K any field

<u>Affine space</u> K^n $\qquad \mathbb{A}^n_K$

A point $P \in \mathbb{A}^n_K$ $\qquad P = (a_1, \dots, a_n) \quad a_i \in K$

$\mathbb{A}^n_K \quad n \geq 0$

<u>Projective space</u> $\mathbb{P}(V)$ $\quad V$ $\quad$ K-vector space of finite dimension

$$\mathbb{P}(V) = \frac{V - \{0\}}{\sim}$$

$v, v' \in V$

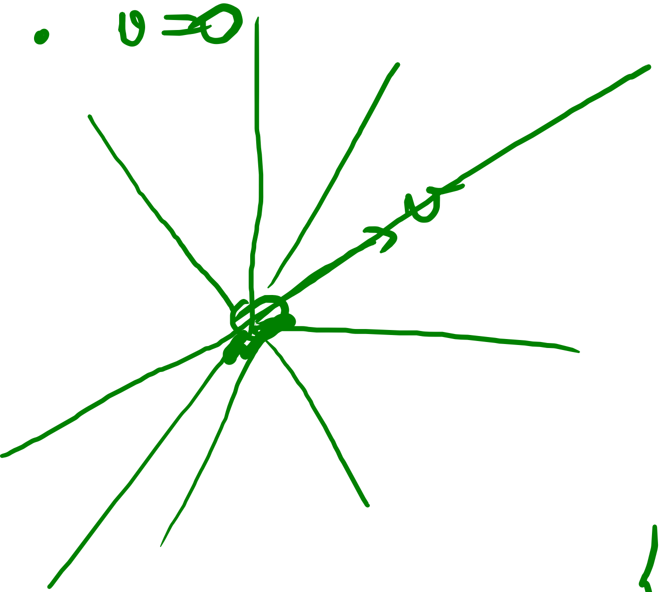
$v \sim v' \iff \exists\, \lambda \in K - \{0\} = K^*$ st. $v' = \lambda v$

$\sim$ proportionality

$\sim$ is an equivalence relation

Equivalence classes:

- $v \neq 0$ $\quad [v] = \{\lambda v \mid \lambda \in K^*\} = \langle v \rangle \setminus \{0\}$
- $v = 0$ $\quad [0] = \{0\}$



$$\mathbb{P}(V) = \frac{V \setminus \{0\}}{\sim}$$

$$\Updownarrow$$

$\{$vector subspaces of V of dim $1\}$

$\dim V = n+1 \implies \dim \mathbb{P}(V) = n$

canonical surjection $\quad p : V^* \longrightarrow \mathbb{P}(V)$

$\qquad\qquad\qquad\qquad v \longrightarrow [v]$

Fibres of p: $\quad Q \in \mathbb{P}(V) \quad p^{-1}(Q) = \langle v \rangle - \{0\} \quad Q = [v]$

$\mathcal{B}$ basis of V $\quad \mathcal{B} = (v_0, v_1, \ldots, v_n)$

$\Longrightarrow$ homogeneous coordinates in $\mathbb{P}(V)$

$Q \in \mathbb{P}(V) \qquad Q = [v] = [\lambda v] \quad \lambda \in K^*$

$v = x_0 v_0 + \cdots + x_n v_n \qquad x_0, \ldots, x_n$

$\lambda v = (\lambda x_0) v_0 + \cdots + (\lambda x_n) v_n \qquad \lambda x_0, \ldots, \lambda x_n$

$Q \rightsquigarrow$ homog. coordinates of Q $\quad Q[x_0, \ldots, x_n]$

To Q we give homog. coordinates which are defined only up to proportionality, $[x_0, \ldots, x_n] \neq [0, \ldots, 0]$

$[x_0, \ldots, x_n] \qquad (x_0 : \cdots : x_n)$

$\mathbb{P}(V)$ $\quad \dim V = n+1 \quad V \simeq K^{n+1}$

$\mathbb{P}(V) \longleftrightarrow \mathbb{P}(K^{n+1}) = \mathbb{P}^n_K$ numerical projective space

Fundamental points $E_0 = [1, 0, \dots, 0] = [\lambda, 0, \dots, 0]$ $\lambda \in K^*$

$E_1 = [0, 1, 0 \dots, 0]$

$\vdots$

$E_n = [0 \dots, 1]$

$U = [1, \dots, 1] = [\lambda, \dots, \lambda]$ unity point

$\mathbb{A}^n_K$ affine subspaces

$S = p + W$ $\quad W \subset K^n$ direction of S

$\dim S = \dim W$

cartesian equations of S are of the form

$A x = b \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A \ \underline{m} \times \underline{n} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_{\underline{m}} \end{pmatrix}$

$\operatorname{rk} A = n - d, \quad d = \dim S$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n - b_1 = 0 \\ \vdots \\ a_{m1}x_1 - \dots + a_{mn}x - b_m = 0 \end{cases}$$

S is the set of solutions of a system of eq. of degree 1

$\mathbb{P}(V)$ projective subspaces or linear subspaces

$\cup\!\!|$

$\mathbb{P}(W)$ $W \subseteq V$ vector subspace

$\mathbb{P}(W)$ is described by the same systems of equations of W : homog. system of equations

$$\boxed{Ax = 0} \qquad x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \qquad A = \text{matrix of type } m \times (n+1)$$

represents also $\mathbb{P}(W)$

$\operatorname{rg} A = (n+1) - (d+1)$

$\dim W = d+1.$

$\operatorname{rg} A = \dim \mathbb{P}(V) - \dim \mathbb{P}(W)$ the codimension of $\mathbb{P}(W)$ in $\mathbb{P}(V)$

$\mathbb{P}(W)$ is described by homogeneous equations of deg 1

In V: Grassmann relations

$U, W \subseteq V$ $\quad \dim(U \cap W) + \dim(U+W) = \dim U + \dim V$

$\dim \mathbb{P}(U \cap V) + \dim \mathbb{P}(U+W) = \dim \mathbb{P}(U) + \dim \mathbb{P}(V)$

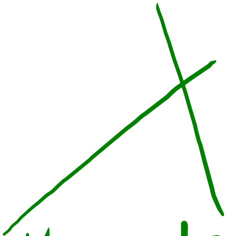
$\mathbb{P}(U \cap V) = \mathbb{P}(U) \cap \mathbb{P}(V)$

$\mathbb{P}(U+W)$

$U+W$ = linear span of $U \cup W$

$\Rightarrow \mathbb{P}(U+W)$ is the minimal proj. subspace of $\mathbb{P}(V)$ containing $\mathbb{P}(U) \cup \mathbb{P}(W)$

2 lines in $\mathbb{P}^2_K$ always meet



How to embed the affine space $\mathbb{A}^n_K$ into $\mathbb{P}^n_K$?

Fix a system of homogeneous coord. in $\mathbb{P}^n_K$:

Fundamental hyperplanes in $\mathbb{P}^n_K$: $\mathcal{B}=(e_0,e_1,\dots,e_n)$

$H_0 = \mathbb{P}(\langle e_1,\dots,e_n\rangle)$ $\quad x_0=0 \quad H_0 \quad E_1,\dots,E_n$

$H_1 = \mathbb{P}(\langle e_0,e_2,\dots,e_n\rangle)$ $\quad x_1=0$

$\vdots$

$H_n = \mathbb{P}(\langle e_0,\dots,e_{n-1}\rangle)$ $\quad x_n=0$

$H_0 \quad \mathbb{P}^n_K \setminus H_0 = U_0 = \{[x_0,\dots,x_n] \mid x_0\neq 0\}$

$j_0: \mathbb{A}^n \longrightarrow U_0 \quad (x_1,\dots,x_n) \longrightarrow [1,x_1,\dots,x_n]$

$\varphi_0: U_0 \longrightarrow \mathbb{A}^n \quad [y_0,y_1,\dots,y_n] = [y_0^{-1}y_0, y_0^{-1}y_1,\dots,y_0^{-1}y_n] =$

$y_0\neq 0 \ : \ \exists\, y_0^{-1} \quad = [1, \frac{y_1}{y_0},\dots,\frac{y_n}{y_0}]$

$\varphi_0([y_0,\dots,y_n]) = \left(\frac{y_1}{y_0},\dots,\frac{y_n}{y_0}\right)$

$$(x_1,\dots,x_n) \xrightarrow{j_0} [1,x_1,\dots,x_n] \xrightarrow{\varphi_0} (x_1,\dots,x_n)$$

$$[y_0,\dots,y_n] \xrightarrow{\varphi_0} \left(\frac{y_1}{y_0},\dots,\frac{y_n}{y_0}\right) \xrightarrow{j_0} \left[1,\frac{y_1}{y_0},\dots,\frac{y_n}{y_0}\right] = [y_0,\dots,y_n]$$

$\Longrightarrow$ j_0,φ_0 give a bijection $\mathbb{A}^n \longleftrightarrow U_0$

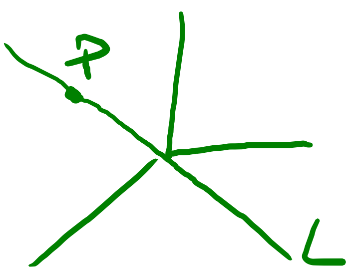
$j_0 : \mathbb{A}^n \hookrightarrow \mathbb{P}^n_k$ we embed $\mathbb{A}^n_k$ in $\mathbb{P}^n_k$ identifying $\mathbb{A}^n$ with U_0 via j_0

H_0 : points at infinity

$\mathbb{A}^n \supseteq L$ through O, $P(a_1,\dots,a_n)$

$$L \left\{ \begin{array}{l} x_0 = 1 \\ x_1 = a_1 t \\ \vdots \\ x_n = a_n t \end{array} \right.$$

we identify $\mathbb{A}^n$ with U_0



$$L \cap U_0 \quad \begin{cases} x_0 = 1 \\ x_1 = a_1 t \\ \vdots \\ x_n = a_n t \end{cases}$$

$t = 0 \Rightarrow$ origin $(0 \dots 0) \in \mathbb{A}^n$
$\updownarrow$
$[1, 0 \dots 0] \in U_0$

$t \neq 0$
$\Downarrow$
t^{-1}

For points of L different from O

$$[1, a_1 t, \dots, a_n t] = \left[\frac{1}{t}, a_1, \dots, a_n\right] \to [0, a_1 \dots a_n]$$

$\uparrow$
H_0

$[0, a_1 \dots a_n]$: the point at infinity of L

$L' \subset \mathbb{A}^n$ any line $\qquad L' = P + W, \quad W = \langle w \rangle$

Check that with the same procedure you get a point at infinity $[0, w]$

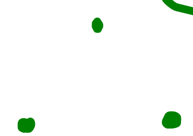
$\mathbb{P}^n_K \qquad P_0, \dots, P_r \qquad P_0 = [v_0], \dots, P_r = [v_r]$

$P_0, \dots, P_r$ are linearly independent points if
$v_0, \dots, v_r$ " " " vectors

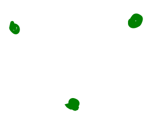
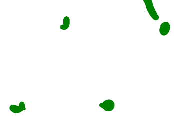
The definition is well posed.

In $\mathbb{P}^n_K$ at most $n+1$ linearly indep. points

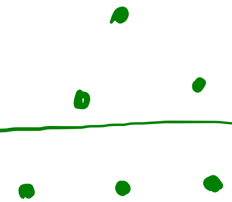
$r \geq n+1$ Def. $P_0, \dots, P_r$ are in <u>general position</u> if any $n+1$ among $P_0, \dots, P_r$ are linearly indep.

Ex. $\mathbb{P}^2$  3 lin. indep. points = not aligned

4 pts in gen. pos. if no 3 of them are aligned

3 pts in $\mathbb{P}^2$



linearly indep.

lin. dep. = aligned

4 pts in $\mathbb{P}^2$ in gen. position = 3 by 3 lin. independent