$\mathbb{P}^n_K$

1) $P_1, \dots, P_r \in \mathbb{P}^n_K$ are linearly indep if $P_i = [v_i]$, $v_1, \dots, v_r$ are lin. indep.

$r \le n+1$

2) $P_1, \dots, P_r$ are in general position if either $r \le n+1$ and lin. indep., or $r > n+1$ and $P_1 \dots P_r$ are $n+1$ by $n+1$ lin. indep.

$r = 3$ lin. indep. $\iff$ not aligned

$r = 4$ " " they generate a sp. of dim 3

not coplanar

$\mathbb{P}^3$ 4 pts $4 = 3+1$

5 pts : gen. pos. : 4 by 4 they generate $\mathbb{P}^3$

<u>Ex</u>: $\mathbb{P}^n$, fix a system of homog. coord.

$E_0, \dots, E_n, U$ $\quad$ $n+2$ pts in general position

$$n+2 \geq n+1$$

1) $E_0, \dots, E_n$ lin. indep. because

$E_i = [e_i]$ $\quad$ $e_0, \dots, e_n$ basis

2) $E_0, \dots, \hat{E_i}, \dots, E_n, U$: $e_0, \dots, \hat{e_i}, \dots, e_n, e_0 + \dots + e_n$ lin. indep.

$\mathbb{P}^n$ $P_0, \dots, P_{n+1}$ $n+2$ pts in gen. position

on $\mathbb{P}(V)$

$\Rightarrow \exists$ a basis in V s.t. $P_0, \dots, P_n$ are the fundamental pts and P_{n+1} unity pt

Pf. $P_i = [v_i]$ $v_0, \dots, v_n$ lin. indep.

$$\boxed{v_{n+1} = \lambda_0 v_0 + \dots + \lambda_n v_n}$$

$\boxed{\lambda_0 v_0, \dots, \lambda_n v_n}$ are lin. indep because all $\lambda_0, \dots, \lambda_n$ are $\neq 0$

If $\lambda_i = 0$, $v_0, \dots, \hat{v}_i, \dots, v_n, v_{n+1}$ are lin. dep. : contradiction.

$\forall i$ $P_i = [\lambda_i v_i]$, $P_{n+1} = [\lambda_0 v_0 + \dots + \lambda_n v_n]$

Algebraic sets

$\mathbb{A}^{n}_{K}$ $\quad K[x_1, \ldots, x_n] \ni F(x_1, \ldots, x_n)$

$P \in \mathbb{A}^n_K$ $\quad P(a_1, \ldots, a_n)$ $\quad$ P is a <u>zero</u> of F

if $F(a_1, \ldots, a_n) = 0$

$S \subseteq K[x_1, \ldots, x_n]$

$$V(S) = \{ P \in \mathbb{A}^{n}_K \mid F(P) = 0 \ \forall F \in S \}$$

set of common zeros of polynomials in S

$Z(S)$

<u>Def</u>. $X \subseteq \mathbb{A}^n_K$ is an <u>affine algebraic set</u>

if $X = V(S)$, for some $S \subseteq K[x_1, \ldots, x_n]$

affine <u>algebraic variety</u>

Examples 1) $P(a_1, \dots, a_n)$ $X = \{P\}$ is an alg. set $P = V(x_1 - a_1, \dots, x_n - a_n)$

$$\begin{cases} x_1 - a_1 = 0 \\ \vdots \\ x_n - a_n = 0 \end{cases}$$

2) Linear subspace $L \subset \mathbb{A}^n_k$ of dim d

$L = V(L_1, \dots, L_{n-d})$

$L_1 = a_{11}x_1 + \dots + a_{1n}x_n - b_1$

$\vdots$

$L_{n-d} = \dots$

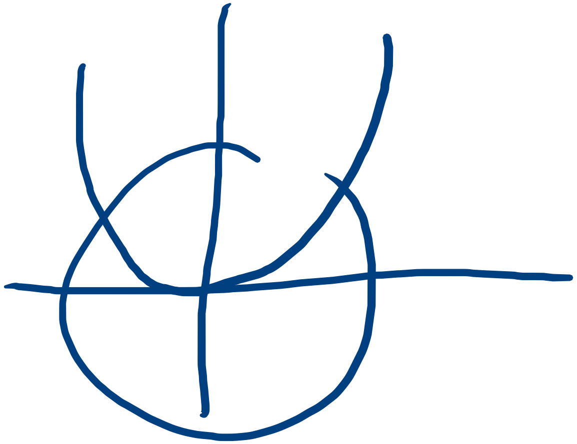
linear equations

3) $S = K[x_1 \dots x_n]$ $V(S) = \emptyset$ $1 \in K[x_1 \dots x_n]$

4) $S = \emptyset \qquad V(\emptyset) = \mathbb{A}^n_K$

5) $\mathbb{A}^2_K \qquad K[x,y] \quad V(x^2 - y) \qquad V(x^2+y^2-1)$

$y - x^2 = 0 \qquad x^2+y^2=1$



6) quadrics $\quad Q = V(F)$

$F \in K[x_1 \ldots x_n] \quad \deg F = 2$

$S \subseteq K[x_1 \ldots x_n] \qquad \langle S \rangle = \alpha$

$$\alpha = \Big\{ \sum_{\text{fin}} H_i \cdot F_i \;\Big|\; F_i \in S,\; H_i \in K[x_1 \ldots x_n] \Big\}$$

Prop. $\boxed{V(\alpha) = V(S)}$

Pf $S \subseteq \alpha \implies V(S) \supseteq V(\alpha)$

Conversely: $P \in V(S)$ $F(P) = 0$ $\forall F \in S$

$\sum H_i F_i$, $F_i \in S$

fin

$$\left(\sum H_i F_i\right)(P) = \sum H_i(P) \underbrace{F_i(P)}_{=0} = 0$$

Rmk X is an algebraic set in $\mathbb{A}^n_k$

$\iff X = V(\alpha)$, α ideal of $k[x_1 \ldots x_n]$

Hilbert's Basis Theorem: R noetherian ring

$\implies R[X]$ is noetherian

Cor. $K[x_1, \dots, x_n]$ is noetherian, K field

$\Rightarrow$ every ideal $\alpha \subseteq K[x_1 \dots x_n]$ is finitely generated.

consequence for algebraic sets:

$X \subseteq \mathbb{A}^n_K$ algebraic set

$X = V(S) = V(\alpha) = V(F_1, \dots, F_r)$

$S \subset K[x_1 \dots x_n]$ $\quad \alpha = \langle S \rangle = \langle F_1, \dots, F_r \rangle$

HBT

$\Rightarrow$ X is the set of common zeros of a finite set of polynomials

Any set of algebraic equations is equivalent to a finite set of equations.

Topology on $\mathbb{A}^n_K$: <u>ZARISKI</u> TOPOLOGY

Castelnuovo

Enriques

The unreal life of Oscar Zariski

PARIKH

<u>Def</u>: Zariski topology on $\mathbb{A}^n_K$: closed sets are the affine algebraic sets

1) $\emptyset = V(1)$, $\mathbb{A}^n_K = V(0)$

2) $X, Y \subseteq A^n_K$ $\quad X = V(F_1, \ldots, F_r)$

$Y = V(G_1, \ldots, G_s)$

$X \cup Y$

$$X \begin{cases} F_1 = 0 \\ \vdots \\ F_r = 0 \end{cases} \qquad Y \begin{cases} G_1 = 0 \\ \vdots \\ G_s = 0 \end{cases}$$

$X \cup Y$?

$$P \begin{cases} x_1 - a_1 = 0 \\ \vdots \\ x_n - a_n = 0 \end{cases} \qquad Q \begin{cases} x_1 - b_1 = 0 \\ \vdots \\ x_n - b_n = 0 \end{cases}$$

$\{P, Q\}$

$FG = 0$

$\Updownarrow$

$V(F) \cup V(G) = V(FG)$ $\quad F = 0$ or $G = 0$

$X = V(\alpha)$, $Y = V(\beta)$ α, β ideals

$\alpha\beta$ product ideal

$X \cup Y = V(\alpha\beta) = V(\alpha \cap \beta)$

$\alpha\beta \subseteq \alpha \cap \beta \subseteq \alpha, \subseteq \beta \implies V(\alpha) \subseteq V(\alpha\cap\beta) \subseteq V(\alpha\beta)$, $V(\beta) \subseteq V(\alpha\cap\beta)$

$V(\alpha) \cup V(\beta) \subseteq V(\alpha \cap \beta) \subseteq \underline{V(\alpha\beta)}$

$P \in V(\alpha\beta)$; we want to prove $P \in V(\alpha) \cup V(\beta)$;

an. $P \notin V(\alpha)$, we want to deduce $P \in V(\beta)$

$\exists F \in \alpha$ s.t. $F(P) \neq 0$, $P \in V(\alpha\beta)$

$\forall G \in \beta$ $FG \in \alpha\beta$, $(FG)(P) = 0 =$

$= \underbrace{F(P)}_{\neq 0} G(P) \implies G(P) = 0 \implies P \in V(\beta)$

3)

3) $\{V(\alpha_i)\}_{i\in I}$ $\qquad \alpha_i \subseteq k[x_1 \dots x_n]$

$$\bigcap_{i\in I} V(\alpha_i) = V\Big(\sum_{i\in I}\alpha_i\Big)$$

$\sum \alpha_i$ in the ideal generated by $\bigcup_{i\in I}\alpha_i$
$\shortparallel$
$\{$finite sums of elements in the α_i's$\}$

$\alpha_j \subseteq \sum_{i\in I}\alpha_i \quad \forall j \qquad V(\alpha_j) \supseteq V(\sum\alpha_i) \Rightarrow$

$\Rightarrow V(\sum\alpha_i) \subseteq \bigcap_{i\in I} V(\alpha_i)$

$P \in \bigcap V(\alpha_i) \qquad F \in \sum\alpha_i \quad F = F_{i_1} + F_{i_2} + \dots + F_{i_n}$

P is a zero of each summand $\quad \checkmark$

In the Zariski topology, each point is closed.

$\mathbb{A}^1$ $\quad K[x] \quad$ PID

A closed set in $\mathbb{A}^1$ is of the form $V(F)$

1) $F=0$ $\qquad V(F)=V(0)=\mathbb{A}^1$

2) $F=c\in K\setminus\{0\}$ $\qquad V(c)=\emptyset$

3) $\deg F \geq 1$ $\qquad$ Look at the factoriz. in linear factors of F in $\bar{K}[x]$,

d

where $\bar{K}$ = algebraic closure of K

At most d zeros of F belong to K

Every finite set is closed: the open sets are the complementaries of finite sets.

$K = \mathbb{R}$ Compare Zariski top. τ_Z with Euclidean topology τ_E : $\tau_E > \tau_Z$

$\mathbb{A}^n_{\mathbb{R}}$ $U \subseteq \mathbb{A}^n_{\mathbb{R}}$ open in the Zariski top.

1) $U = \mathbb{A}^n_{\mathbb{R}} \setminus V(F_1, \dots, F_k)$

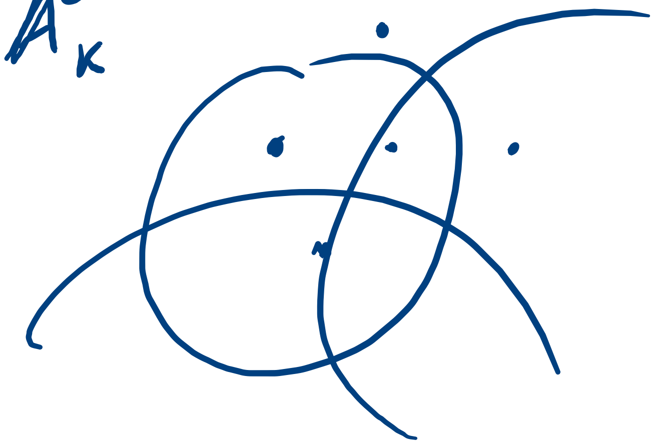
$\exists P \in U$, F_i : $F_i(P) \neq 0$, F_i is a continuous fn.: $F_i(P) \neq 0 \Rightarrow F_i$ is $\neq 0$ on a nbhd of P in τ_E. $\Rightarrow U$ is open in τ_E

2) $\exists$ U open in τ_E but not open in τ_Z :

$\mathbb{A}^1_{\mathbb{R}} = \mathbb{R}$

intervals are not open

$\mathbb{A}^2_K$



$V(F)$, $\deg F > 0$

plane algebraic curves

Šafarevič

Projective space

$K[x_0, x_1, \dots, x_n]$

$P \in \mathbb{P}^n_K$ $\quad P[a_0, a_1, \dots, a_n]$

$P = [v] = \langle v \rangle \setminus \{0\}$ $\quad K^{n+1}$

set of linear equations in K^{n+1} for $\langle v \rangle = W$

$W \subset K^{n+1}$ $\quad W = \langle \underbrace{w_0, \dots, w_d}_{\text{basis}} \rangle$

Cartesian equations for W?

$\mathbb{R}^4$ $\qquad W = \langle w_1, w_2 \rangle$

$w_1 = (1, 1, 0, 1)$

$w_2 = (2, 1, 1, 2)$ $\qquad \cancel{w_1 \times w_2}$

$v(x_1, x_2, x_3, x_4) \in W \iff w_1, w_2, v$ are lin. dependent $\iff$

$$\operatorname{rk}\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix} < 3 \iff \text{all } 3\times 3$$

minors are 0 $\Rightarrow$ equations!

$x_1 - x_2 - x_3 = 0 \qquad -x_1 + x_3 + x_4 = 0$

rk $\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ a_0 & a_1 & \cdots & a_n \end{pmatrix} < 2$: these are equations

for P in $\mathbb{P}^n_K$ and for $\langle v \rangle \subseteq K^{n+1}$

$$\left\{ \begin{array}{l} a_1 x_0 - a_0 x_1 = 0 \\ \vdots \\ a_i x_j - a_j x_i = 0 \\ \vdots \end{array} \right.$$

this expresses that $x_0, \ldots, x_n$ are proportional to $a_0, \ldots, a_n$

$$\left\{ \begin{array}{l} x_0 - a_0 = 0 \\ \vdots \\ x_n - a_n = 0 \end{array} \right.$$

$\mathbb{P}^n_K \qquad K[x_0 \ldots x_n]$

Rmk $F \in K[x_0 \ldots x_n]$

$P \in \mathbb{P}^n_K$: $F(P)$ is not well defined $\quad P[a_0 \ldots a_n] \qquad F(a_0, \ldots, a_n) \neq F(\lambda a_0, \ldots, \lambda a_n)$ in general

$S = v + W \subseteq V$ in the vector space

$\{v + w \mid w \in W\}$ affine subspace

Line $= \dim 1$ in projective space $\mathbb{P}(V)$ = projective subspace of $\dim 1$ $= \mathbb{P}(W)$, $\dim W = 2$

$W \subseteq V$ vector subspace of V

Def. $P[a_0 - a_n]$ is a zero (projective zero) of F is $F(\lambda a_0 - , \lambda a_n) = 0 \quad \forall \lambda \in K \setminus \{0\}$

F is homogeneous polynomial of deg d if F is a linear combination of monomials all of degree d

Rm. F homg of deg d $\Longrightarrow$

$$\boxed{F(tx_0, - , tx_n) = t^d F(x_0, -, x_n)}$$

t new variable

An. F is a monomial: $F = x_0^{i_0} x_1^{i_1} - - x_n^{i_n}$

$i_0 + i_1 + - - + i_n = d$

$$F(tx_0 - . tx_n) = (tx_0)^{i_0} (tx_1)^{i_1} - - . (tx_n)^{i_n} =$$

$$= t^{i_0 + \cdots + i_n} x_0^{i_0} - - x_n^{i_n} = t^d F(x_0 - x_n)$$

F homog. of deg d $\quad P[a_0 \dots a_n] \in \mathbb{P}^n$

$F(a_0 \dots a_n) = 0 \iff F(\lambda a_0, \dots, \lambda a_n) = 0$
$\forall \lambda \neq 0$

P is a projective zero of $F \iff F(a_0 \dots a_n) = 0$ for a particular $(n+1)$-tuple of coord. of P

F homog. $\quad V_P(F) \subseteq \mathbb{P}^n_K$
$\quad = \{$projective zeros of $F\}$

$S \subseteq K[x_0 \dots x_n]$ set of homog. polynomials
$V_P(S)$ set of common zeros of pol. in S

<u>Def</u>. $X \subseteq \mathbb{P}^n_K$ is a projective algebraic set or projective variety if $\exists$ S set of homog. polyn. s.t. $X = V_P(S)$