SEISMOLOGY I

Laurea Magistralis in Physics of the Earth and of the Environment

Wave phenomena

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The linear wave equation

Earlier we introduced the concept of a wavefunction to represent waves travelling on a string.

All wavefunctions $y(x,t)$ represent solutions of the

LINEAR WAVE EQUATION

The wave equation provides a complete description of the wave motion and from it we can derive the wave velocity

Superposition of waves

When two waves meet in space their individual disturbances (represented by their wavefunctions) superimpose and add together.

The principle of superposition states:

If two or more travelling waves are moving through a medium, the resultant wavefunction at any point is the algebraic sum of the wavefunctions of the individual waves

Waves that obey this principle are called **LINEAR WAVES**

Waves that do not are called **NONLINEAR WAVES**

Generally LW have small amplitudes, NLW have large amplitude

Superposition and standing waves

Already looked at interference effects - the combination of two waves travelling simultaneously through a medium.

Now look at superposition of harmonic waves.

Beats

Standing waves

Modes of vibration

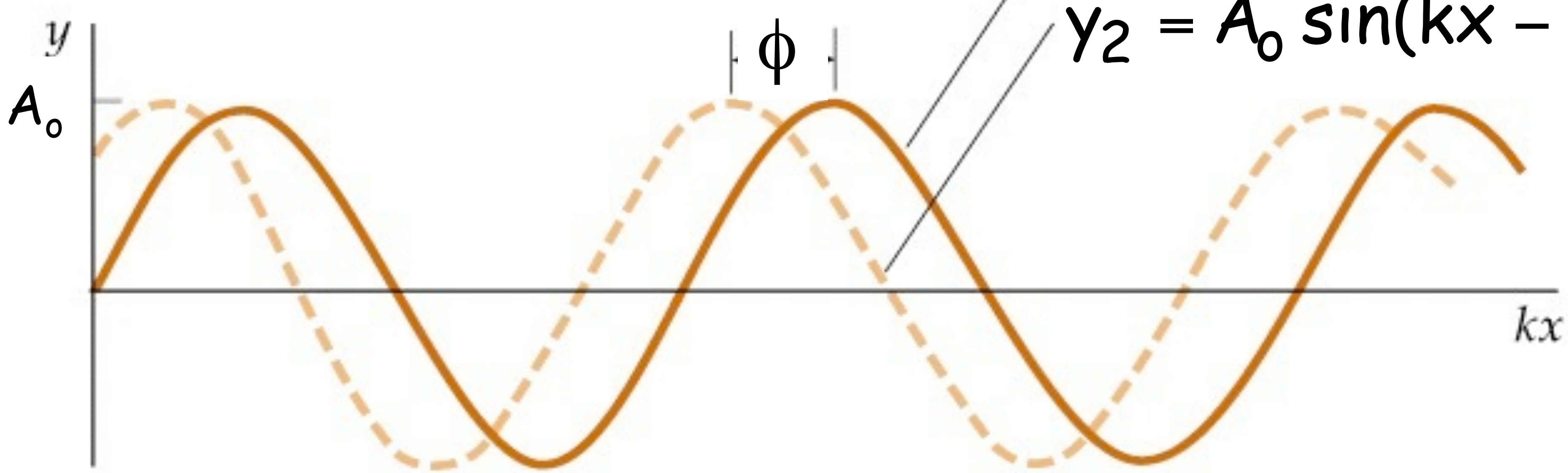
Superposition of Harmonic Waves

Principle of superposition states that when two or more waves combine the net displacement of the medium is the algebraic sum of the two displacements.

Consider two harmonic waves travelling in the same direction in a medium

$$y_1 = A_o \sin(kx - \omega t)$$

$$y_2 = A_o \sin(kx - \omega t - \phi)$$



Tipler

Fig 16-2

$$y_1 = A_0 \sin(kx - \omega t) \qquad y_2 = A_0 \sin(kx - \omega t - \phi)$$

The resultant wave function is given by

$$y = y_1 + y_2 = A_0 \sin(kx - \omega t) + A_0 \sin(kx - \omega t - \phi)$$
$$= A_0 \left[\sin(kx - \omega t) + \sin(kx - \omega t - \phi)\right]$$

This can be simplified using

$$\sin A + \sin B = 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

with $A = (kx - \omega t)$ and $B = (kx - \omega t - \phi)$

$$y = A_o \lfloor \sin(kx - \omega t) + \sin(kx - \omega t - \phi) \rfloor$$

$$= 2A_o \cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

The resulting wavefunction is harmonic and has the same frequency and wavelength as the original waves.

Amplitude of the resultant wave = $2A_o\cos(\phi/2)$

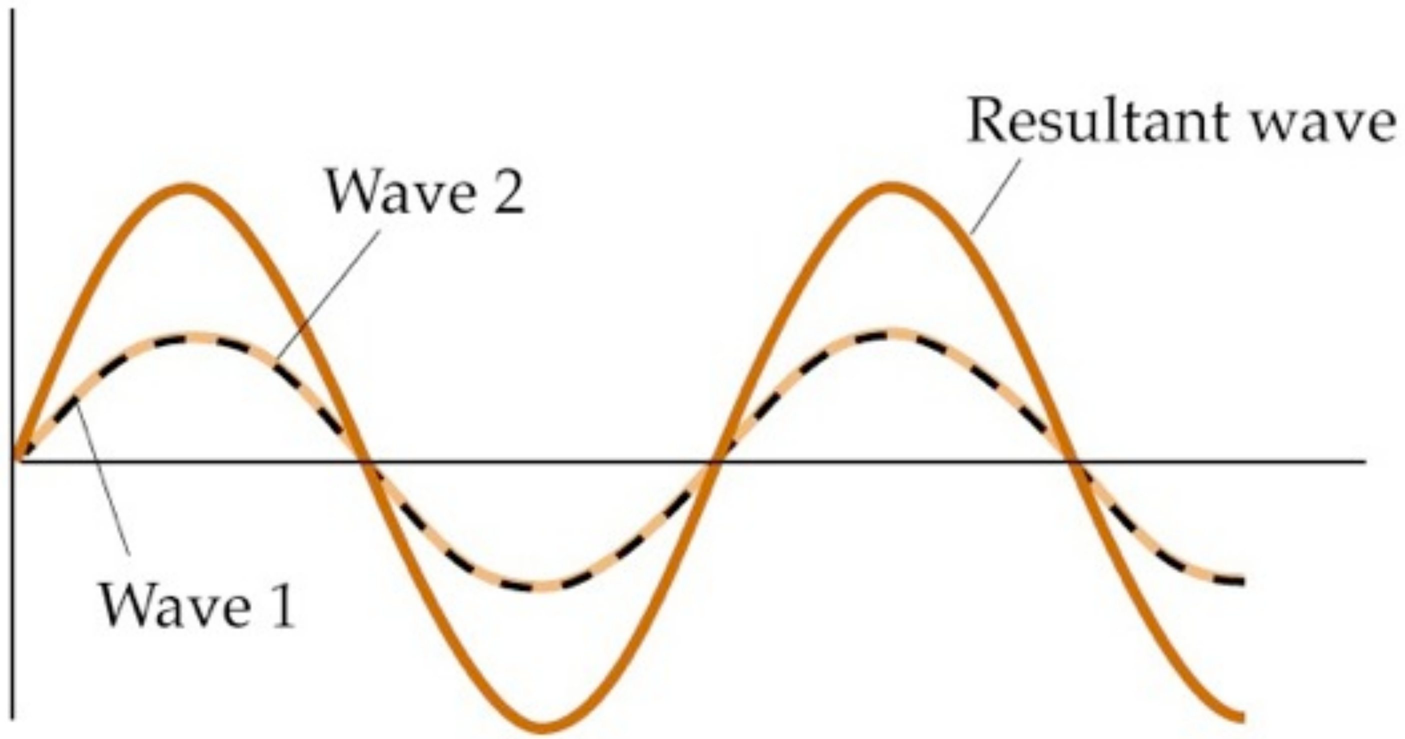
Phase of the resultant wave = $(\phi/2)$

$$y = 2A_0 \cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

when $\phi = 0$ $\cos(\phi/2) = 1$

the amplitude of the resultant wave = $2A_o$

The waves are **in phase** and interfere **constructively.**



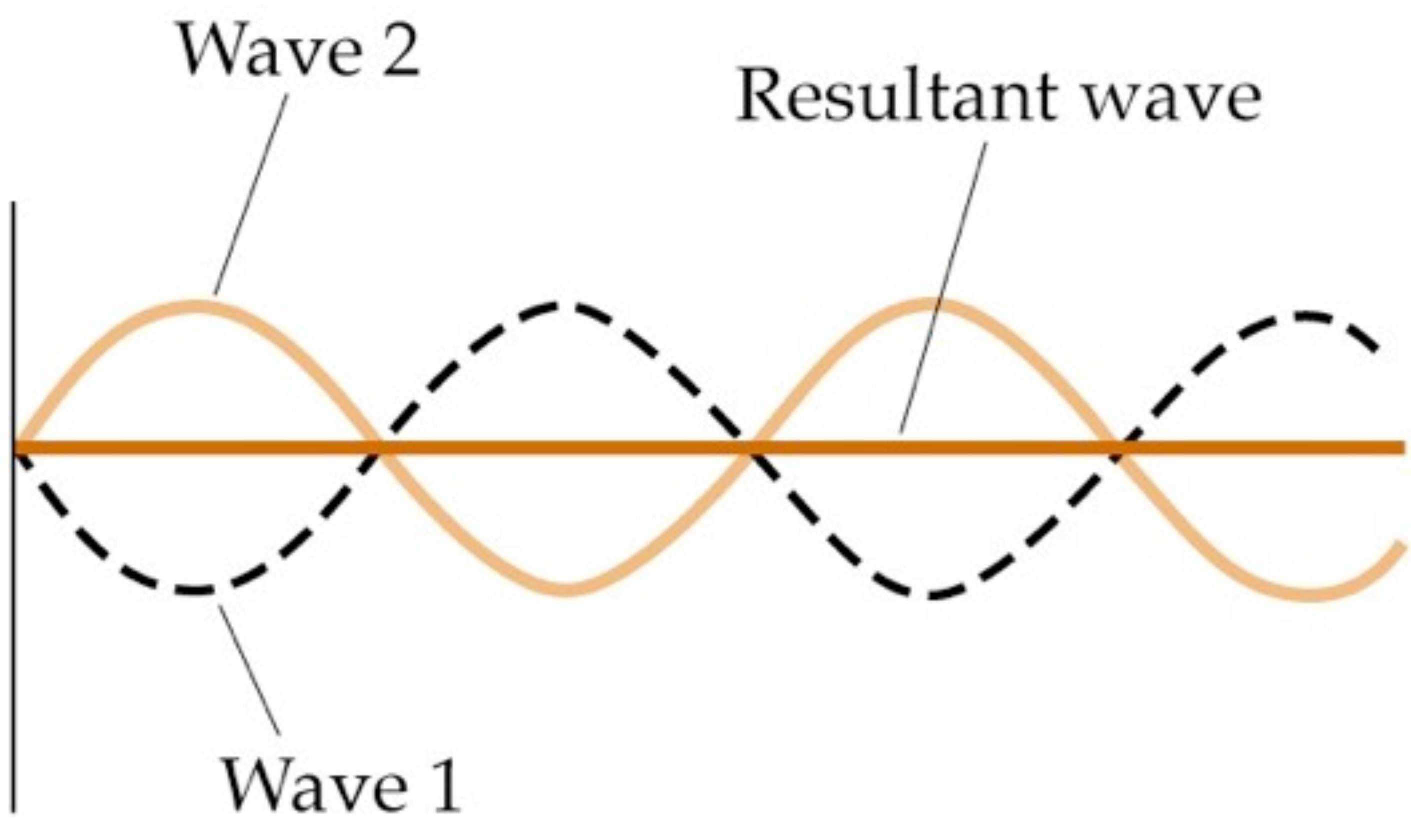
Tipler Fig 16-3

$$y = 2A_0 \cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

when $\phi = \pi$ or any odd multiple of π $\cos(\phi/2) = 0$

the amplitude of the resultant wave = 0

The waves are **out of phase** and interfere **destructively.**

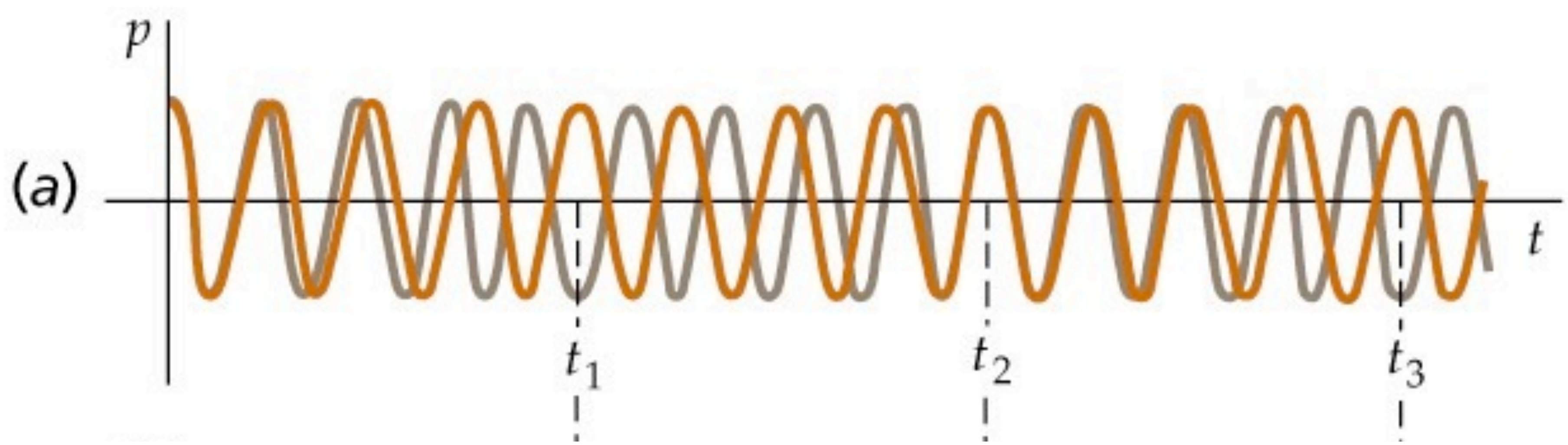


Tipler Fig 16-4

Beats

What happens if the wave have different frequencies ?

Consider two waves travelling in the same direction but with slightly different frequencies



Tipler Fig 16-5a

$$y_1 = A_o \cos(2\pi f_1 t) \qquad y_2 = A_o \cos(2\pi f_2 t)$$

Using the principle of superposition we can say

$$y = y_1 + y_2 = A_o\left[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)\right]$$

$$y = y_1 + y_2 = A_0[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)]$$

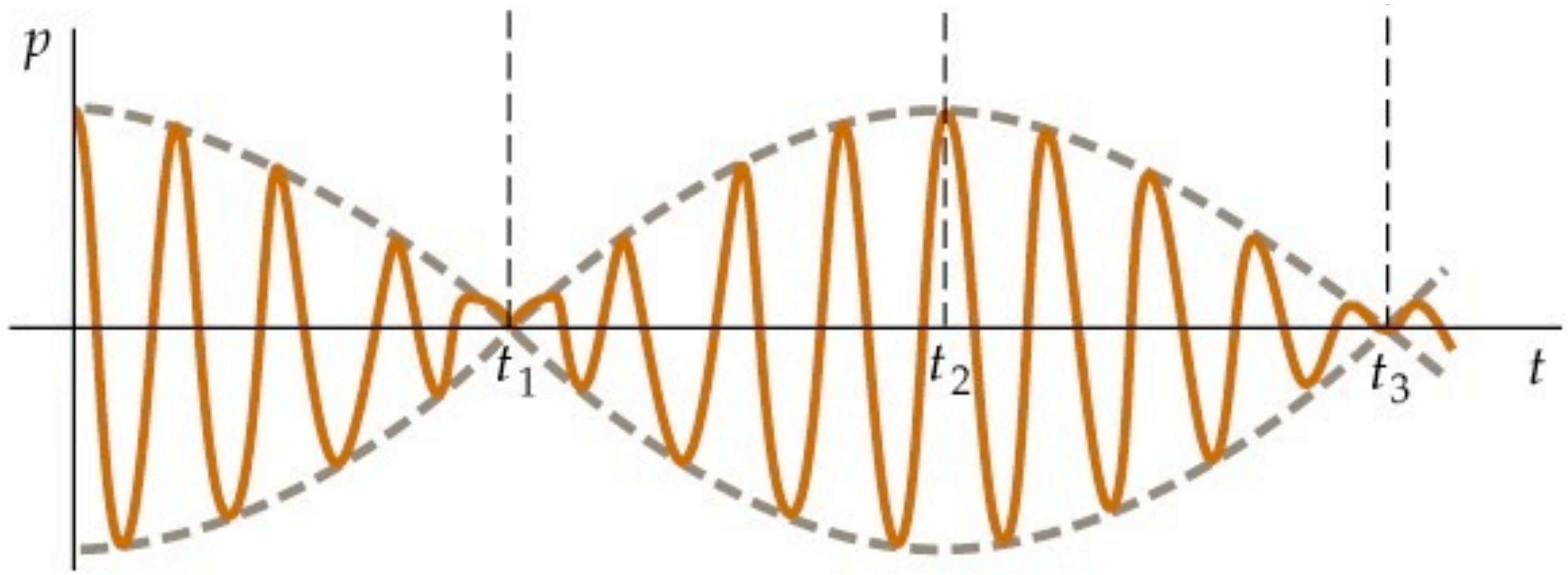
This can be simplified using

$$\cos A + \cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

with $A = (2\pi f_1 t)$

and $B = (2\pi f_2 t)$

$$\therefore \quad y = 2A_0 \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right] \cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right]$$



Tipler Fig 16-5b

$$y = 2A_0 \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right] \cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right]$$

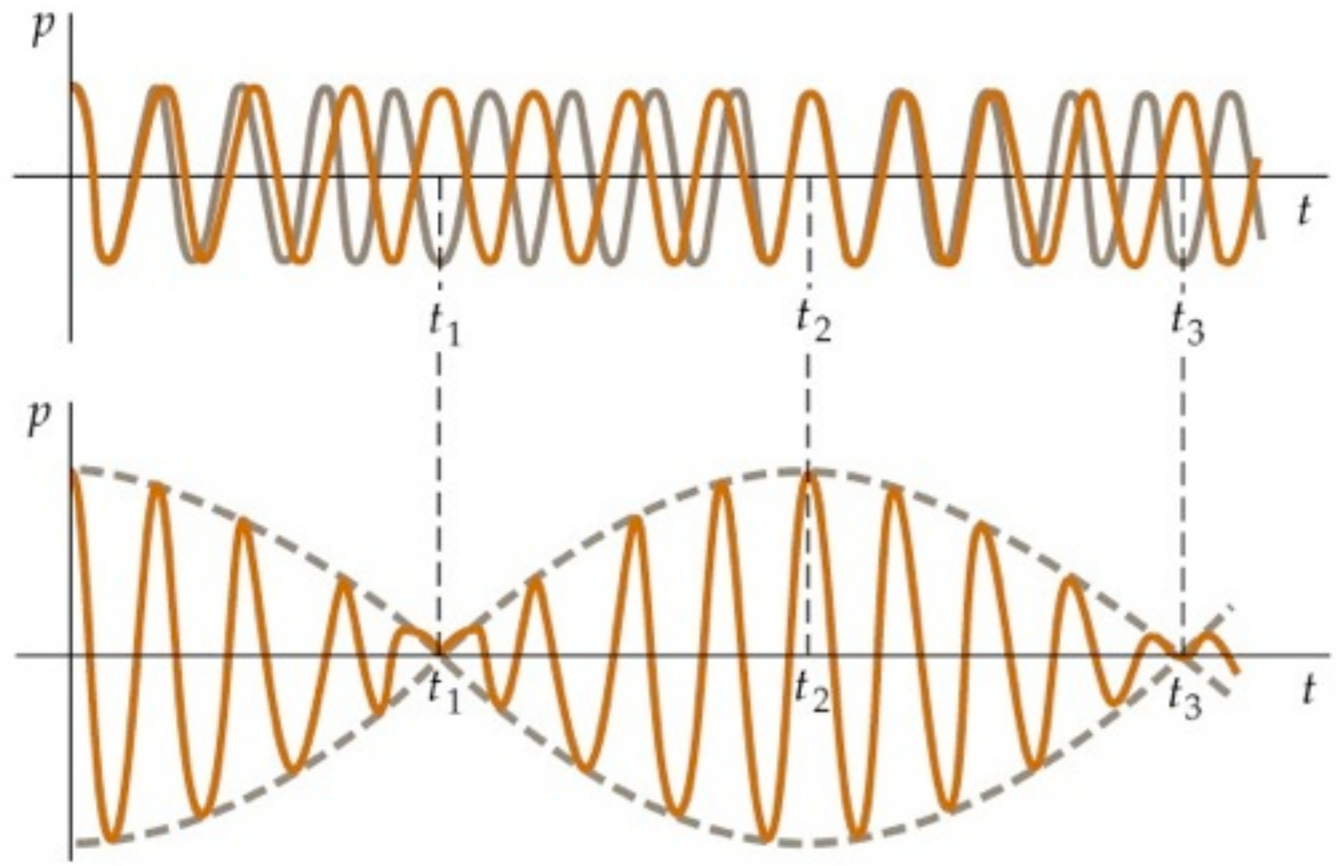
Compare this to the individual wavefunctions:

$$y_1 = A_0 \cos(2\pi f_1 t) \qquad y_2 = A_0 \cos(2\pi f_2 t)$$

Resultant vibration has an effective frequency $(f_1 + f_2)/2$ and an amplitude given by

$$2A_0 \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right]$$

The amplitude varies with time with a frequency $(f_1 - f_2)/2$



Tipler
fig 16-5

A beat is detected when $\cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right] = \pm 1$

There are two beats per cycle

Beat frequency = $2\,(f_1 - f_2)/2 = f_1 - f_2$

What will we hear ?

Consider two tuning forks vibrating at frequencies of 438 and 442Hz.

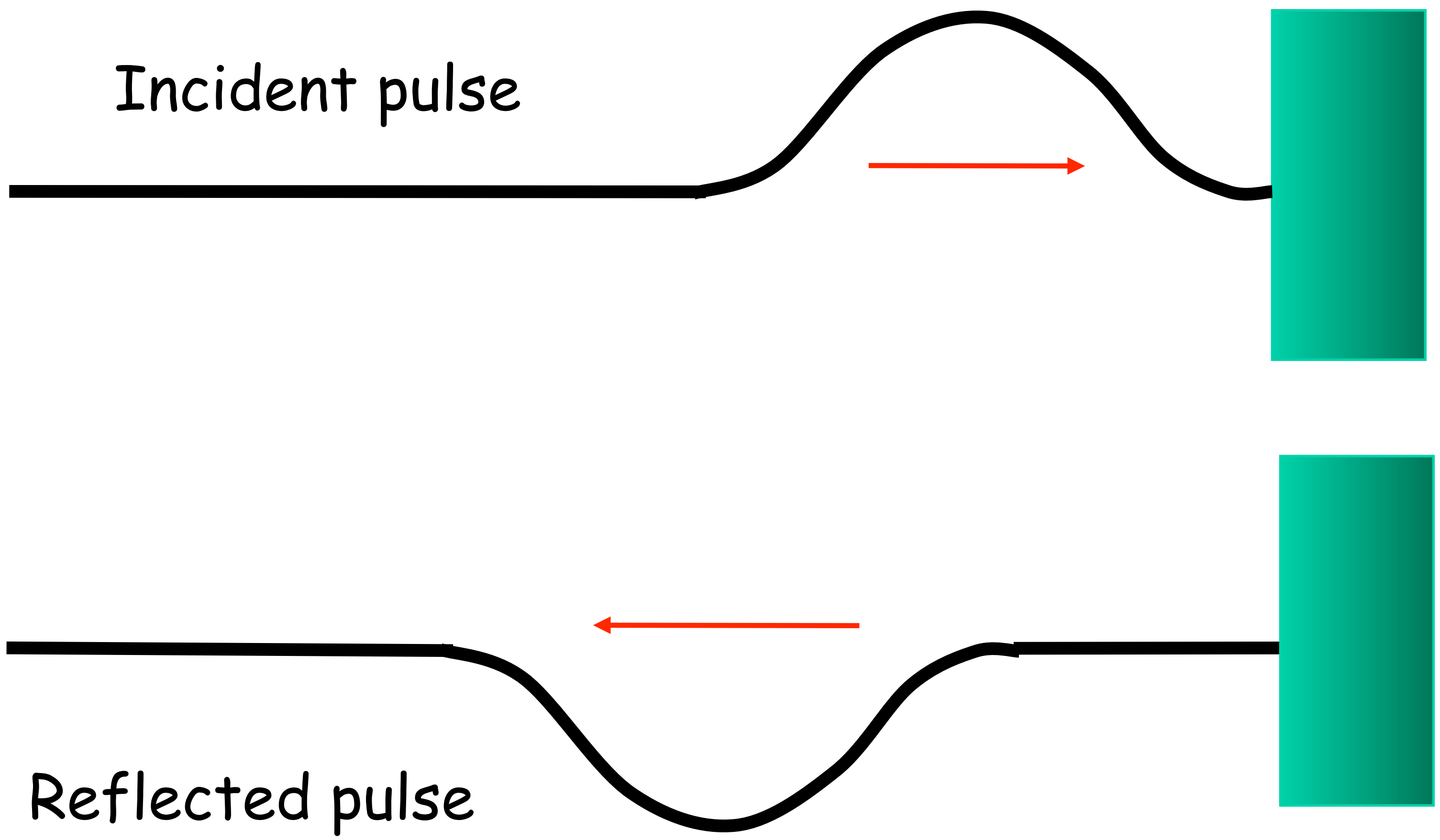
The resultant sound wave would have a frequency of
(438+442) / 2 = 440Hz (A on piano)

and a beat frequency of 442 - 438 = 4Hz

The listener would hear the 440Hz sound wave go through an intensity maximum four times per second

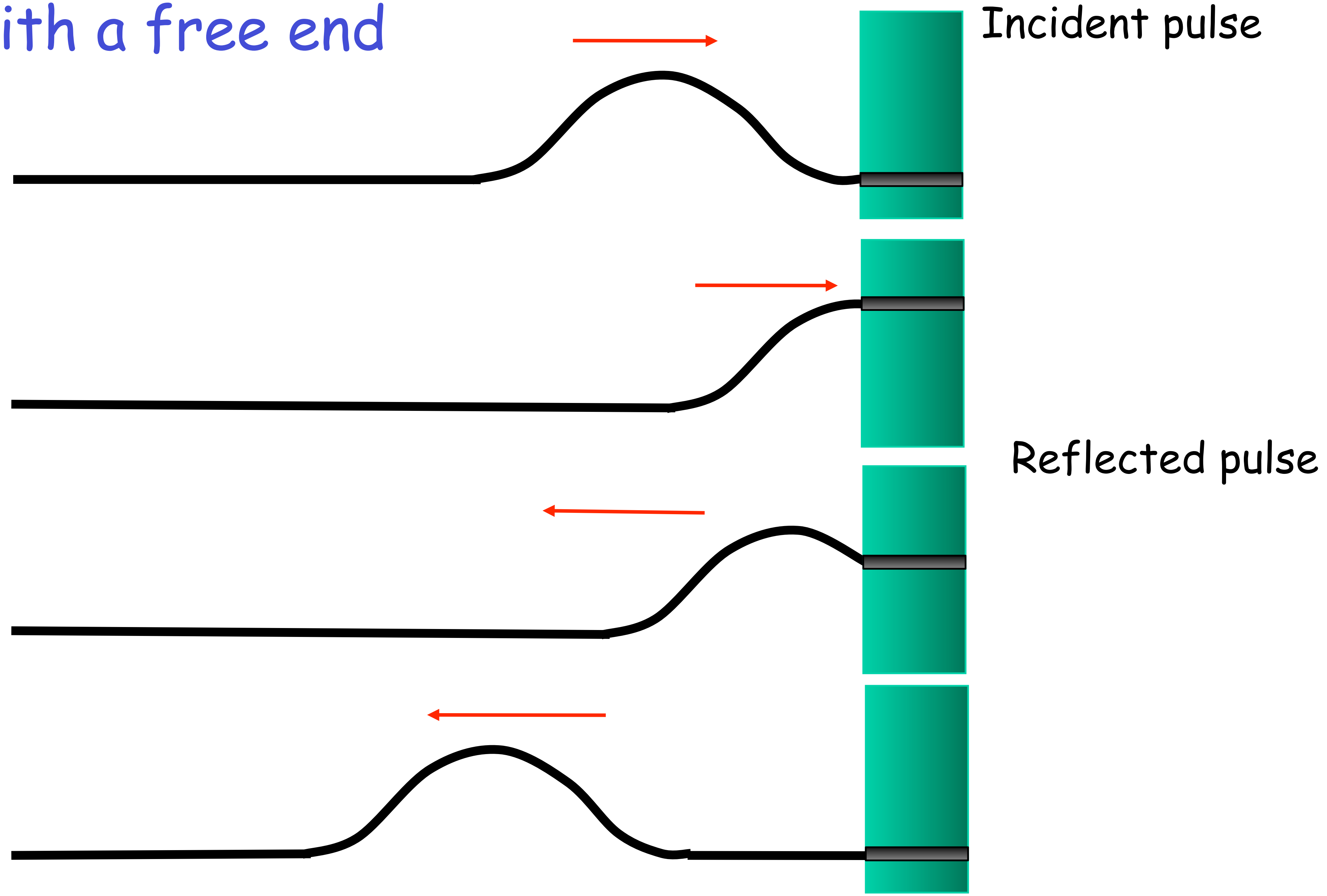
Musicians use beats to tune an instrument

A pulse travelling a string fixed at one end

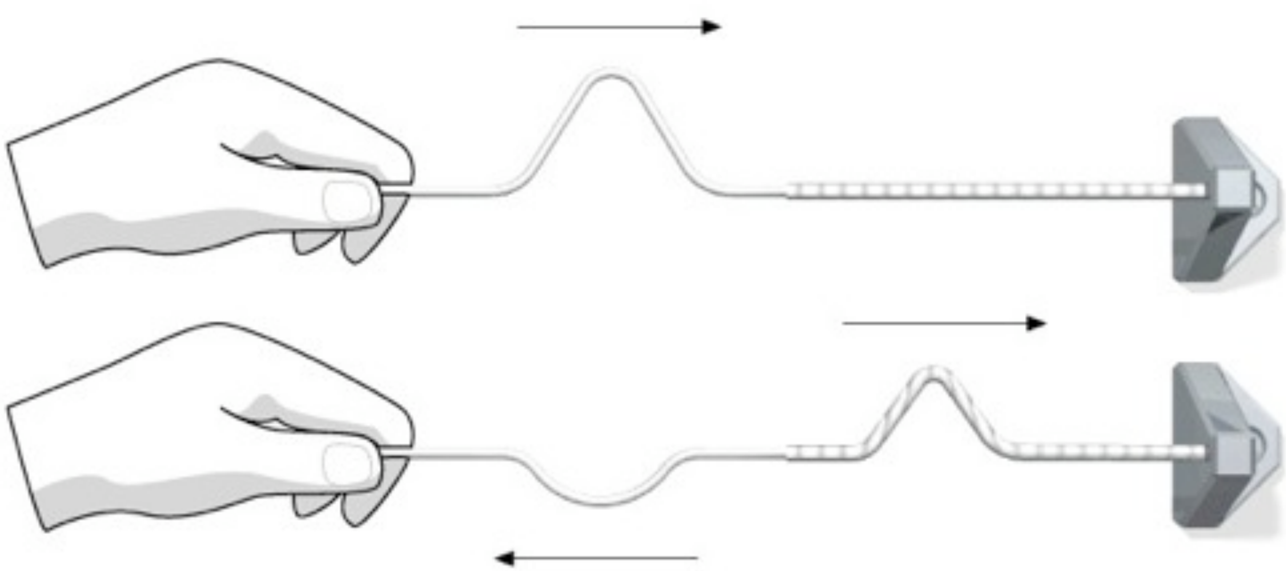


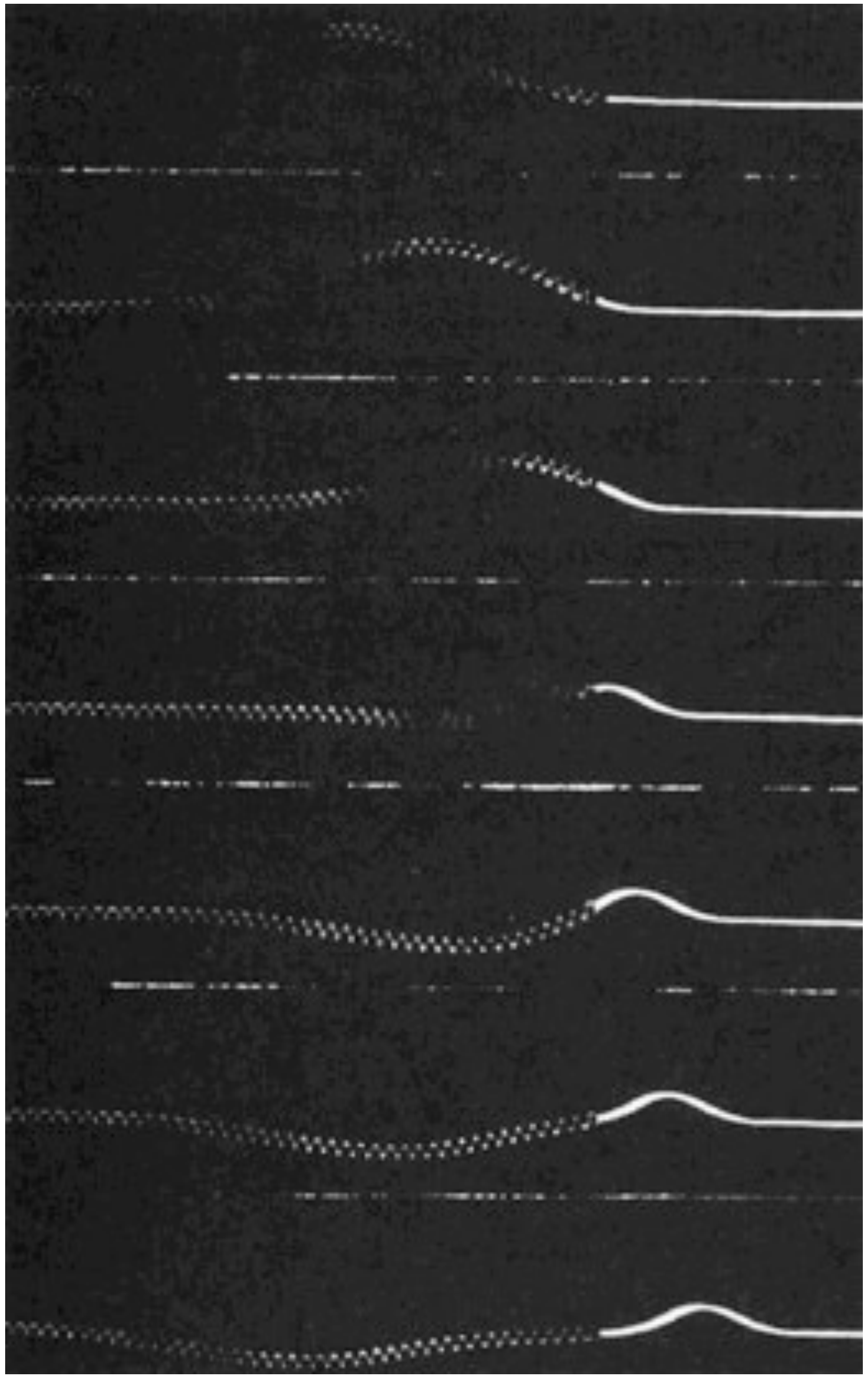
NB. we assume that the wall is rigid and the wave does not transmit any part of the disturbance to the wall

A pulse travelling on a string with a free end

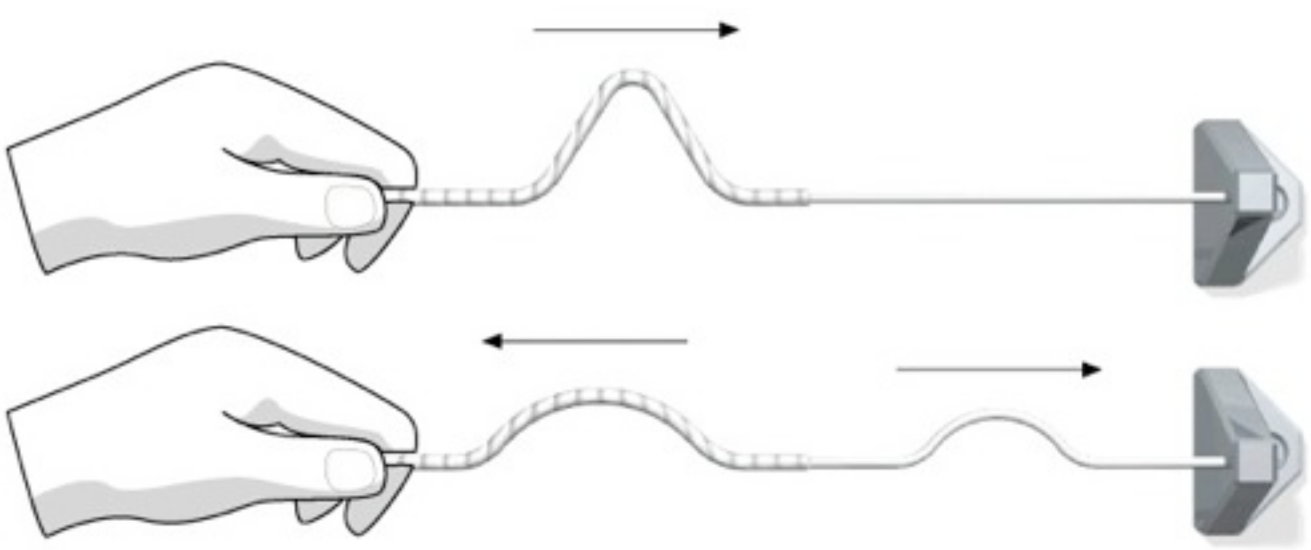


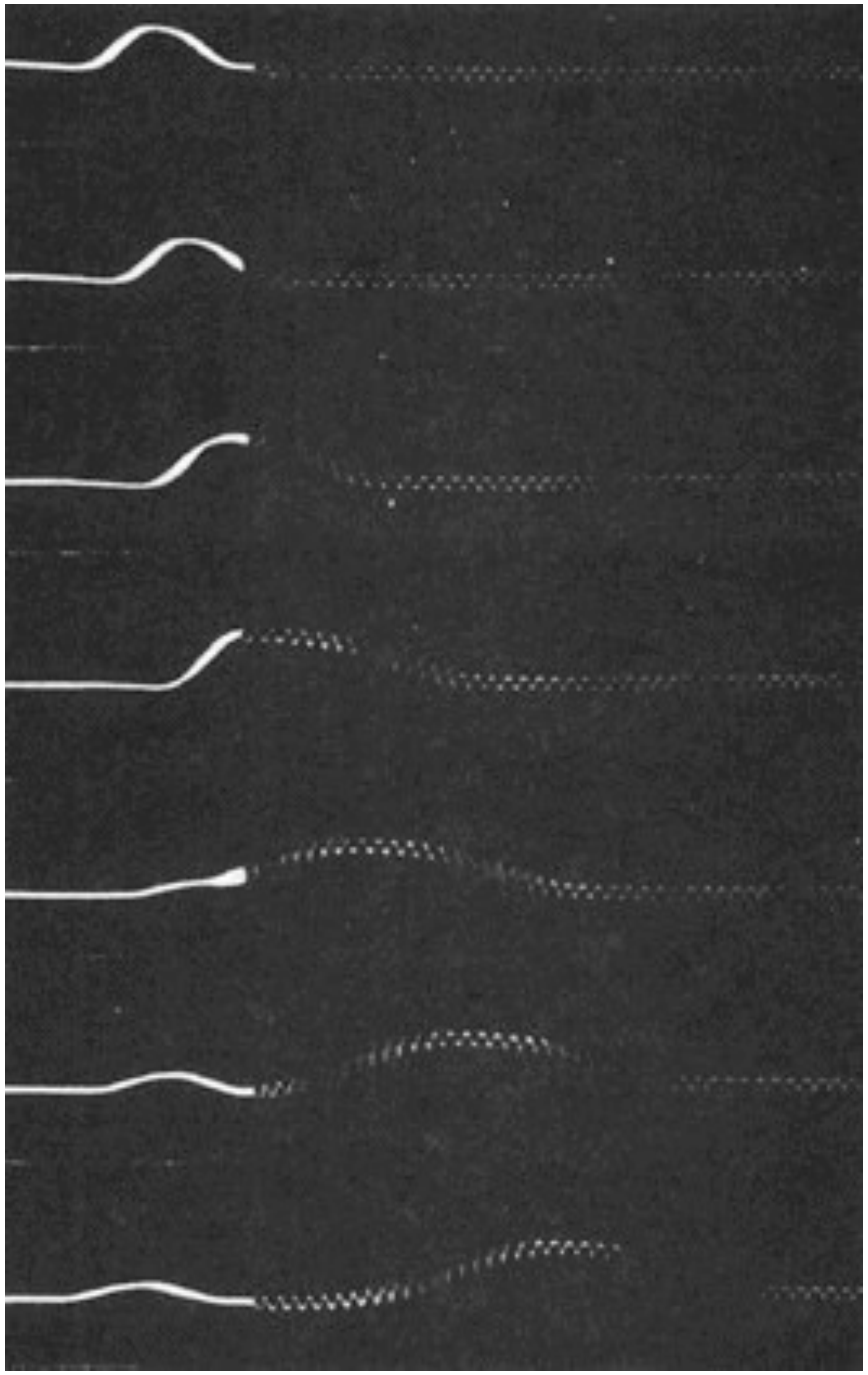
A pulse travelling on a light string attached to a heavier string





A pulse travelling on a heavy string attached to a lighter string



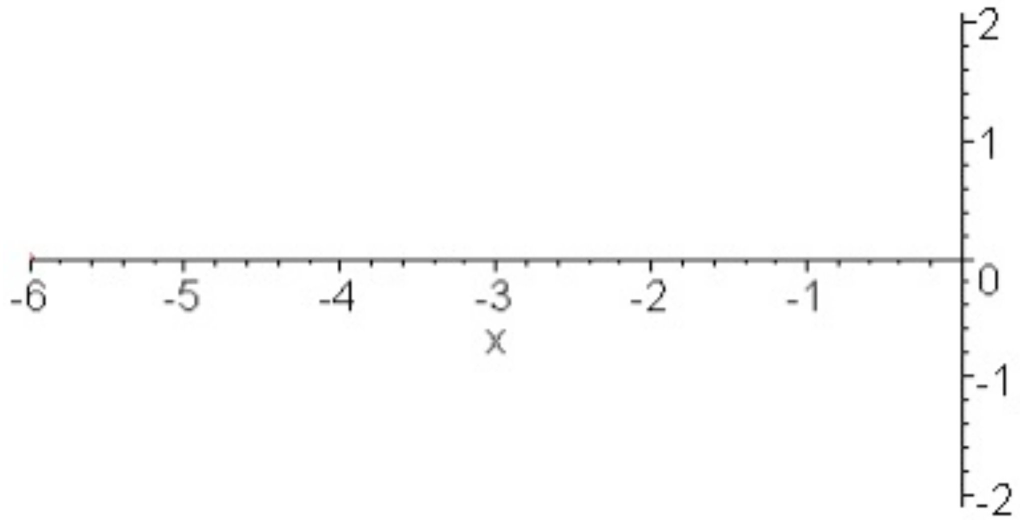


Standing waves

If a string is clamped at both ends, waves will be reflected from the fixed ends and a standing wave will be set up.

The incident and reflected waves will combine according to the principle of superposition

Essential in music and quantum theory !



Wavefunction for a standing wave

Consider two sinusoidal waves in the same medium with the same amplitude, frequency and wavelength but travelling in opposite directions

$$y_1 = A_o \sin(kx - \omega t) \qquad\qquad y_2 = A_o \sin(kx + \omega t)$$

$$y = A_o \lfloor \sin(kx - \omega t) + \sin(kx + \omega t) \rfloor$$

Using the identity $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$y = 2A_o \sin(kx)\cos(\omega t)$$

This is the wavefunction of a standing wave

$$y = 2A_0 \sin(kx) \cos(\omega t)$$

Amplitude = $2A_0 \sin(kx)$

Angular frequency = ω

Every particle on the string vibrates in SHM with the same frequency.

The amplitude of a given particle depends on x

Compare this to travelling harmonic wave where all particles oscillate with the same amplitude and at the same frequency

Antinodes

$$y = 2A_o \sin(kx)\cos(\omega t)$$

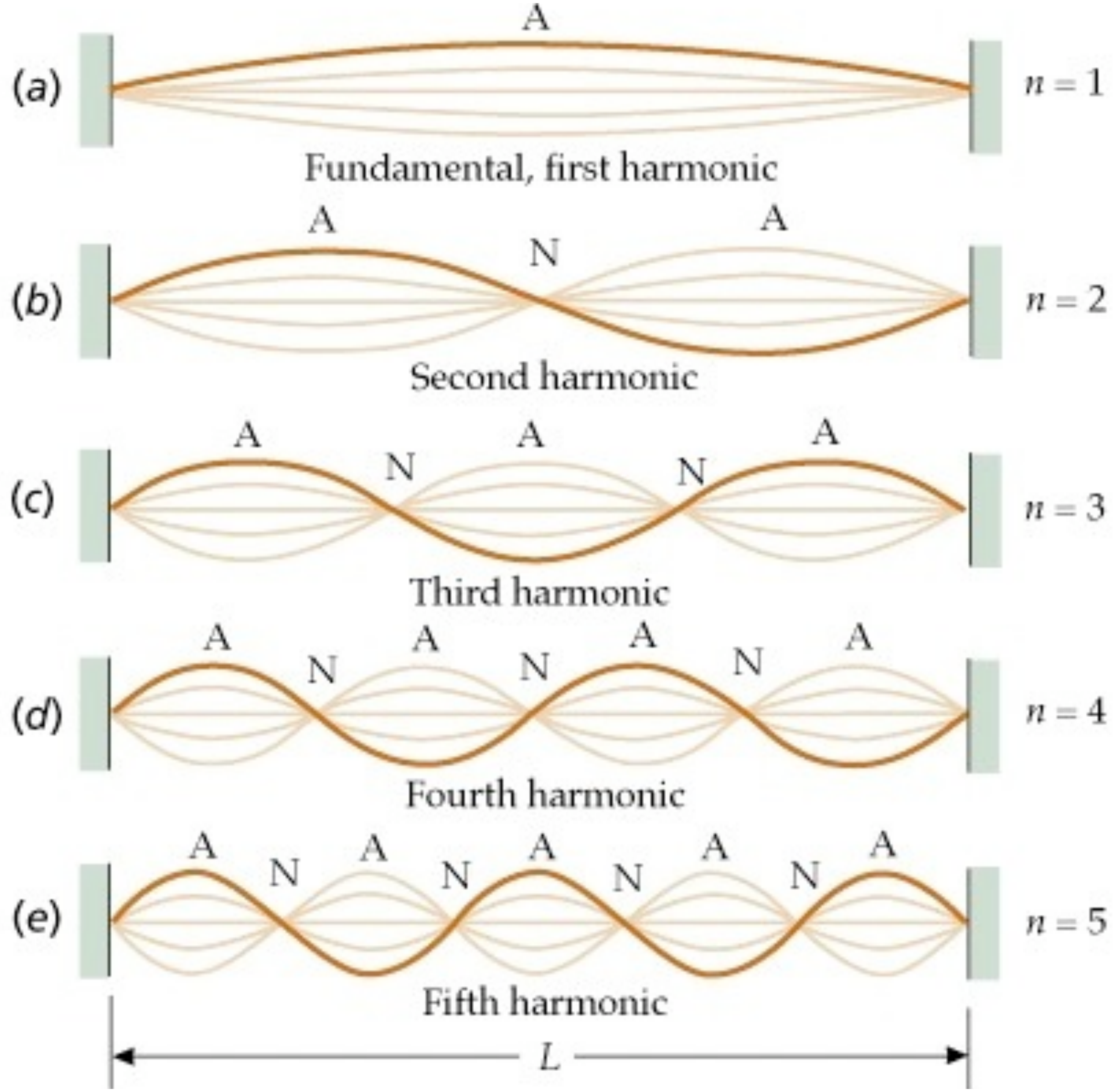
At any x maximum amplitude ($2A_o$) occurs when sin(kx) = 1

$$\text{or when} \quad kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots$$

but $k = 2\pi / \lambda$ and positions of maximum amplitude occur at

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots\ldots = \frac{n\lambda}{4} \quad \text{with } n = 1, 3, 5, \ldots.$$

Positions of maximum amplitude are **ANTINODES** and are separated by a distance of $\lambda/2$.



Tipler Fig 16-11

Nodes

$$y = 2A_o \sin(kx)\cos(\omega t)$$

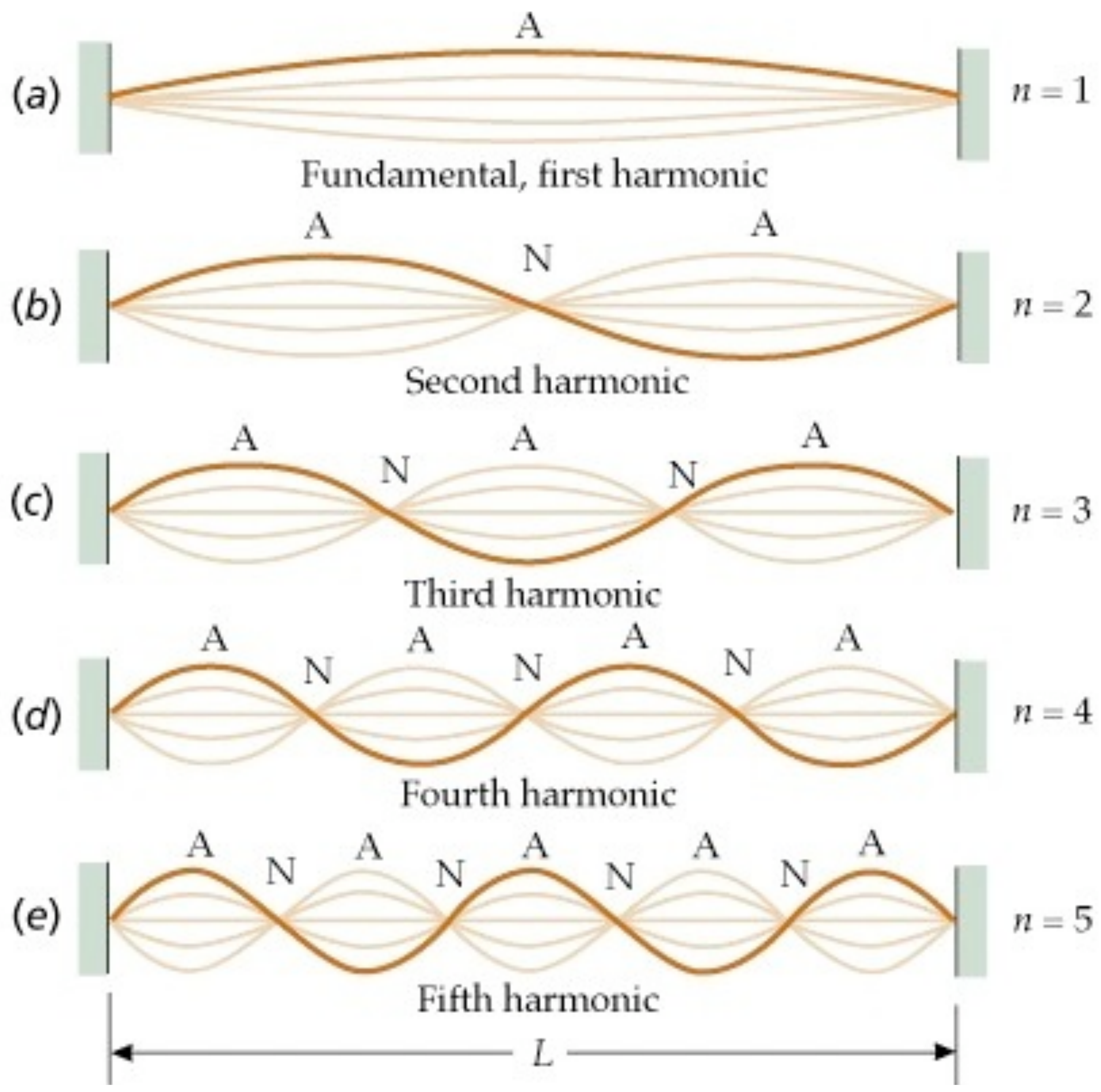
Similarly zero amplitude occurs when sin(kx) = 0

or when $kx = \pi,\ 2\pi,\ 3\pi.......$

$$x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}....... = \frac{n\lambda}{2} \quad \text{with } n = 1, 2, 3,....$$

Positions of zero amplitude are **NODES** and are also separated by a distance of $\lambda/2$.

The distance between a node and an antinode is $\lambda/4$

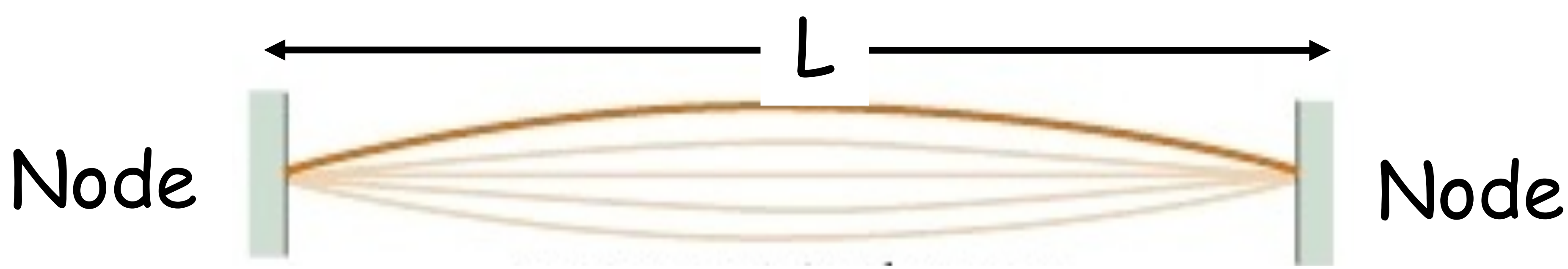


Tipler Fig 16-11

Consider a string of length L and fixed at both ends

The string has a number of natural patterns of vibration called **NORMAL MODES**

Each normal mode has a characteristic frequency which we can easily calculate



When the string is displaced at its mid point the centre of the string becomes an antinode.

For first normal mode $L = \lambda_1 / 2$

The next normal mode occurs when the length of the string L = one wavelength, ie $L = \lambda_2$

The third normal mode occurs when $L = 3\lambda_3 /2$

Generally normal modes occur when $L = n\lambda_n /2$

ie $\lambda_n = \frac{2L}{n}$ where $n = 1,2,3........$

The natural frequencies associated with these modes can be derived from $f = v/\lambda$

$$f = \frac{v}{\lambda} = \frac{n}{2L} v \quad \text{with } n = 1,2,3.....$$

For a string of mass/unit length μ, under tension F we can replace v by $(F/\mu)^{\frac{1}{2}}$

$$f = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \quad \text{with } n = 1,2,3.....$$

The lowest frequency (**fundamental**) corresponds to n = 1

ie $f = \frac{1}{2L} v$ or $f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$

Musical Interpretation

The frequencies of modes with $n = 2, 3, \ldots$ (**harmonics**) are integral multiples of the fundamental frequency, $2f_1$, $3f_1$……

These higher natural frequencies together with the fundamental form a **harmonic series**.

The fundamental f_1 is the first harmonic, $f_2 = 2f_1$ is the second harmonic, $f_n = nf_1$ is the nth harmonic

In music the allowed frequencies are called **overtones** where the second harmonic is the first overtone, the third harmonic the second overtone etc.

We can obtain these expressions from the wavefunctions

Consider wavefunction of a standing wave

$$y(x,t) = 2A_0 \sin(kx)\cos(\omega t)$$

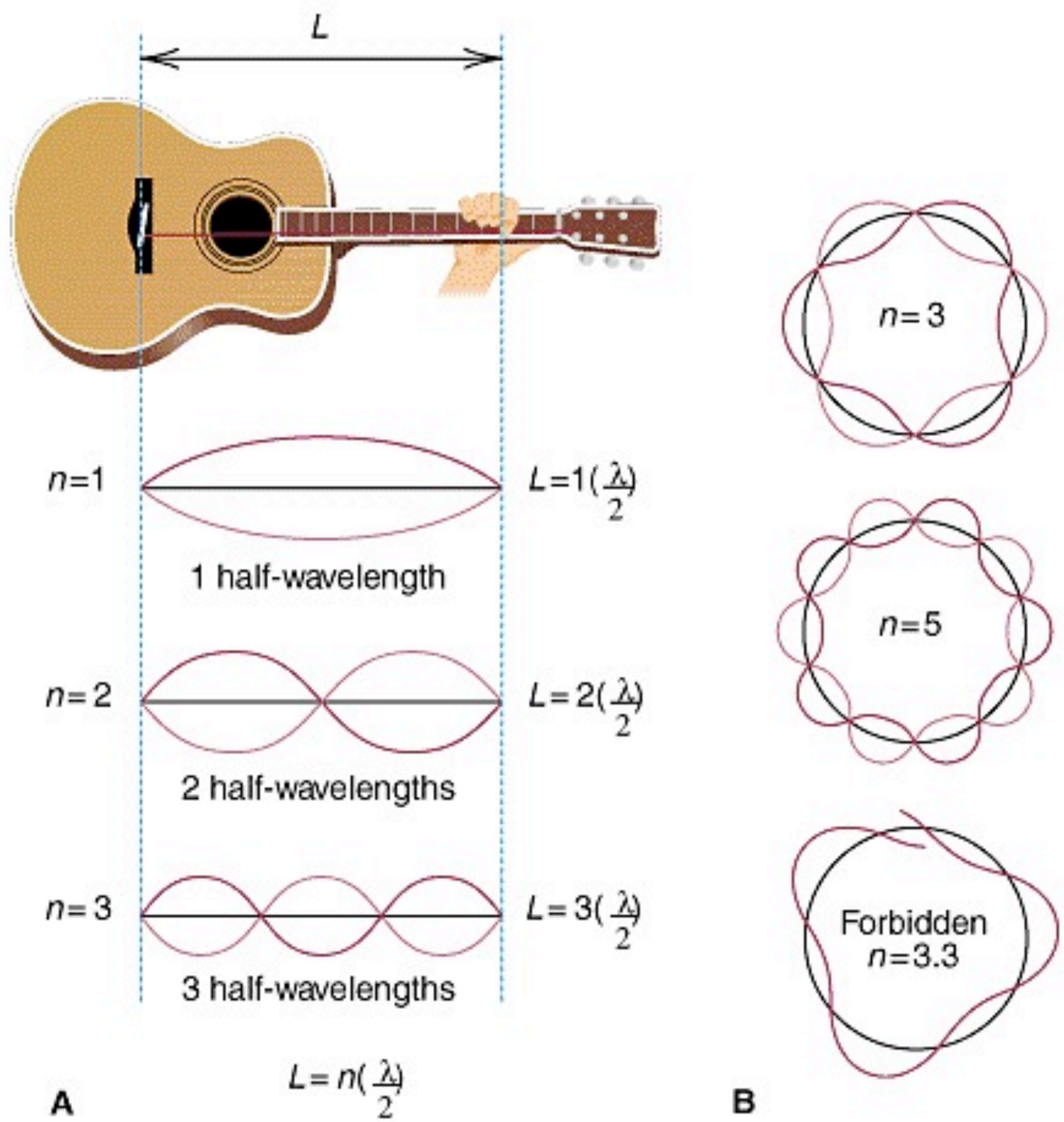
String is fixed at both ends $\therefore$ $y(x,t) = 0$ at $x = 0$ and L

$y(0,t)=0$ when $x = 0$ as $\sin(kx) = 0$ at $x = 0$

$y(L,t) = 0$ when $\sin(kL) = 0$ ie $k_n L = n\pi$ $n=1,2,3....$

but $k_n = 2\pi / \lambda$ $\therefore (2\pi / \lambda_n)L = n\pi$ or $\boxed{\lambda_n = 2L/n}$

Guitars and Quantum mechanics



Fourier Series

- Linear Superimposition of Sinusoids to build complex waveforms

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_n t + \phi_n)$$

- If periodic (repeating)

$$\omega_n = n\omega_1$$

Jean Baptiste Joseph Fourier
1768-1830

Fourier Series

- Decompose our complex periodic waveform into a series of simple sinusoids
- Where

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

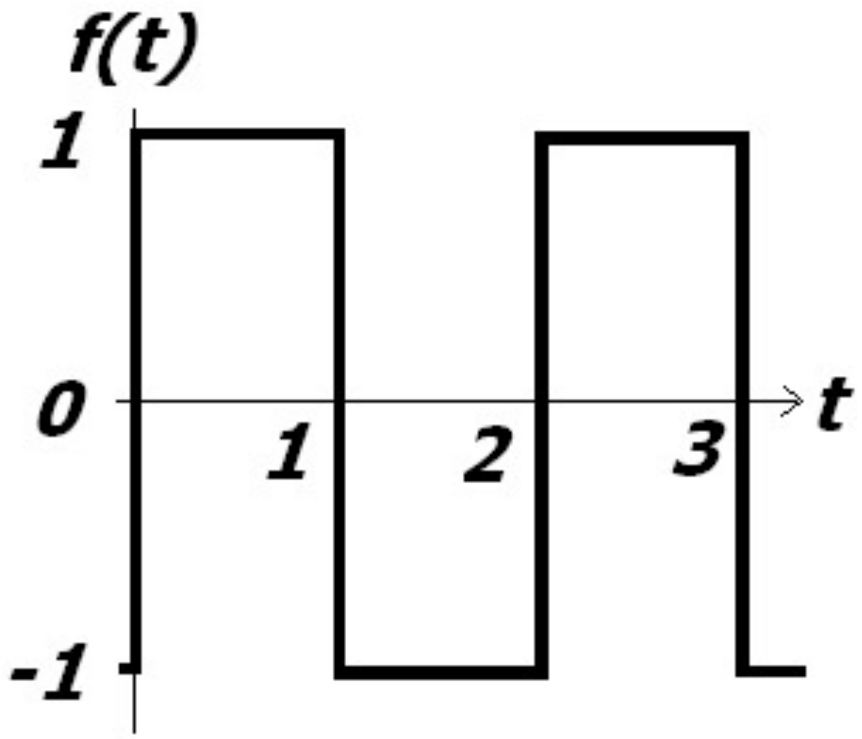
$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

$$\omega = \frac{2\pi}{T}$$

Square Wave Example

- Consider



$$f(t) = 1 \qquad 0 < t < 1$$

$$= -1 \qquad 1 < t < 2$$

- Clearly the period is T=2 hence $\omega = \pi$
- When we integrate we need to do so over sections: t= 0 to 1 and t=1 to 2

$$a_0 = \int_0^2 f(t)dt$$

- So to find the series, we have to calculate coefficients a_0, a_n and b_n

$$= \int_0^1 dt - \int_1^2 dt = 1 - 1 \Rightarrow a_0 = 0$$

$$a_n = \int_0^2 f(t)\cos(n\pi t)dt$$

$$= \int_0^1 \cos(n\pi t)dt - \int_1^2 \cos(n\pi t)dt$$

$$= \frac{1}{n\pi}\left[\sin(n\pi t)\Big|_0^1 - \sin(n\pi t)\Big|_1^2\right] \Rightarrow a_n = 0$$

When evaluating a_n note that the sin function is 0 when angle is every multiple of π

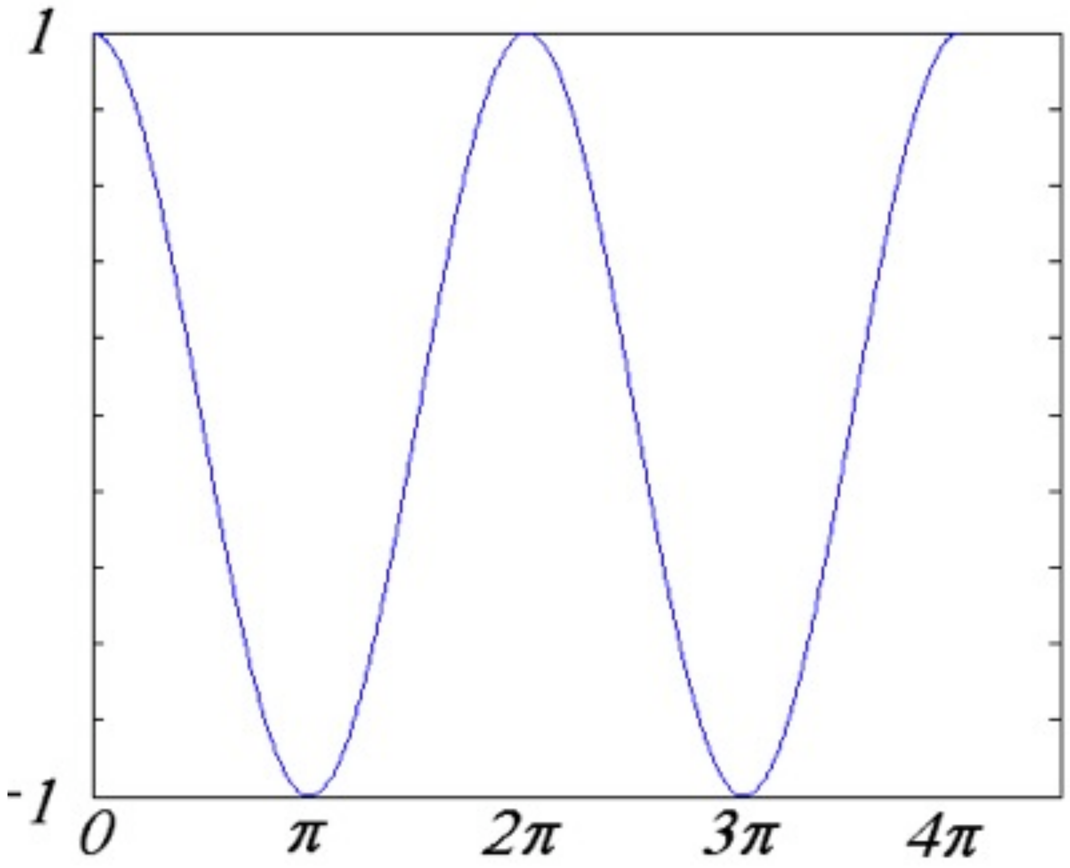
$$b_n = \int_0^2 f(t)\sin(n\pi t)dt = \int_0^1 \sin(n\pi t)dt - \int_1^2 \sin(n\pi t)dt$$

$$= -\frac{1}{n\pi}\left[\cos(n\pi t)\Big|_0^1 - \cos(n\pi t)\Big|_1^2\right]$$

$$= -\frac{1}{n\pi}\Big((\cos(n\pi) - 1) - (\cos(n\pi 2) - \cos(n\pi))\Big)$$

$$= -\frac{1}{n\pi}(2\cos(n\pi) - 1 - \cos(n2\pi))$$

- Knowing $$b_n = -\frac{1}{n\pi}(2\cos(n\pi) - 1 - \cos(n2\pi))$$
- We need to consider the cos function to determine values of b_n for n=1,2,3,…. etc



$$n = 1,\quad b_1 = -\frac{1}{n\pi}(2(-1) - 1 - 1) = \frac{4}{n\pi}$$

$$n = 2,\quad b_2 = -\frac{1}{n\pi}(2(1) - 1 - 1) = 0$$

$$n = 3,\quad b_3 = -\frac{1}{n\pi}(2(-1) - 1 - 1) = \frac{4}{n\pi}$$

$$n = 4,\quad b_4 = -\frac{1}{n\pi}(2(1) - 1 - 1) = 0$$

$$n = 5,\quad b_2 = -\frac{1}{n\pi}(2(-1) - 1 - 1) = \frac{4}{n\pi}$$

- We found coefficients to be

$$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{4}{n\pi} \text{ when n} = 1,3,5...$$

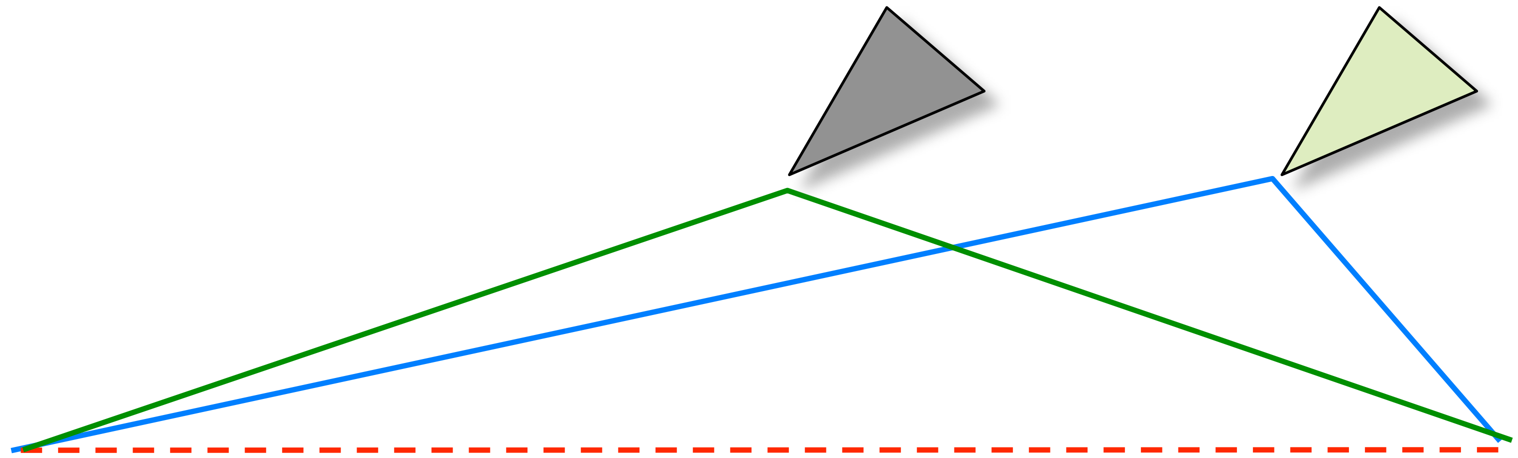
- Hence Fourier Series for a square wave is

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$= \frac{4}{\pi}\ [\ \sin(\pi\ t) + \tfrac{1}{3}\sin(3\pi\ t) + \tfrac{1}{5}\sin(5\pi\ t) + \ldots\ ]$$

Plucked string

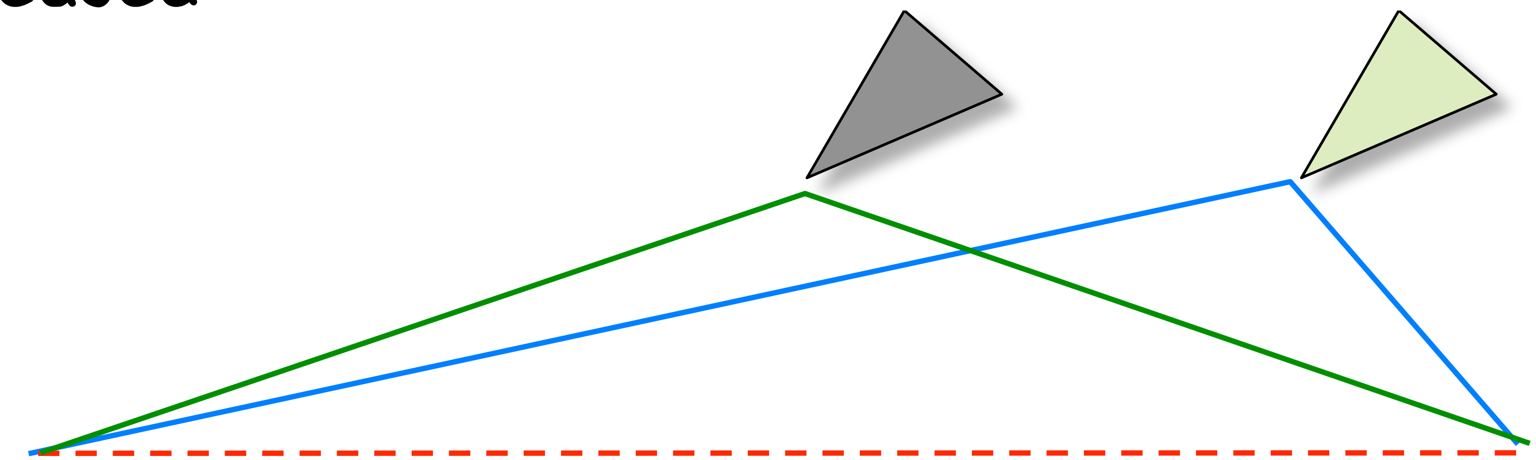
- Can one predict the amplitude of each mode (overtone/harmonic?) following plucking?



- Using the procedure to measure the Fourier coefficients it is possible to predict the amplitude of each harmonic tone.

Plucked string

- You know the shape just before it is plucked.
- You know that each mode moves at its own frequency
- The shape when released
- We rewrite this as



$$f(x, t = 0)$$

$$f(x, t = 0) = \sum_{m=0}^{\infty} A_m \sin \frac{2\pi m x}{2L}$$

Plucked string

Each harmonic has its own frequency of oscillation, the m-th harmonic moves at a frequency $f_m = mf_0$ or m times that of the fundamental mode.

$$f(x,t=0) = \sum_{m=0}^{\infty} A_m \sin\frac{2\pi mx}{2L} \quad \text{initial condition}$$

$$f(x,t) = \sum_{m=0}^{\infty} A_m \sin\frac{2\pi mx}{2L} \cos 2\pi m f_0 t$$

http://cnyack.homestead.com/files/afourse/fs1dwave.htm

Modal summation on a string

Recall modes on a string:

$$u(x, t) = \sum_{n=0}^{\infty} A_n U_n(x, \omega_n) \cos(\omega_n t)$$

This is the sum of standing waves or *eigenfunctions,* $U_n(x, \omega_n)$, each of which is weighted by the amplitude A_n and vibrates at its *eigenfrequency* ω_n.

The eigenfunctions and eigenfrequencies are constants due to the physical properties of the string.

The amplitudes depend on the position and nature of the source that excited the motion.

The eigenfunctions were constrained by the boundary conditions, so that

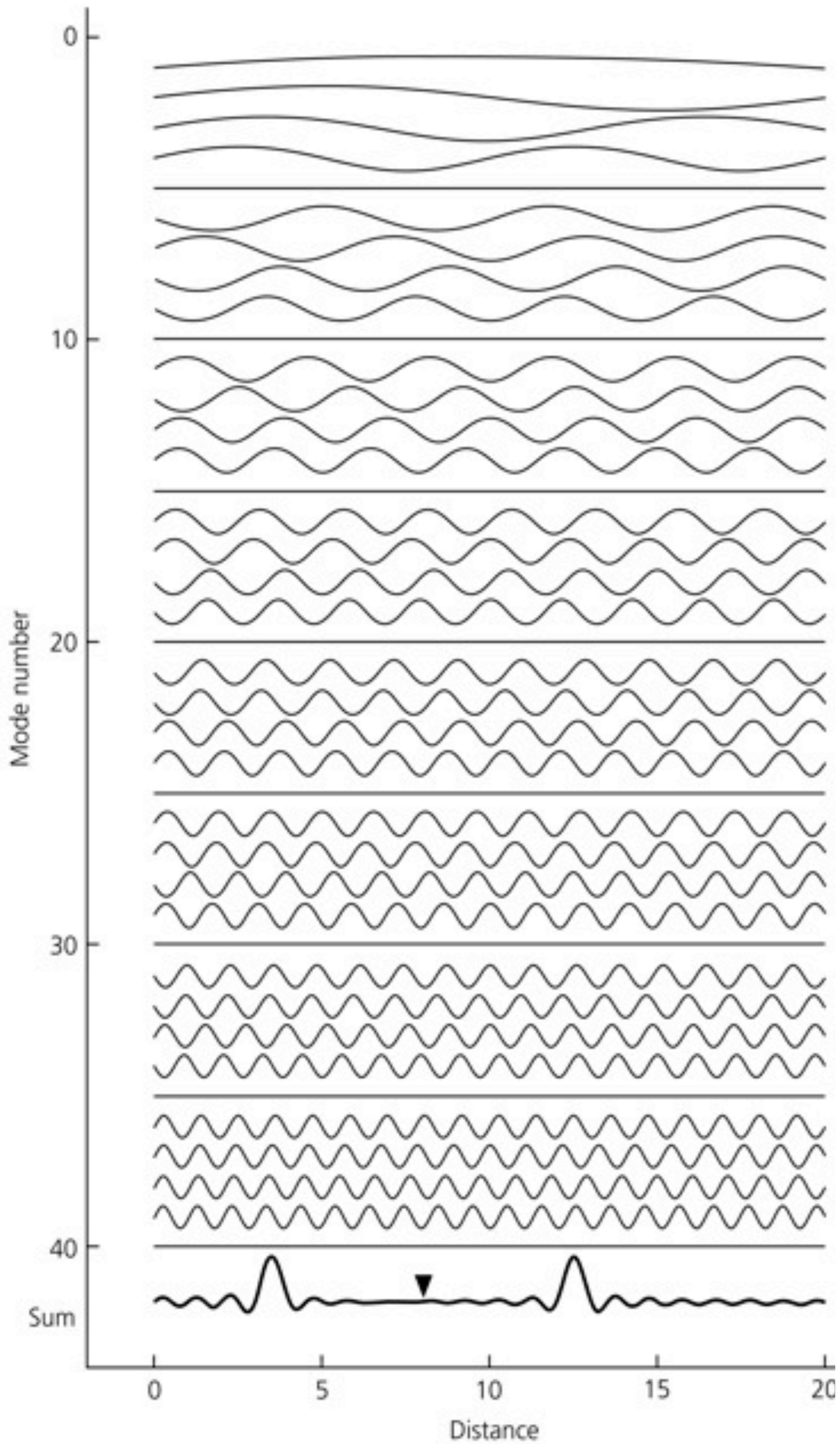
$$U_n(x, \omega_n) = \sin(n\pi x/L) = \sin(\omega_n x/\mathrm{v}) \qquad \omega_n = n\pi \mathrm{v}/L = 2\pi \mathrm{v}/\lambda$$

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\, F(\omega_n)\, \sin(n\pi x/L)\, \cos(\omega_n t)$$

The source, at $x_s = 8$, is described by

$$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$$

with $\tau = 0.2$.



Example: Violin