can be modeled by fully constrained loading as shown in

It is made of UHMWPE in a constraining frame of CoCr.

Given a load of 30 kN which is evenly spread across the tibial plateau, what are the stresses and strains that develop in this implant? Use E = 1 GPa and ν = 0.4 for UHMWPE and assume that the CoCr acts as a rigid constraint around the polymer.

$$\varepsilon_x = 0 = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = 0 = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Solve to get $\sigma_x = \sigma_y = 16.7$ MPa, $\varepsilon_z = 0.012$.

Isotropy/anisotropy

The equations in three-dimensional Hooke's Law assume that the material is **homogeneous and isotropic**, meaning that the deformation in response to load is invariant with respect to direction.

If a material is anisotropic, the general form of Hooke's Law is given as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

$S_{ij} = S_{ji}$ in the compliance matrix S.

For the **anisotropic** case, there are 21 independent constants needed to fully define the interactions between stress and strain.

Many standard engineering materials are considered to be isotropic.

The isotropic case can be defined by two independent constants, *E* and ν.

This special case is shown in the matrix form as:

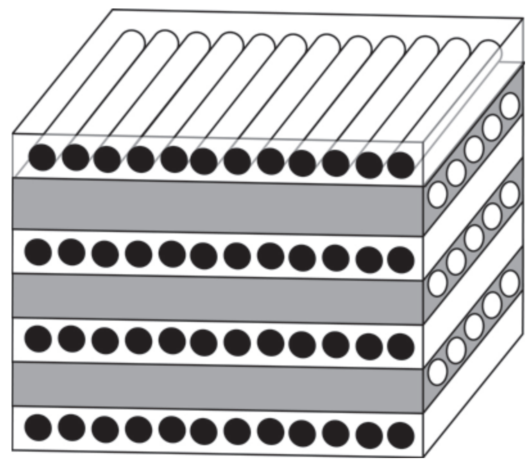
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}.$$

For simplicity, the shear modulus, *G*, is used instead of its representation using *E* and ν.

Two other special cases are of interest when considering biomaterials.

The first is **orthotropy**.

An orthotropic material possesses symmetry about three orthogonal planes, such as a composite material with fibers of different strengths laid 90° to each other, as illustrated in Figure.



(a)

Figure 6.16

Special cases of symmetry: (a) orthotropy and (b) transverse isotropy.

In this case, there will be three elastic moduli, E_x, E_y, and E_z, each associated with one plane of symmetry.

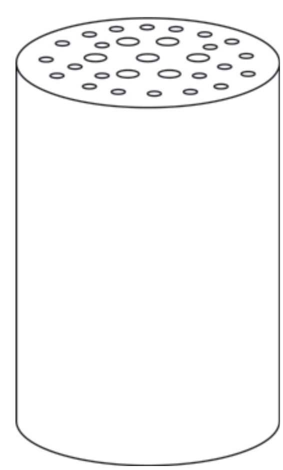
Although this example uses x, y, and z for the planes of symmetry, the planes do not need to be tied to the coordinate axis system.

There will also be three shear moduli, G_{xy}, G_{yz}, and G_{zx} and three Poisson's ratios, ν_{xy}, ν_{yz}, and ν_{zx}. It is important to remember that $\nu_{xy} = \nu_{yx}$ due to symmetry, **which is how the total number of independent constants for an orthotropic material is reduced to nine.**

The matrix form of Hooke's Law for an orthotropic material is given below:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}.$$

Another special case to be considered here is **transverse isotropy**.



(b)

In transverse isotropy, the mechanical properties are the same in a single plane (for example, the *x-y* plane) and different in the *z* direction. An example of this is shown in Figure.

Transversely isotropic materials have five independent constants: in this example, they are E_x and ν_{xy} for the *x-y* plane, and E_z, ν_{xz}, and G_{zx} for the z direction.

The matrix form is :

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_x} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_x} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{xz}}{E_x} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu_{xy})}{E_x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Finding strains in cortical bone

Cortical bone can be thought of as a transversely isotropic material. The compliance matrix for dry human femur is as follows (Yoon and Katz, 1976), with values in GPa

$$\underline{\underline{S}} = \begin{bmatrix} 0.053 & -0.017 & -0.010 & 0 & 0 & 0 \\ -0.017 & 0.053 & -0.010 & 0 & 0 & 0 \\ -0.010 & -0.010 & 0.037 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.115 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.115 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.139 \end{bmatrix}.$$

If the femur is loaded such that it experiences 5 MPa in compressive stress along the *z*-axis, what are the resulting strains?

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0.053 & -0.017 & -0.010 & 0 & 0 & 0 \\ -0.017 & 0.053 & -0.010 & 0 & 0 & 0 \\ -0.010 & -0.010 & 0.037 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.115 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.115 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.139 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.005 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.00005 \\ 0.00005 \\ -0.000185 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The bone will experience slight expansion in the x-y plane, and an even smaller compression in the z-direction.

Stress-strain curves

Bending stresses and beam theory

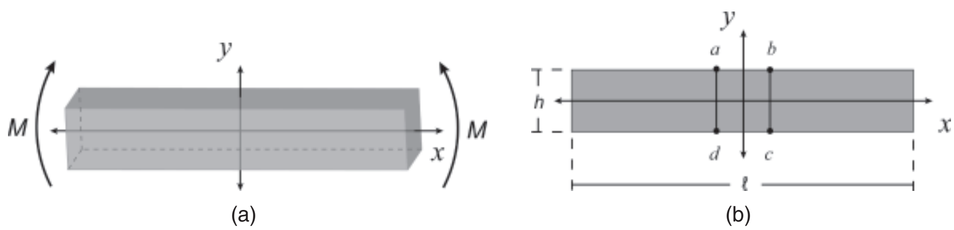


Figure 6.26

(a) A beam under bending moment with (b) marked segments that are perpendicular to the axis.

The stresses which develop due to the bending moment will be tensile on the convex side of the beam and compressive on the concave side of the beam. The plane at which the stresses induced by M_z are zero is known as the **neutral axis.**

$$\sigma_x = \frac{M_z y}{I_z}.$$

I = area moment of inertia

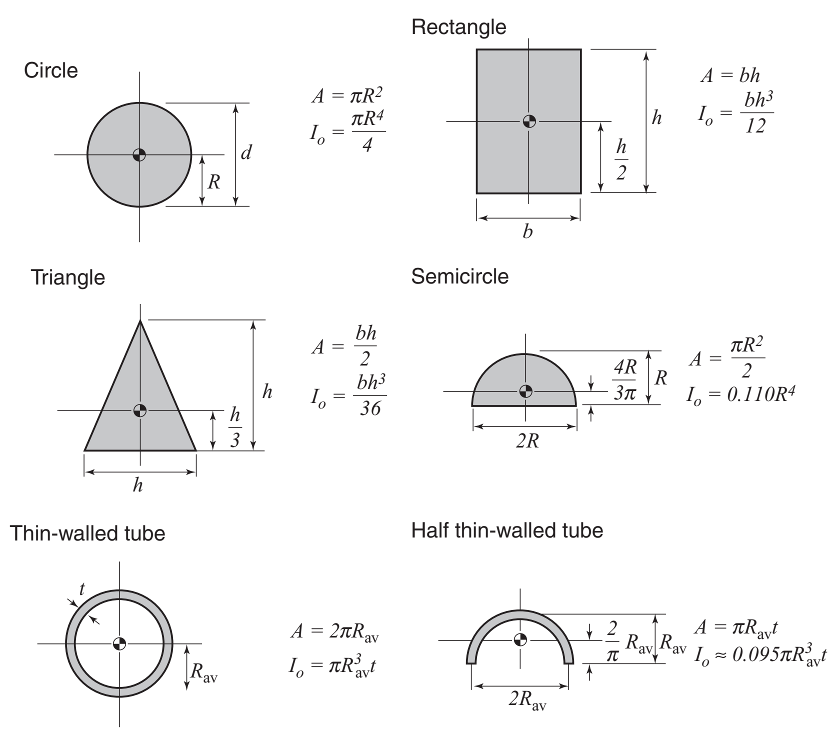


Figure A.1

Areas and moments of inertia around centroidal axes for basic geometries.

Designing a hip stem cross-section

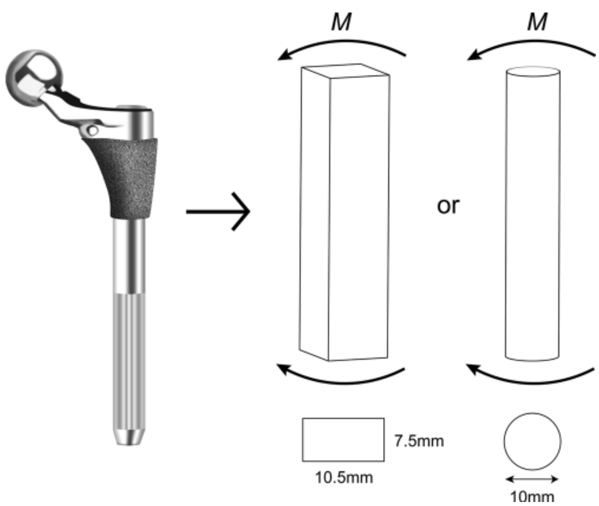


Figure 6.29

Circular and rectangular cross-sections for the hip stem described in Example 6.8.

Assess if the maximum stresses are different under a 50 N-m bending moment.

In order to keep the cross-sectional area the same, she decides to use the cross-sections shown in Figure.

Assuming they are both made of Ti-4Al-6V with E = 114 GPa, what are the maximum stresses in each hip stem?

Using the dimensions given $I_{rect} = 7.18\ m^4$ and $I_{circle} = 4.91\ m^4$. Substituting these values along with the bending moment and maximum distance from the neutral axis, the result is $\sigma_{max,rect} = 365$ MPa and $\sigma_{max,circle} = 509$ MPa.

The maximum stress is larger in the circular cross-section. **Other things to consider in selecting a cross-sectional geometry include ease of manufacture, reduced stress-concentrations, and stabilization inside the femur.**

Finding the neutral axis of a composite beam

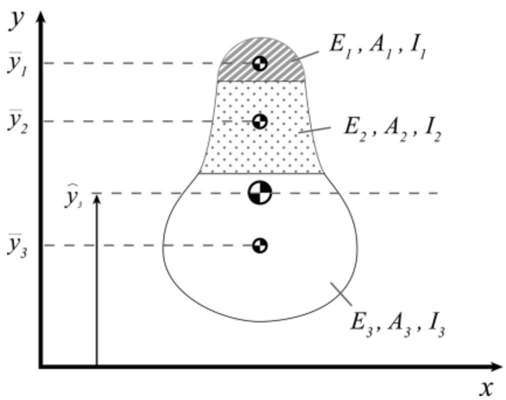


Figure 6.30

A generic composite beam, used to demonstrate the method for determining the location of the neutral axis.

it is assumed that all n sections are firmly bonded and that the cross-section is still symmetric in the z-plane

$\bar{y}_i$ represents the distance to the centroid of the individual sections, and t to represent general vertical distance from the point of interest to the neutral axis.

To find the neutral axis, use the following formula:

$$\hat{y} = \frac{\sum_{i=1}^{n} E_i \bar{y}_i A_i}{E_i A_i}.$$

Using $\bar{I}_i$ to represent the moment area of inertia for the section around its own centroid, the following equation, known as the Parallel Axis Theorem, gives the moment area of inertia around the neutral axis:

$$\hat{I}_j = \bar{I}_j + (\bar{y}_j - \hat{y})^2 A_j.$$

the stress induced by the bending moment within a particular section of the composite beam is given as

$$\sigma_{i,\text{bending}} = \frac{M t E_i}{\sum_{j=1}^{n} E_j \hat{I}_j}.$$

Bending of a composite beam

Consider a composite beam which consists of an UHMWPE and a Ti-6Al-4V beam perfectly bonded together for a custom implant that requires a polymer surface on one side and a metal surface on the other. Both beams have length and width $h = 20.0$ mm and are bonded as shown in Figure.

$E_{UHMWPE} = 1$ GPa and $E_{Ti\text{-}6Al\text{-}4V} = 114$ GPa.

What is the maximum stress in the UHMWPE beam under bending moment $M = 100.0$ N-m?

What is the maximum stress in the Ti-6Al-4V beam

Figure 6.31

A composite beam for a custom implant described in Example 6.9. The upper half is UHMWPE while the lower half is Ti-6Al-4V.

First, use Equation (6.55) to find the neutral axis.

$$\hat{y} = \frac{E_1 \bar{y}_1 A_1 + E_2 \bar{y}_2 A_2}{E_1 A_1 + E_2 A_2} = \frac{E_1 \left(\frac{h}{2}\right) h^2 + E_2 \left(-\frac{h}{2}\right) h^2}{E_1 h^2 + E_2 h^2} = -9.83\,\text{mm}$$

Substitute this value into Equation (6.57).

$$\sigma_{UHMWPE,\max} = \frac{M\,(h - \hat{y})\,E_1}{E_1 \left[\frac{h\,(h^3)}{12} + \left(\frac{h}{2} - (\hat{y})\right)^2 h^2\right] + E_2 \left[\frac{h\,(h^3)}{12} + \left(-\frac{h}{2} - (\hat{y})\right)^2 h^2\right]}$$
$$= -1.76\,\text{MPa}$$

$$\sigma_{Ti-6Al-4V,\max} = \frac{M\,(-h - \hat{y})\,E_2}{E_1 \left[\frac{h\,(h^3)}{12} + \left(\frac{h}{2} - (\hat{y})\right)^2 h^2\right] + E_2 \left[\frac{h\,(h^3)}{12} + \left(-\frac{h}{2} - (\hat{y})\right)^2 h^2\right]}$$
$$= -68.5\,\text{MPa}.$$

If the composite beam is under purely axial loading, the formula for calculating stress in a composite beam is

$$\sigma_{i,axial} = \frac{F E_i}{\sum_{j=1}^{n} E_j A_j}. \tag{6.58}$$

Composites
Finding upper and lower limit of E

When comppposites are composed of unidirectional fibers, they exhibit **orthotropic behavior**. ***Of particular interest in these cases are the upper and lower bounds for elastic modulus.***

Assume that the fibers are perfectly bonded such that there is no delamination and that the matrix is an isotropic material.

The total cross-sectional area will be referred to as *A*, while the total cross-sectional area of the fibers is *Af* and the total cross-sectional area of the matrix is *Am*. Thus

$$A = A_f + A_m. \qquad \varepsilon = \varepsilon_f = \varepsilon_m \qquad F = F_f + F_m.$$

$$\sigma A = \sigma_f A_f + \sigma_m A_m$$

$$E_{\text{upper}} \varepsilon A = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m.$$

$$V_f = \frac{A_f}{A} \text{ and } V_m = \frac{A_m}{A},$$

$$E_{\text{upper}} = E_f V_f + E_m V_m.$$

To find the lower bound, imagine transverse loading

$$\sigma = \sigma_f = \sigma_m.$$

The total length in this direction can be written as the sum of the total lengths of the fibers and matrix:

$$l = l_f + l_m. \tag{6.66}$$

Furthermore, the total change in length in this direction can be written as the sum of the changes in the fibers and in the matrix:

$$\Delta l = \Delta l_f + \Delta l_m. \tag{6.67}$$

As described earlier, the definitions for strain can be written

$$\varepsilon = \frac{\Delta l}{l}\ \varepsilon_f = \frac{\Delta l_f}{l_f} \varepsilon_m = \frac{\Delta l_m}{l_m} \tag{6.68}$$

Combining Equations (6.67) and (6.68) gives

$$\varepsilon = \frac{\varepsilon_f l_f + \varepsilon_m l_m}{l}. \tag{6.69}$$

Using Equations (6.13) and (6.69) and volume fraction definitions $V_f = \dfrac{l_f}{l}$ and $V_m = \dfrac{l_m}{l}$, the following result is found:

$$\frac{1}{E_{\text{lower}}} = \frac{V_f}{E_f} + \frac{V_m}{E_m} \text{ or } E_{\text{lower}} = \frac{E_f E_m}{E_f V_m + E_m V_f}. \tag{6.70}$$

Developing optimal thicknesses in layered devices

An engineer is designing a hydroxyapatite-coated polymer sleeve to assist with bone ingrowth in a total joint replacement. The cross-section of the proposed device is shown in Figure. Although the inner and outer diameters of the device are fixed at 6 mm and 12 mm, respectively, the thickness of the two layers needs to be optimized such that the overall elastic modulus of the device matches that of bone. Using these parameters, find the optimal thickness for each layer, using the following values: $E_{polymer}$ = 10 GPa, E_{HA} = 27 GPa, E_{bone} = 17 GPa.

Assume that the overall elastic modulus follows the rule of mixtures, that is:

$$E_{\text{device}} = \frac{E_{\text{polymer}} A_{\text{polymer}} + E_{HA} A_{HA}}{A_{\text{polymer}} + A_{HA}}.$$

The area of the polymer cross-section and the HA cross-section both depend on the desired value, the thickness of the polymer layer, t.

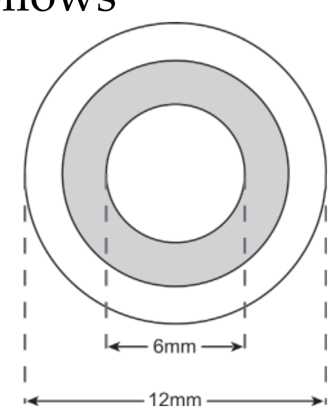


Figure 6.34

The cross-section of a hydroxyapatite-coated polymer sleeve

$$A_{\text{polymer}} = \pi \lfloor (r_{\text{polymer}} + t)^2 - r^2_{\text{polymer}} \rfloor$$

$$A_{HA} = \pi [r^2_{HA} - (r_{\text{polymer}} + t)^2].$$

Plug these into the equation for E of the device and set equal to 17 GPa. Solve for t to find t = 2 mm. **The optimal thickness of the polymer layer is 2 mm, leaving 1 mm for the HA layer.**

Modifying material and cross-section to reduce bone absorption

A primary consideration is to ensure that the bone is carrying enough load that it does not suffer from **stress shielding**.
In this case study, two factors in the design of a hip stem will be considered: **material choice and hip stem diameter.**

The materials :

CoCr alloy or Ti.

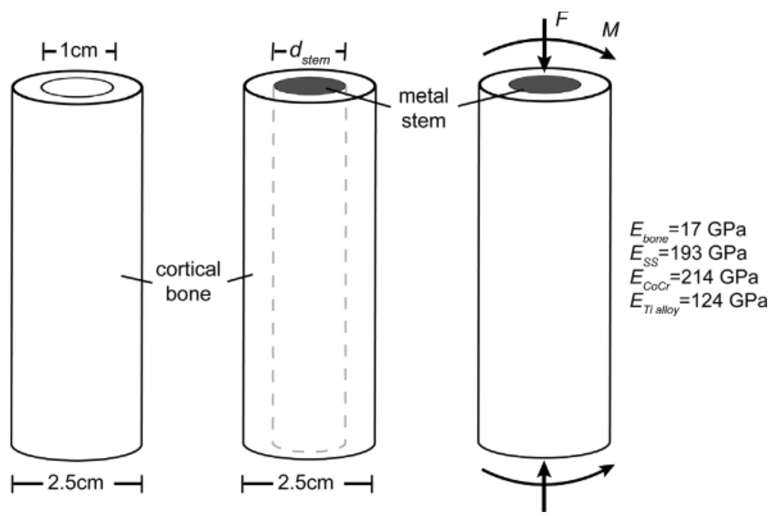


Figure 6.35

A schematic representation of a hip stem fixed into a femur.

The outer diameter of the bone is assumed to be 2.5 cm, and the inner diameter of the bone is 1.0 cm.
The hip stem will have an outer diameter of 1.1 cm (d_1) or 1.5 cm (d_2).
Using the geometry and material properties given in Figure, the stresses in the bone with each stem can be calculated by evaluating the stresses for an axial load of 2851 N (4 × Body Weight for a 160 lb person) and a bending moment of 30 N-m separately and then summing them.

$$\sigma_{\text{bone,axial}} = \frac{E_{\text{bone}} F}{E_{\text{bone}} A_{\text{bone}} + E_{\text{stem}} A_{\text{stem}}}$$

$$\sigma_{\text{bone,bending}} = \frac{E_{\text{bone}} M \left(\frac{d_{\text{bone}}}{2}\right)}{E_{\text{bone}} I_{\text{bone}} + E_{\text{stem}} I_{\text{stem}}}$$

Stress due to axial load [MPa]		
bone alone	6.91	
	d_1	d_2
bone with SS	1.93	1.23
bone with CoCr	1.79	1.12
bone with Ti	2.62	1.78

Stress due to bending load [MPa]		
bone alone	20.07	
	d_1	d_2
bone with SS	14.09	8.35
bone with CoCr	13.63	7.82
bone with Ti	15.82	10.77

Total stress [MPa]		
bone alone	26.98	
	d_1	d_2
bone with SS	16.02	9.58
bone with CoCr	15.43	8.94
bone with Ti	18.44	12.55

worst-case scenario from a stress-shielding point of view is the CoCr stem with the 1.5 cm diameter. The best-case scenario is the titanium alloy stem with the 1.1 cm diameter,

Failure theories

How would you safely design a tibial insert of a total knee replacement that is known to experience a complex loading state with a normal stress component that is on the order of the uniaxial strength for this material?

The inquiry posed above represents a realistic design challenge that one might face in the field of orthopedics. Many of the **tibial** components used in total knee arthroplasty utilize UHMWPE with a uniaxial **yield stress** on the order of 20 MPa; yet, the contact pressures for many of the clinical designs exceed this value.

In order to assess the likelihood for failure owing to yield or plastic deformation, it is important to calculate the **effective stress** that provides a scalar representation of the multiaxial stress state acting on the implant.

It is the effective stress that must be compared to the uniaxial yield strength as an assessment for the **factor of safety** against failure. Furthermore, localized plastic damage due to the presence of a notch or stress concentration can serve as a nucleation site for cracks if the component undergoes cyclic loading conditions.

In general, ductile materials *yield before fracture* while brittle materials *fracture before yield*.

The **yield strength** of a material is defined as the stress at which plastic (permanent) deformation begins. The **modulus of resilience** for a material is defined as the energy that is stored in a material until the onset of yielding.

The tensile yield strength provides the stress level at which permanent deformation will occur in an isotropic material subjected to a one- dimensional (axial) tensile stress.

This material property serves as an important design parameter, as it represents the upper limit of stress that can be applied without incurring plastic deformation to the component.

Ductile materials generally deform through shear in response to generalized states of stress.
Consequently the yield criteria developed for ductile metals are based on localized maximum shear stress (planes of maximum shear stress) or distortional energy

Henri Tresca developed the first criterion for yield in 1864 – ***this theory utilizes the maximum shear stress as the predictor of plastic deformation in metals***.

The other well-known criterion was established in 1913 by **Richard von Mises**, ***who utilized the distortional energy as a basis for yield in ductile materials***.
The Tresca and von Mises yield criteria are commonly employed to this day.

Brittle materials, are weak in tension and their failure modes utilize normal stresses rather than shear stresses. In brittle materials, the generalized failure criterion is based on the normal stress, or principal stresses, reaching the ultimate strength of the material.

A **yield surface** is the surface within the space of stresses that defines the boundary between elastic and plastic behavior for a material

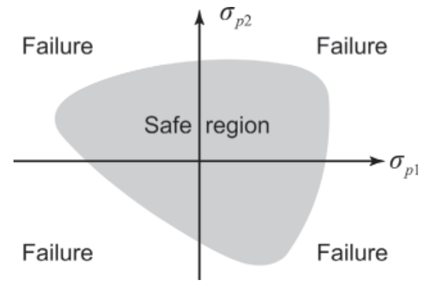


Figure 8.3

Hypothetical yield surface for a planar (two-dimensional) stress state.

The yield surface is generally represented in principal stress space or space defined by the stress invariants. The stress invariants used to describe the yield surface are given as:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$
$$J_3 = \det(s) = s_1 s_2 s_3$$

where σ is the Cauchy stress (true stress), σ_1, σ_2, σ_3 are the principal values of σ , and ***s*** is the deviatoric part of the stress whose principal values are s_1, s_2, s_3.

These stress invariants are utilized in the fundamental failure theories discussed below – Tresca employs the use of the maximum shear stress, von Mises makes use of the deviatoric part of the stress (J_2 invariant), and the normal stress criteria utilize the principal normal stresses.

Maximum shear stress (Tresca yield criterion)

This criterion is based on the notion that yielding (slip) occurs when the maximum shear stress reaches the yield stress determined from the uniaxial tensile test

The relationship between the principal stresses and planes of maximum shear stress are readily visualized using the Mohr circle

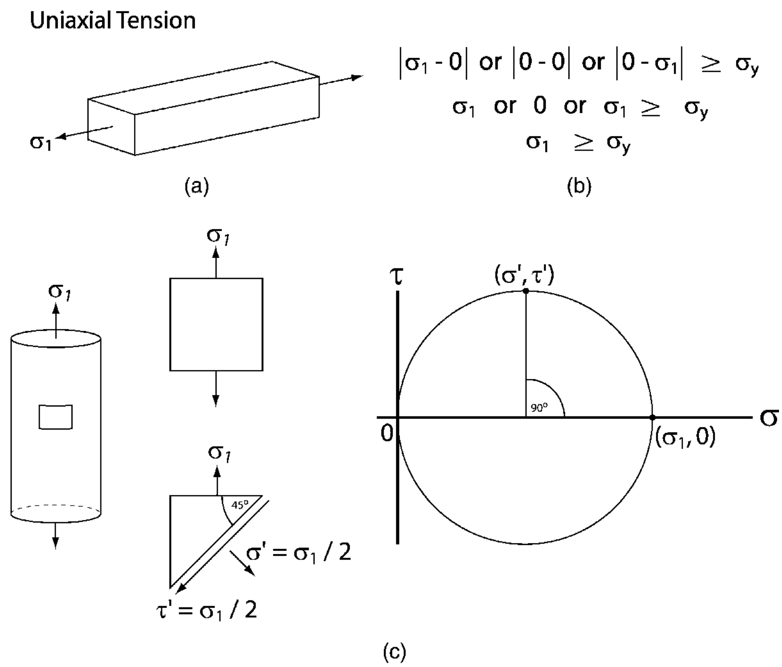


Figure 8.4

Illustration shows the Mohr circle for uniaxial loading and the relationship between maximum shear stress and the principal (normal) stress. Yielding occurs when the principal stress is greater than or equal to the yield strength measured in the tensile test. The figure shows (a) uniaxial tension, (b) resulting equations, and (c) relationship depicted using Mohr's circle.

The planes of maximum shear stress are oriented at $\theta = 45^{\circ}$ to the principal stresses (recall that the Mohr circle plots shear as an angular function of normal stress using 2θ and that maximum planes of shear stress are oriented at 90° to the principal stress direction). For uniaxial loading, the maximum shear stress occurs at $\sigma y/2$, and thus we can write:

$$\tau_{\max} = \frac{|\sigma_1 - 0|}{2} = \frac{\sigma_y}{2}$$

for

$$\sigma_1 = \sigma_y, \sigma_2 = \sigma_3 = 0$$

The general three-dimensional Tresca yield criterion is founded upon the notion that the plane of greatest shear stress dictates the maximum overall shear stress, and is given as:

$$\tau_{\max} = \tau_f = \frac{\sigma_y}{2} = MAX\left\{\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right\}.$$

Determining the Tresca stress in a spinal implant

Determine the Tresca stress for the infinitesimal element in an artificial spinal disk that is loaded as shown in Figure.

If the uniaxial yield strength of the implant material is 8 MPa, is the device safe from yielding?

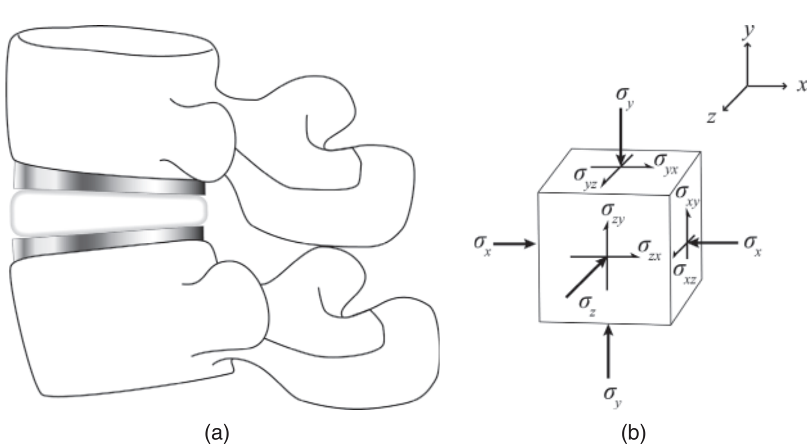


Figure 6.12

(a) An artificial intervertebral disk, and (b) the stress tensor associated with its loading.

$$\sigma_x = -2.2\,\text{MPa}, \sigma_y = -1.1\,\text{MPa}, \sigma_z = -0.58\,\text{MPa}$$
$$\tau_{xy} = -0.57\,\text{MPa}, \tau_{xz} = -0.79\,\text{MPa}, \tau_{yz} = -0.33\,\text{MPa}$$

Mohr's circle was used in Example 6.3 to find the principal stresses:

$$\sigma_1 = -0.25\,\text{MPa}, \sigma_2 = -0.86\,\text{MPa}, \sigma_3 = -2.77\,\text{MPa}$$

Using the expression for the Tresca yield criterion (Equation 8.4), the effective Tresca stress is:

$$\begin{aligned}
\tau_{\max} = \frac{\sigma_y}{2} &= MAX\left\{\frac{|\sigma_1-\sigma_2|}{2}, \frac{|\sigma_2-\sigma_3|}{2}, \frac{|\sigma_1-\sigma_3|}{2}\right\} \\
&= MAX\left\{\frac{|-0.25-(-0.86)|}{2}, \frac{|-0.86-(-2.77)|}{2}, \frac{|-0.25-|(-2.77)}{2}\right\} \\
&= MAX\,\{0.305\,\text{MPa},\ 0.95\,\text{MPa},\ 1.26\,\text{MPa}\} = 1.26\,\text{MPa} \\
\sigma_{\text{Tresca}} &= 2\tau_{\max} = 2(1.26) = 2.52\,\text{MPa}
\end{aligned}$$

Yielding occurs when the Tresca stress reaches the uniaxial yield strength. If the Tresca stress is less than the yield strength, then the material is safe from yield:

$$\begin{aligned}
\sigma_{\text{Tresca}} &= \sigma_{\text{yield}}\,\text{(Yields)} \\
\sigma_{\text{Tresca}} &= 2.52\,\text{MPa} < \sigma_{\text{yield}} = 8\,\text{MPa}\,\text{(No Yielding)}
\end{aligned}$$

The factor of safety (FS) against yielding is defined as the ratio of the yield stress normalized by the effective (Tresca) stress:

$$\text{FS} = \frac{\sigma_{\text{yield}}}{\sigma_{\text{Tresca}}} = \frac{8\,\text{MPa}}{2.52\,\text{MPa}} = 3.17.$$

The factor of safety for this material utilized in the spine application is 3.

von Mises yield criterion

Maximum distortional energy

This criterion is based on the view that yielding occurs when the maximum distortional energy associated with the combined stress state reaches the uniaxial yield strength.

The symmetric stress tensor comprises both normal stress components and shear stress components and **can be decomposed into dilatational and distortional components**. The dilatational portion of stress is responsible for volume change and is controlled by the normal stresses (imagine a cube that becomes a larger cube under the action of hydrostatic stresses). The distortional portion of stress results in shape change but no volume change and is controlled by the shear stresses (imagine a deck of cards that is sheared).

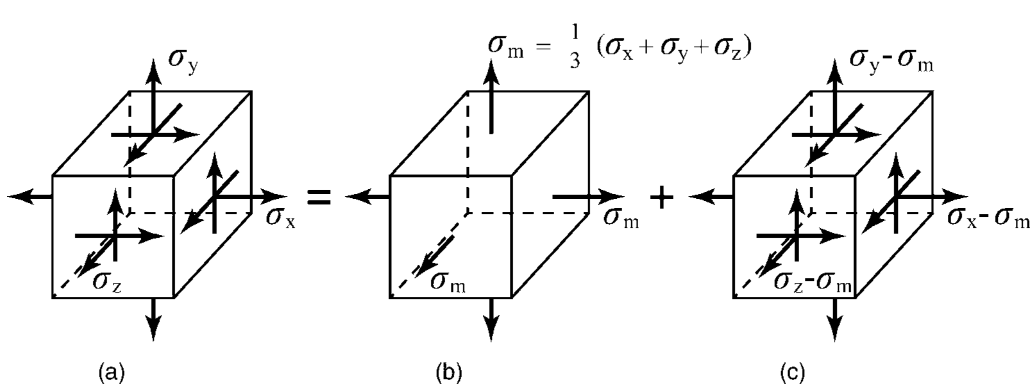


Figure 8.7

Illustration depicting the decomposition of (a) state of stress into (b) dilatational stresses and (c) distortional or deviatoric stresses.

This failure stress is often denoted as the von Mises effective stress and can be written also as:

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.$$

Another advantage of the von Mises yield criterion is that it can also be written in general stress component terms:

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}. \quad (8.19)$$

the maximum shear stress criterion is more conservative than the criterion using the distortional energy; however, the von Mises yield criterion is known to better match experimental data for many alloy systems and is commonly employed in design for the calculation of an effective stress for a component subjected to multiaxial loading.

Determining the von Mises effective stress in a tibial plateau

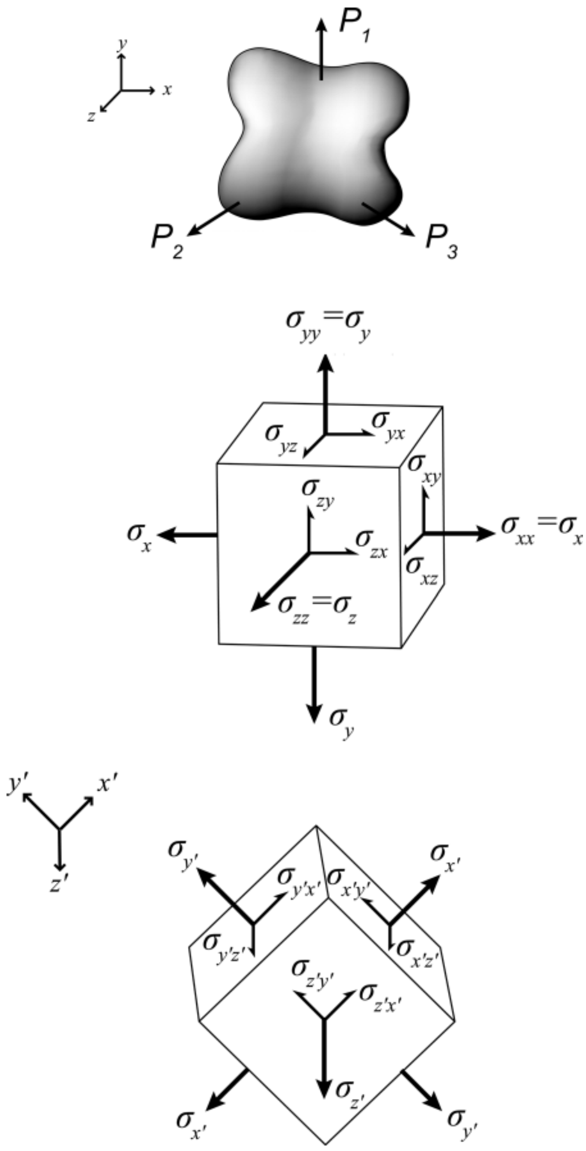


Figure 6.5

An infinitesimal cube inside an object under loading (top), before (middle), and after (bottom) rotation.

This implant is made of UHMWPE in a constraining frame of CoCr and loaded with a force of **30 kN** that is evenly spread across the tibial plateau. **UHMWPE has a uniaxial yield strength of 22 MPa**. Is the device safe from yielding?

Applying the 3D Hooke's Law and geometric constraints provides the stresses and strains

$$\sigma_x = \sigma_y = 16.7\,\text{MPa}$$
$$\sigma_z = 25\,\text{MPa}$$
$$\tau_{xz} = \tau_{xy} = \tau_{yz} = 0\,\text{MPa}$$
$$\varepsilon_x = \varepsilon_y = 0$$
$$\varepsilon_z = 0.012$$

The effective von Mises stress is found by employing Equation (8.19):

$$\begin{aligned}
\sigma_{eff} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)} \\
&= \frac{1}{\sqrt{2}}\sqrt{(16.7-16.7)^2+(16.7-25)^2+(25-16.7)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{2\,(8.3)^2} = 8.3\,\text{MPa}
\end{aligned}$$

The material is safe from yielding if the effective (von Mises) stress is less than the uniaxial strength:

$$\begin{aligned} \sigma_{eff} &< \sigma_y \\ 8.3\ \text{MPa} &< 22\ \text{MPa} \end{aligned}$$

The factor of safety (FS) against yielding is found by normalizing the yield strength with the effective stress:

$$\text{FS} = \frac{\sigma_y}{\sigma_{eff}} = \frac{22\ \text{MPa}}{8.3\ \text{MPa}} = 2.65.$$

The primary benefit of developing a general yield criterion, either through maximum shear stress or maximum octahedral stress, is that a scalar representation of the three-dimensional stress state can be used in combination with the uniaxial yield strength to predict the likelihood of failure. The **factor of safety (FS)** can be calculated from the ratio of the uniaxial yield strength to the effective von Mises stress

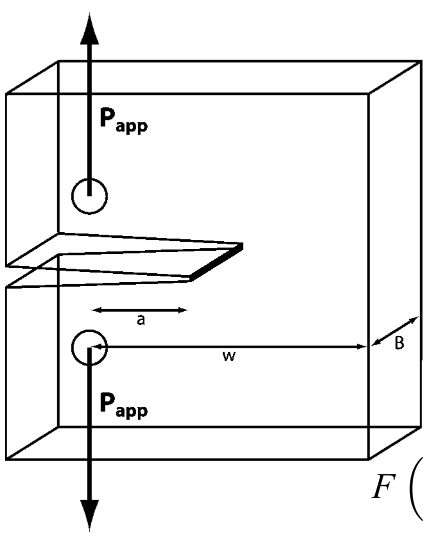
In general, the factor of safety should be at least a factor of 2–3 lower than the uniaxial yield strength. For safety-critical applications, however, this is often insufficient, and factors of safety are often extended and additional analysis pertaining to fracture, fatigue, and wear are employed for complete structural analysis.

Notches and stress concentrations

$$K = Y\sigma_{\infty}\sqrt{\pi a}$$

K is the stress intensity factor

K is a *single parameter* that describes completely the severity of the stress in the singularity region near a crack tip, which is of primary interest in fracture mechanics.



$$F\left(\frac{a}{W}\right) = \frac{2+(a/W)}{(1-(a/W))^{3/2}}[0.866 + 4.64(a/W) - 13.32(a/W)^2 + 14.72(a/W)^3 - 5.60(a/W)^4]$$

Figure 9.3

Compact tension (CT) specimen geometry with crack length *a* and specimen length *W*, as measured from the load line. The specimen thickness (out of the page) is denoted *B*.

Estimating fracture toughness from fractography of a tibial implant

This implant is made of UHMWPE in a constraining frame of CoCr and loaded with a force of 30 kN that is evenly spread across the tibial plateau.

UHMWPE has a yield strength of 22 MPa and an elastic modulus of 1 GPa.

A fractured component reveals an embedded penny-shaped flaw that served as the initiation site for fast fracture. The fractography reveals that the radius of the flaw is 25 μm.

The maximum far field stress is 25 MPa

Estimate the fracture toughness for this material using the fractography (fracture surface image) and knowledge of stresses on the system.

from previous example:

$\sigma_x = \sigma_y = 16.7$ MPa; $\sigma_z = 25$MPa

$\tau_{xz} = \tau_{xy} = \tau_{yz} = 0$ MPa

$\varepsilon_x = \varepsilon_y = 0$; $\varepsilon_z = 0.012$

The effective von Mises stress was found $\sigma_{eff} = 8.3$ MPa.

$$K_I = \sigma^{\infty}\sqrt{\pi a} = 25\,\text{MPa}\sqrt{\pi}\sqrt{25 \times 10^{-6}(\text{m})} = 0.221\,\text{MPa}\sqrt{\text{m}}.$$

FATIGUE

Let's define the primary mechanical variables associated with the cyclic loading of a component. These factors include **mean stress**, peak stress, load ratio, waveform, frequency, and amplitude variation (**spectrum loading**).

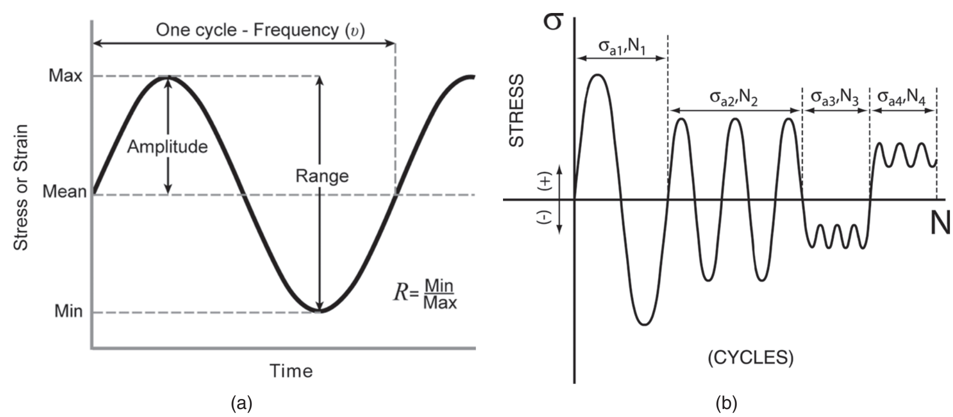


Figure 10.1

Basic definitions of stress or strain range, stress or strain amplitude and load ratio, *R*, for cyclic conditions imposed using a (a) sinusoidal waveform and (b) spectrum (variable amplitude) loading.

the **stress range** is typically the dominant factor in the progression of fatigue damage, and failure, especially in metals, and is defined as:

$\Delta\sigma = \sigma_{max} - \sigma_{min}$ and correspondingly the **stress amplitude** and the **mean stress** of the loading cycle are

$\sigma_a = (\sigma_{max} - \sigma_{min})/2$

$\sigma_m = (\sigma_{max} + \sigma_{min})/2$

The ratio of the minimum stress normalized by the maximum stress is defined as the stress ratio or ***R*-ratio**:

$R = \sigma_{min}/\sigma_{max}$

One critical aspect of the design process is the decision as to whether the component's fatigue life will be dominated by the ***initiation* or *propagation*** process of a critical flaw.

The **total life** design methodology assumes that the component is initially free of any flaws that are sufficiently sized for growth or ideally that the component is "defect-free." This methodology is based on the notion that fatigue failure is a consequence of crack nucleation and subsequent growth to a critical size and that the majority of the life is spent in the nucleation (initiation) phase.

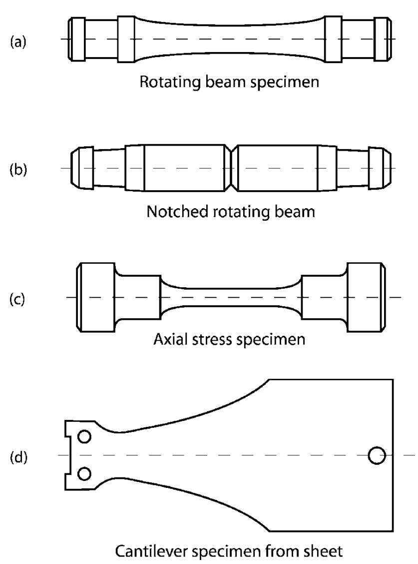
This design philosophy is distinct from the **defect-tolerant** approach in which the fatigue life of a component is based on the number of loading cycles needed to propagate an existing crack to a critical dimension for the material.

The initial size of the flaw is assumed to correspond to the resolution of an inspection test. The critical dimension of the flaw is directly correlated to the fracture toughness of the material.

The defect-tolerant philosophy is more commonly employed in safety-critical applications such as heart valve design.

TOTAL LIFE PHILOSOPHY

The fatigue characterization of a material based on the total life philosophy is based on either a *stress-based* test that examines the conditions for failure for a range of stress amplitudes and mean stress, or a *strain-based* test that examines the fatigue behavior under cyclic strain amplitudes.



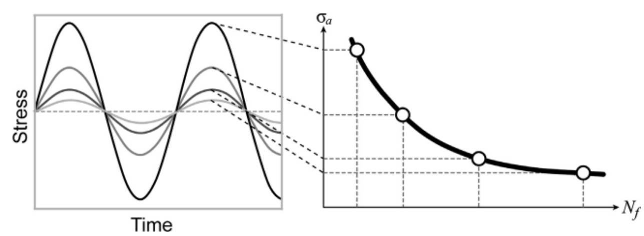
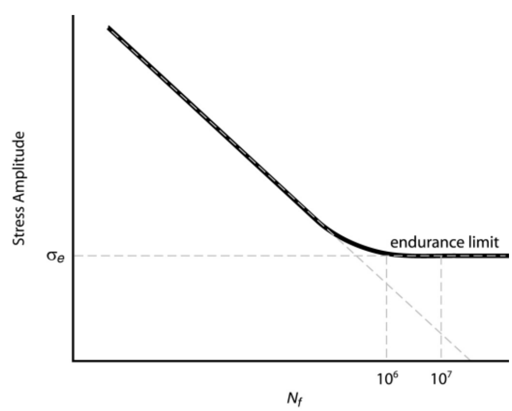


Figure 10.4
Schematic of typical S-N plot on linear-logarithmic scale showing the stress amplitude (σ_a) and concomitant cycles to failure for that loading range.



The endurance limit of most steels is 35–50% of the ultimate tensile strength.

The endurance limit can be affected by several factors such as surface finish, stress concentrations, heat treatment, environment, and component design

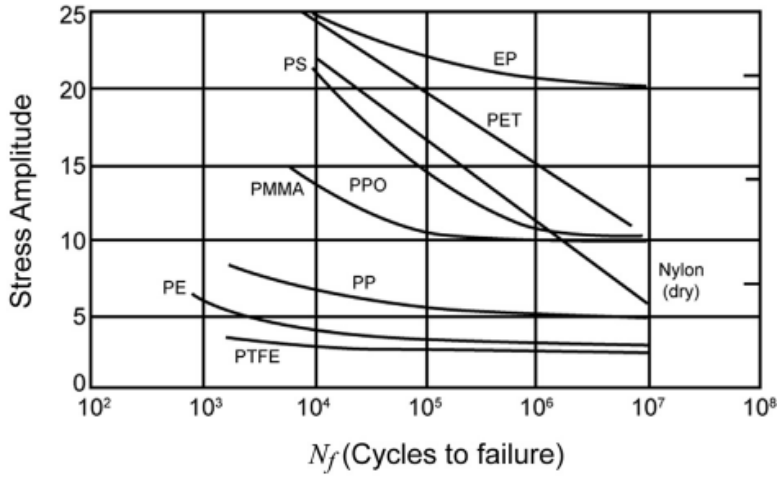


Figure 10.6

S-N fatigue data for several polymer systems. Note that PET and nylon do not have endurance limits.

nylon and polyethylene terephthalate (PET) do not exhibit an endurance limit.

On the other hand, polymers such as polyethylene (PE), polypropylene oxide (PPO), polystyrene (PS), polytetrafluoroethylene (PTFE), polypropylene (PP), polymethylmethacrylate (PMMA), and epoxy (EP) clearly exhibit an endurance limit below which failure does not occur in less than 10^7 cycles .

The relationship between the stress amplitude and the number of cycles to failure is known as the **Basquin equation** (Basquin, 1910) and is given as:

$$\sigma_a = \sigma'_f \ (N_f)^b$$

σ_a is the stress amplitude, σ'_f is the fatigue strength coefficient and is comparable to the true fracture strength for the material, **N_f is the number of cycles to failure**, and b is the Basquin exponent. b is between −0.05 and −0.10.

Determining the cycles to failure for a spinal implant

Consider the artificial spinal disk from Example 6.3. The stresses on the implant as shown in Figure 6.11 are:

$$\sigma_x = -2.2 \text{ MPa}, \sigma_y = -1.1 \text{ MPa}, \sigma_z = -0.58 \text{ MPa}$$
$$\tau_{xy} = -0.57 \text{ MPa}, \tau_{xz} = -0.79 \text{ MPa}, \tau_{yz} = -0.33 \text{ MPa}$$

and the principal stresses:

$$\sigma_1 = -0.25 \text{ MPa}, \sigma_2 = -0.86 \text{ MPa}, \sigma_3 = -2.77 \text{ MPa}$$

The Tresca yield criterion (Equation 8.4) was employed in Example 8.1 to determine the effective stress on the system:

$$\sigma_{\text{Tresca}} = 2.52 \text{ MPa}.$$

The Basquin exponent for the material is – 0.1.

Let's determine the stress amplitude for the system. The largest stress range in this system is determined by the principal stress difference, $\sigma_1 - \sigma_3 = \sigma_{\text{Tres}} = 2.52$ MPa.

One can assume that the effective stress range serves as the effective stress amplitude in this case. The Basquin exponent for the material is –0.1 and the yield strength is 8 MPa. If the yield stress is employed as the failure stress, then the Basquin equation takes the form: $\sigma_a = \sigma_f'(N_f)^b \Rightarrow 2.52 = 8(N_f)^{-.1}$

$$N_f = \left(\frac{\sigma_a}{\sigma_f'}\right)^{\frac{1}{b}} = \left(\frac{2.52}{8}\right)^{-\frac{1}{0.1}} = 103{,}966.$$

Hence, for the effective stress range on the spinal implant, the number of cycles to failure is predicted to be 103,966 loading cycles.

The use of the Basquin equation assumes that the mean stress is zero – that is, that the specimen or component is undergoing **fully reversed loading** and that each cycle represents two reversals. ***If the mean stress is not zero, then these effects must be considered in predicting the life of the component.*** The mean stress has a dramatic effect on the fatigue behavior of a material, and this is schematically illustrated in Figure.

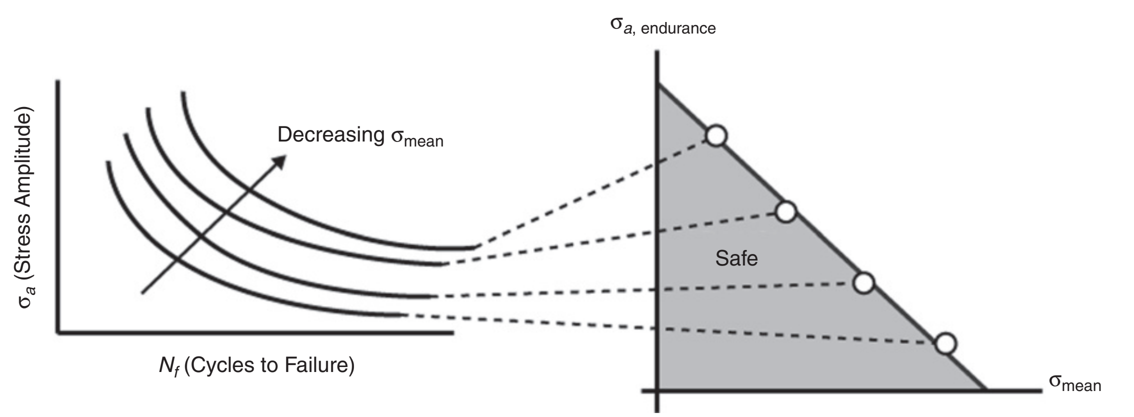


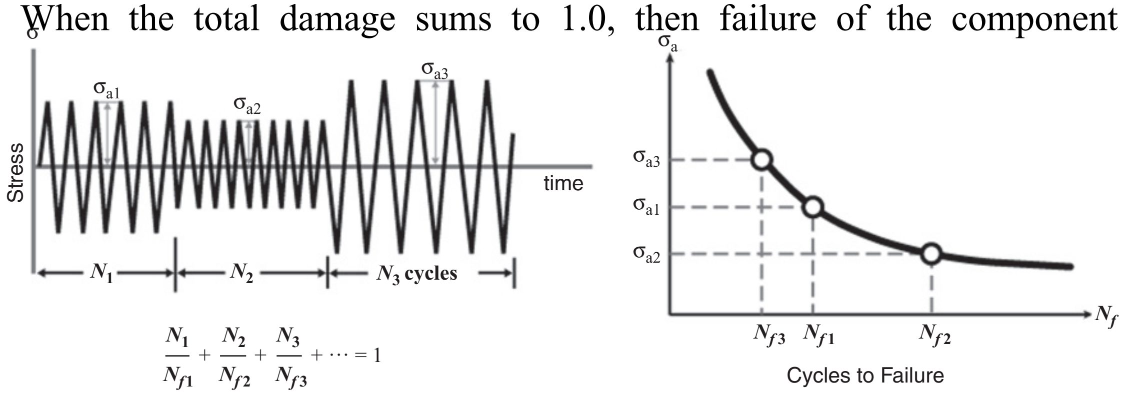
Figure 10.7

The effect of mean stress on the fatigue life of a material and the concomitant linearly decreasing endurance limit plotted as a function of mean stress.

The Palmgren-Miner accumulated damage model

$$D = \sum \frac{N_i}{N_{f_i}}. \tag{10.7}$$

When the total damage sums to 1.0, then failure of the component is predicted

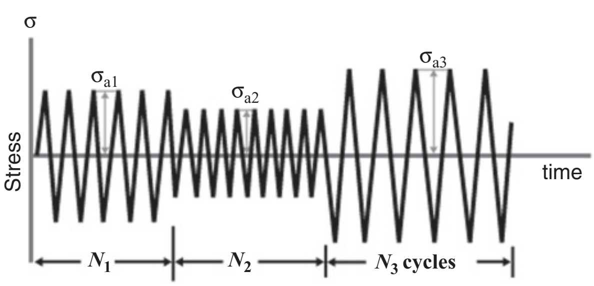


Fatigue life using Palmgren-Miner's rule

Consider a metallic implant that undergoes a series of variations in hourly loading as shown schematically in Figure. For this loading block, the variations in load are 6 reversals at σ_{a1}= 290MPa, 10 reversals at σ_{a2}= 200MPa, and 5 reversals at σ_{a3}= 400 MPa.

The form of the Basquin equation for this alloy is

$\sigma_a = 1758\,(N_f)^{-0.098}$.



(i) How many loading blocks can be sustained before fracture?

$$\frac{N_1}{N_{f1}} + \frac{N_2}{N_{f2}} + \frac{N_3}{N_{f3}} + \cdots = 1$$

Stress amplitude, σ_a (MPa)	Cycles to failure (N_f)	Cycles at this amplitude (N)	Damage, $d = N/N_f$
400	136,000	5	3.67×10^{-5}
290	1,540,000	6	3.89×10^{-6}
200	4.29×10^{9}	10	2.33×10^{-9}

The total damage in one loading block is $\sum N/N_f = 4.04\times10^{-5}$; the number of loading blocks that can be sustained is 24,757.
(ii) Is this a sufficient number of cycles for a fracture fixation device that must last 6 months? The device offers 24,757 safe loading blocks. There are 24 hours in a day. The device offers 1,031 days of service. This is more than sufficient to support loads for 6 months

DEFECT - TOLERANT PHILOSOPHY

The defect-tolerant philosophy is based on the implicit assumption that structural components are intrinsically flawed and that the fatigue life is based on propagation of an initial flaw to a critical size.

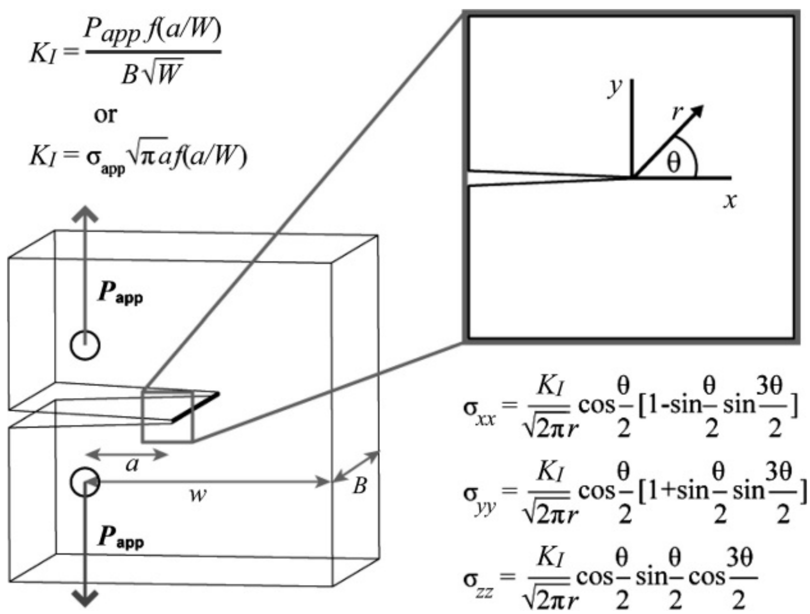


Figure 10.12

Illustration of the compact tension specimen and the associated stress fields at the crack tip.

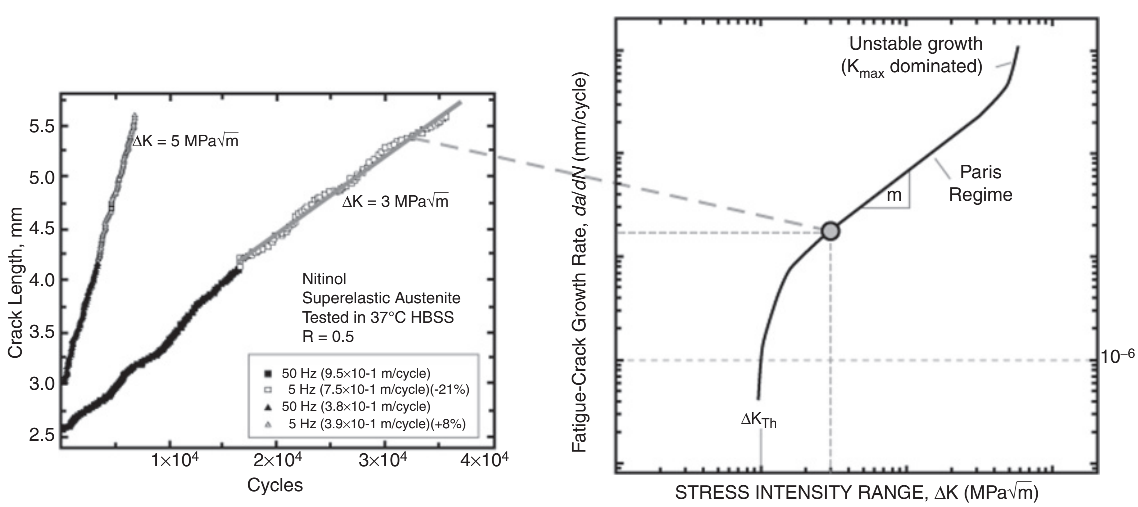
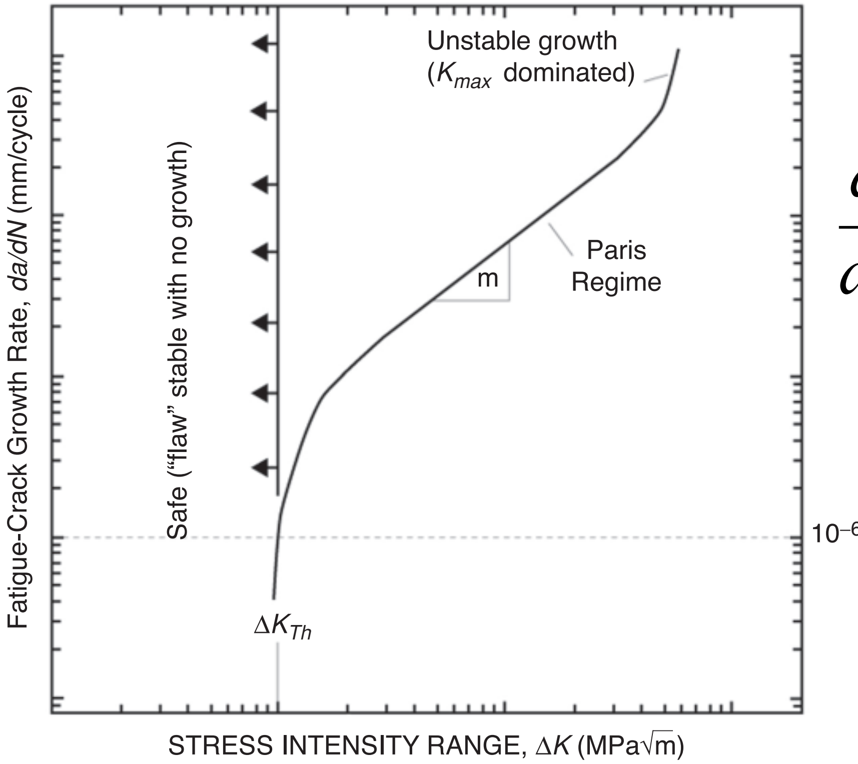


Figure 10.14

Illustration of linear crack growth as a function of loading cycles and how this crack velocity (slope) is used to generate data on a *da*/*dN* versus ΔK plot. The data on the left are shown for a Nitinol (Ni-Ti) shape memory alloy.



Paris (1964)

$$\frac{da}{dN} = C \cdot \Delta K^m$$

Figure 10.15

Illustration of the sigmoidal fatigue crack propagation plot on log-log scale. This schematic depicts the three primary regimes of crack growth: near-threshold, linear, and fast fracture where peak intensity drives the fracture process.

Initial flaw size is typically determined from **non-destructive evaluation** (NDE) techniques such as electron microscopy, X-ray spectroscopy, or ultra- sound. In the event that no defect is found, an initial defect whose size is the limit of resolution of the NDE method is assumed to exist as a worst-case scenario.

$$N_f = \frac{2}{(m-2)Cf(\alpha)^m(\Delta\sigma)^m\pi^{m/2}} \cdot \left[\frac{1}{a_i^{(m-2)/2}} - \frac{1}{a_c^{(m-2)/2}}\right] \text{ for } m \neq 2.$$

Fatigue design of a Nitinol stent

The typical strut thickness is 500 μm

The fatigue crack propagation constants for Nitinol (tube) are $C = 2 \times 10^{-11}\,(\text{MPa}\sqrt{\text{m}})^{m}$, $m = 4.2$, and $\text{D}K_{\text{th}} = 2.5\ \text{MPa}\sqrt{\text{m}}$

The geometric parameter, $F(a/W)$ (note that $F(a/W)$ is the same as $F(\alpha)$) for the strut is known from finite element analysis to be 0.624. The maximum allowable flaw size in a medical grade Nitinol alloy is 39 μm as per ASTM F2063. The physiological stress range on the implant is 294 MPa.

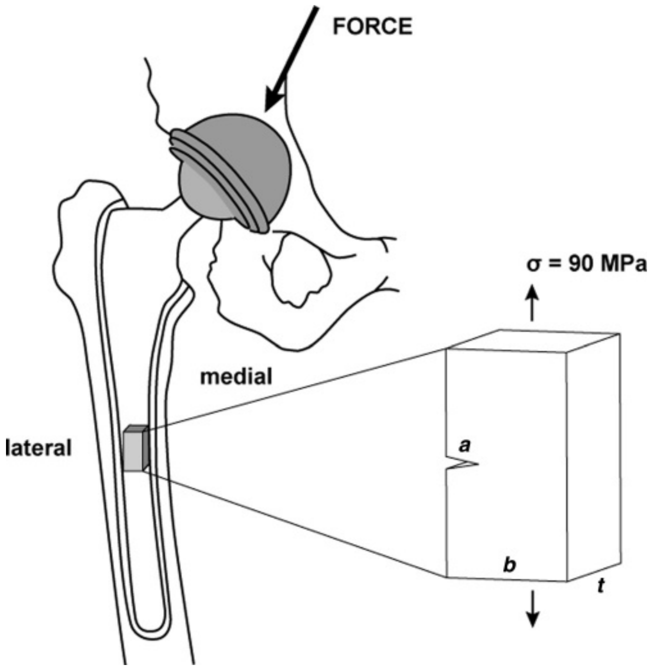
In this case it is essential that any flaw present is incapable of propagating. For this reason, the threshold stress intensity factor is used to determine the critical flaw size that initiates the onset of crack growth

$$\Delta K_{th} = \Delta\sigma\sqrt{\pi a} \cdot F\left(\frac{a}{W}\right);$$

$$a_{cr} = \frac{1}{\pi}\left[\frac{\Delta K_{th}^2}{F\left(\frac{a}{W}\right)^2 \Delta\sigma^2}\right] = 59\ \mu\text{m}$$

Since the critical flaw size for the inception of crack growth is 59 μm and the maximum allowable flaw size for the Nitinol alloy is 39 μm, ***the device is safe against fatigue crack growth***

Fatigue crack propagation in a flawed hip implant



Years ago, laser etching was used on the lateral side of the stem to mark a serial number. The material removed by the laser etching is essentially a small edge-notched crack that is 1 mm deep on the side of the stem. It is estimated that the tensile bending stress on the stem is approximately 90 MPa for a typical active male weighing 200 pounds.

The femoral stem is made of a CoCr alloy with a fracture toughness of 9.5 MPa, and fatigue crack propagation constants $C = 6 \times 10^{-11}$ (MPa√m)^{m} and $m = 4$. The form of the stress intensity factor for a single edge-notched geometry is

$K_I = 1.10\, \sigma \sqrt{a}\sqrt{\pi}$.

What is the critical flaw size for this alloy? How many fatigue cycles will this system last? Is this acceptable for a hip implant?

$$\Delta K = 1.12 \Delta\sigma \sqrt{\pi a}$$

$$a_{cr} = \frac{1}{\pi}\left[\frac{K_{IC}^2}{1.12^2 \Delta\sigma^2}\right] = 2.8 \text{ mm.}$$

$$
\begin{aligned}
N_f &= \frac{2}{(m-2)Cf(\alpha)^m(\Delta\sigma)^m\pi^{m/2}} \cdot \left[\frac{1}{a_i^{(m-2)/2}} - \frac{1}{a_c^{(m-2)/2}}\right] \\
&= \frac{1}{6\times10^{-11}(1.12)^4(90)^4\pi^2} \cdot \left[\frac{1}{(1\times10^{-3})^2} - \frac{1}{(2.8\times10^{-3})^2}\right] \\
&= 14.2\times10^6 \text{cycles} = 7.1 \text{ years}
\end{aligned}
$$

The average hip implant must lasts 15–20 years. The laser etching in this case results in a fatigue life that is more than halved due to the initial stress concentration.

Fatigue crack propagation in metals

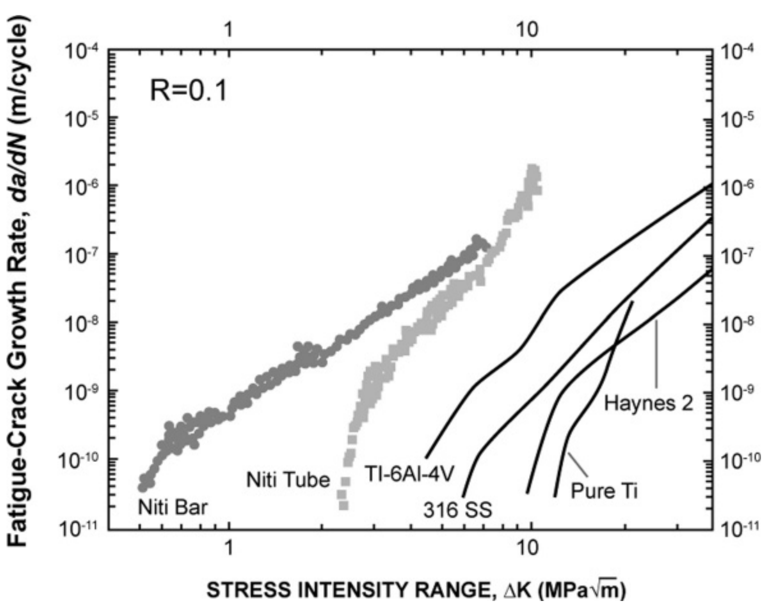


Figure 10.22

Fatigue crack growth rate as a function of stress intensity range for several medical grade alloys including stainless steel, Nitinol (Ni-Ti), Haynes 2, pure titanium, and Ti-6Al-4V. (After Ritchie, 1999.)

Fatigue crack propagation in polymers

Polymers are generally known to be susceptible to creep and strain rate effects; and thus it is important to understand the viscoelastic nature of polymers under cyclic loading conditions. The dissipation of energy in cyclic loading of polymers results in heat generation; the amount of temperature elevation depends strongly on frequency deformation amplitude and the damping properties of the specific polymer

The fatigue behavior of polymers depends on many factors.
In general, polymers with higher molecular weight, chain entanglement density, and crystallinity are more resistant to crack propagation while increased crosslinking decreases resistance to fatigue crack propagation

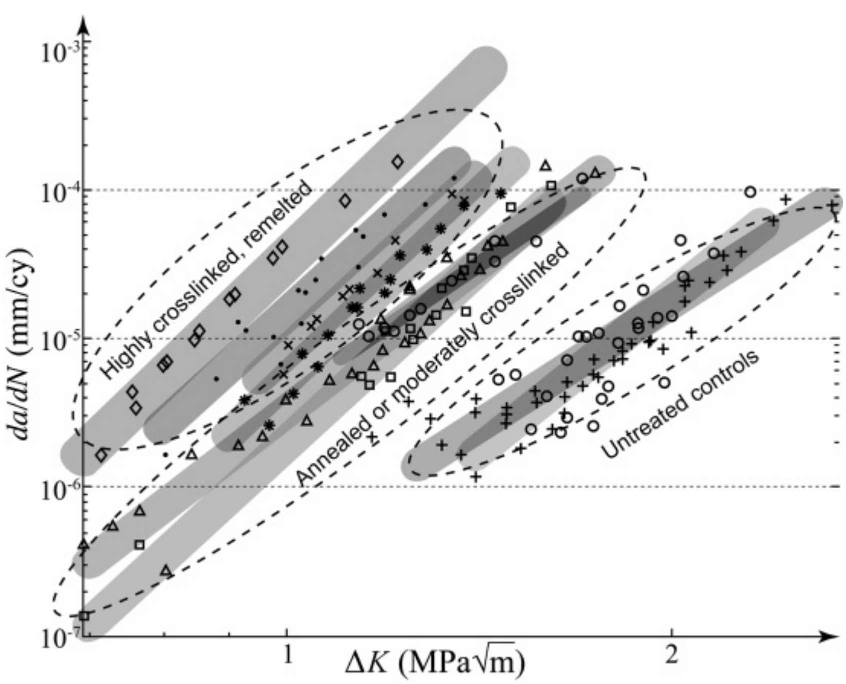


Figure 10.24

Fatigue crack growth rate as a function of stress intensity range for a range of clinically relevant formulations of UHMWPE. Untreated controls, with no crosslinking and greatest level of crystallinity, offer the greatest resistance to fatigue crack propagation (adapted from Atwood *et al.*, 2009).