UNITS - DIPARTIMENTO FISICA

APPUNTI LEZIONI DI APPROFONDIMENTI DI ELETTRODINAMICA

TENUTE DAL PROF. F. PARMIGIANI

A.A. 2020-2021

A SHORT INTRODUCTION TO THE ELECTRO-MAGNETIC LAGRANGIAN

THE ACTION PRINCIPLE WAS CENTRAL TO THE MECHANICAL PHYSICS OF THE 19th CENTURY. THIS WAS THE MAIN REASON WHY MAXWELL HAS DEDICATED AN ENTIRE CHAPTER OF HIS "TREATISE ON ELECTRICITY AND MAGNETISM" TO THE EQ. OF LAGRANGE AND HAMILTON. ON THE OTHER HAND THE USEFULNESS OF A LAGRANGIAN-HAMILTONIAN CONCEPTS AND FORMALISM TO SOLVE PROBLEMS IN E.M. WAS VERY LIMITED. THEREFORE, THE CENTRAL QUESTION HERE IS WHY ADOPTING THE LAGRANGE-HAMILTON FORMALISM? THE IMPORTANCE OF THE LAGRANGE-HAMILTON FORMALISM WILL EMERGE SEVERAL DECADES AFTER MAXWELL WITH THE PHYSICS OF AN ATOM IN AN E.M. (ABSORPTION-EMISSION OF LIGHT) - FIELD QUANTISATION - QUANTUM FIELD THEORY - CHARGES IN AN E.M. FIELD - QUANTUM ELECTRODYNAMICS - QED AND MORE. HISTORICALLY IS DUE TO HELMHOLTZ THE DISCOVERY OF A SINGLE LAGRANGIAN FROM WHICH IS POSSIBLE TO DERIVE BOTH THE MAXWELL'S SYSTEM OF FIELD EQUATIONS AND THE COULOMB-LORENTZ FORCE LAW FOR MOVING CHARGES. HOWEVER, BECAUSE OF ITS COMPLEXITY HELMHOLTZ SOLUTION WAS LARGELY IGNORED.

LATER IN 1892 LORENTZ BY VIRTUAL WORK ARGUMENTS, LATER SET IN THE LAGRANGIAN FORMALISM BY DARRIGOL (2000), WAS THE FIRST TO WRITE THE LAGRANGIAN FOR A CHARGE PARTICLE IN AN E.M. FIELD

$$L(\bar{r},\bar{v}) = \frac{1}{2} m v^2 + Q\bar{v}\cdot\bar{A}(\bar{r},t) - Q\varphi(\bar{r},t)$$

AND USE IT TO DERIVE THE FORCE $\bar{F} = q(\bar{E} + \bar{v}\times\bar{B})$. LORENTZ ALSO DERIVE THE FREE-FIELD LAGRANGIAN

$$L_F = \frac{1}{2}\varepsilon_0 \int_\tau (\bar{E}\cdot\bar{E} - c^2\bar{B}\cdot\bar{B})\, d\tau \equiv \int_\tau \mathcal{L}_F\, d^3r$$

AND USE IT TO DERIVE THE FARADAY LAW TREATING $\bar{E}$ AS GENERALIZED COORDINATE. LATER IN THESE LECTURES WE WILL DERIVED THE GAUSS AND AMPÈRE-MAXWELL LAWS FROM DYNAMICAL PRINCIPLES, NOT DERIVED BY LORENTZ.
THE LAGRANGIAN $L_{EM} = L_P + L_{PF} + L_F$, THAT WE CALL E.M. LAGRANGIAN, WHERE L_P = LAGRANGIAN FOR THE PARTICLES, L_{PF} = LAGRANGIAN FOR THE INTERACTION BETWEEN PARTICLES AND FIELDS, L_F = LAGRANGIAN OF THE FIELDS.

$$L_{EM} = \frac{1}{2}\sum_i m_i v_i^2 + \int_\tau (\bar{J}\cdot\bar{A} - \rho\varphi)\, d\tau +$$

$$+ \frac{1}{2}\varepsilon_0 \int_\tau \left[(\bar{\nabla}\varphi + \partial_t\bar{A})^2 - c^2(\bar{\nabla}\times\bar{A})^2 \right] d\tau .$$

THE IDEA TO USE THE VECTOR AND SCALAR POTENTIALS AS GENERALIZED COORDINATES IS DUE

TO KARL SCHWARZCHILD (1903). TODAY HE IS
BEST KNOWN AS THE FIRST TO OBTAIN AN
EXACT SOLUTION (THE NON-ROTATING BLACK-
HOLE) OF EINSTEIN'EQ. OF GENERAL RELATIVITY.

1 - LAGRANGIAN AND HAMILTONIAN EQUATIONS IN ELECTRODYNAMICS

1.0 THE LAGRANGIAN OF THE E.M. FIELD

THE E.M. FIELD IN THE EMPTY SPACE IS DESCRIBED BY THE MAXWELL EQS_(M.Es)

$$\bar{\nabla} \cdot \bar{E} = \rho / \varepsilon_0 \quad (1) \qquad \bar{\nabla} \cdot \bar{B} = 0 \quad (3)$$

$$\bar{\nabla} \times \bar{E} = -\partial_t \bar{B} \quad (2) \qquad \bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \partial_t \bar{E} \quad (4)$$

THE CONTINUITY EQ. $\partial_t \rho + \bar{\nabla} \cdot \bar{J} = 0$

FOLLOWS FROM (1) AND (4), WHILE THE CLASSICAL DESCRIPTION OF THE E.D. PHENOMENA REQUIRES TO COMPLETE THE FOUR M.Es WITH THE LORENTZ FORCE $\bar{F} = q(\bar{E} + \bar{v} \times \bar{B})$ (5). ALTHOUGH THESE FIVE EQs REPRESENT A COMPLETE SET TO DESCRIBE THE CLASSICAL E.D. FIELDS AND THEY INTERACTION WITH CHARGES THE FORMULATION OF THE E.D. FIELDS BY POTENTIALS ϕ (SCALAR ELECTRIC POTENTIAL) AND $\bar{A}$ (VECTOR

MAGNETIC POTENTIAL.

$$\bar{B} = \bar{\nabla} \times \bar{A}; \quad \bar{E} = -\bar{\nabla}\phi - \partial_t \bar{A}$$

. WHILE

THE EQ. $\bar{\nabla} \cdot \bar{B} = \emptyset$ AND $\bar{\nabla} \times \bar{E} = -\partial_t \bar{B}$ ARE DIRECTLY SATISFIED, FOR THE REMAINING WE NEED TO KNOW THE SOURCES OF THE FIELDS, $\bar{\rho}$ AND $\bar{J}$. FURTHERMORE, USING THE GAUGE RELATIONS

$\bar{A}' = \bar{A} + \bar{\nabla}\lambda$ AND $\phi' = \phi - \partial_t \lambda$ WE CAN ADOPT FORMAL SEMPLIFICATIONS THAT LEAD TO THE SAME FIELDS

$$\bar{B} = \bar{\nabla} \times A' = \bar{\nabla} \times \bar{A}, \quad \bar{E} = -\bar{\nabla}\phi' - \partial_t \bar{A}' = -\bar{\nabla}\phi - \partial_t \bar{A}$$

EXAMPLES OF GAUGE ARE GIVEN BY

$$\phi = \emptyset; \quad \bar{\nabla} \cdot \bar{A} = \emptyset; \quad \bar{\nabla} \cdot \bar{A} = -\frac{1}{c^2} \partial_t \phi.$$

THE GENERAL PROBLEM WE ARE FACING IN E.D. IS GIVEN BY THE DESCRIPTION OF THE PHYSICS RELATED TO AN ELECTRICAL CHARGE INTERACTING WITH AN E.M. FIELD, OR A SYSTEM OF CHARGES INTERACTING WITH AN E.M. FIELD. THIS IS A PROBLEM OF FORMIDABLE COMPLEXITY, THAT WE CAN SOMETIMES SOLVE CONSISTENTLY IN THE CONTEST OF THE CLASSICAL E.D. - FOR EXAMPLE THE PHYSICS OF ACCELERATORS, OTHER TIMES WE HAVE TO EVOLVE THE CLASSICAL E.D. INTO THE QUANTUM PHYSICS AND THE QUANTUM E.D. ULTIMATELY, WE SHOULD EVOLVE THE CLASSICAL

EQUATION OF MOTION (NEWTON'S LAW) INTO EQS OF MOTION THAT TAKE INTO ACCOUNT RADIATIVE EFFECTS, QUANTISATION EFFECTS, WHEN IMPOSED BY THE PROBLEM CONDITIONS, AND RELATIVISTIC EFFECTS. WE KNOW THAT WE CAN TAKE A GREAT ADVANTAGE BY DESCRIBING THE MECHANICS OF THE PROBLEM BY FORMULATING THE PROBLEM USING THE ACTION PRINCIPLE FOR IT INVOLVES THE SAME PARAMETERS AS FORMULATING NEWTON'S EQUATIONS. THIS LEADS US TO THE PRINCIPLE OF THE LEAST ACTION HENCE TO THE DERIVATION OF THE EULER-LAGRANGE EQ. AN EVENTUALLY TO THE SYMMETRY AND CONSERVATION LAWS AND TO THE HAMILTONIAN MECHANICS ALONG WITH TIME-TRANSLATION INVARIANCE.

ASSUMING THIS BASIC NOTIONS OF CLASSICAL MECHANICS RETRIVED WE NOW EXTEND THESE CONCEPT TO THE E.M. FIELDS AND TO THE ELECTRODYNAMICS PROCESSES.

- DERIVING THE MAXWEL FROM THE HAMILTON PRINCIPLE

WE START FROM ASSUMING THE FOLLOWING FORM FOR THE LAGRANGIAN

$$\boxed{L = L_{PART.} + L_{INTER.} + L_{FIELD}} \quad (1)$$

THE LAGRANGIAN FOR THE FREE PARTI-

CLES L_{PART} FOR THE NON-RELATIVISTIC CASE IS GIVEN BY

$$L^{NR}_{PART.} = \sum_i{}' \frac{1}{2} m_i \dot{\bar{r}}_i^2 \,,$$ BEING m_i AND $\bar{r}_i$ THE MASS AND THE COORDINATE OF THE i-ESIME PARTICLE, WHEREAS, FOR THE RELATIVISTIC CASE WE HAVE

$$L^{REL}_{PART.} = \sum_i{}' - m_i c^2 \sqrt{1 - \frac{\dot{\bar{r}}_i^2}{c^2}}$$

i IDENTIFIES THE i-PARTICLE.

WE CAN DERIVE NOW THE MOTION EQ., CONSIDERING THE LORENTZ FORCE, FOR CHARGED PARTICLES IN AN EXTERNAL EM FIELD. FOR THIS CASE WE ASSUME THAT THE PARTICLES DOES NOT SIGNIFICANTLY AFFECT THE EXTERNAL FIELDS, THE INTERACTION LAGRANGIAN IS GIVEN BY (MKSA UNITS)

CHARGE PARTICLE i-

$$L_{INTER.} = \sum_i{}' \left[Q_i \bar{A}(\bar{r}_i) \cdot \dot{\bar{r}}_i - Q_i \phi(\bar{r}_i) \right]$$

TO THE END OF DERIVING THIS EQUATION WE MUST REMEMBER THAT: (1) A CONSERVATIVE FORCE IS RELATED TO A POTENTIAL $V(r,t)$ BY THE RELATION $\bar{F} = -\bar{\nabla} V(r,t)$. EXAP. FOR THE CASE OF THE ELECTRIC COMPONENT OF THE LORENTZ FORCE

$\bar{F} = e\bar{E}$, $\bar{E} = -\nabla\phi$ $(V = \phi) \Rightarrow \bar{F} = -e\bar{\nabla}\phi$ $(e = q)$.

(2) THE CLASSICAL MECHANICS FORM OF THE LAGRANGIAN IS GIVEN BY

$$L(\bar{r}, \dot{\bar{r}}, t) = T - V$$

WHERE $\bar{r}$ IS A SPACE COORDINATE, $\dot{\bar{r}}$ IS ITS TIME DERIVATIVE, T IS THE KINETIC ENERGY AND V THE POTENTIAL ENERGY

(3) THE ACTION S IS GIVEN BY

$$S = \int_{t_0}^{t_1} L(\bar{r}, \dot{\bar{r}})\, dt \quad *$$

(4) THE EULER-LAGRANGE EQ. FOR A 1D CASE $(x, \dot{x})$ IS GIVEN BY

E-L EQ. $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$ (2)

THAT IN 3D CARTESIAN SYSTEM BECOMES

$$\frac{d}{dt}\left(\bar{\nabla}_{\dot{x}_j \equiv \dot{v}_j} L\right) - \bar{\nabla} L = 0 \qquad x_j \quad j \equiv 1, 2, 3 \equiv x, y, z$$

WHERE $\bar{\nabla}_{\dot{q}_i}$ MEANS THE GRADIENT WITH RESPECT THE VELOCITY COORDINATES.

NOW WE GENERALIZE $V(x, t)$ TO $U(x, \dot{x}, t)$ THIS IS POSSIBLE AS LONG AS $L = T - U$ GIVES THE CORRET EQ. OF MOTION.

- LORENTZ FORCE FROM POTENTIALS

* LET ME REMIND HERE THAT FOR THE LAGRANGIAN-HAMILTONIAN FORMALISM $\bar{r}_i \equiv (x_i, y_i, z_i)$ REFERS TO THE i-PARTICLE.

$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B})$. IF THE SOURCES OF THE FIELDS ARE FAR AWAY FROM THE POINT WHERE THE FIELDS ARE ACTING, THAN $\bar{E}$ AND $\bar{B}$ SOLVE CORRECTLY THE HOMOGENEOUS MAXWELL EQS. LET'S DERIVE THE LORENTZ LAW AS FUNCTION OF THE POTENTIALS USING THE GAUSSIAN UNITS. THE CALCULUS IN MKSA IS LEFT AS EXECISE.

$$\bar{F} = q\left(\bar{E} + \frac{\bar{v}}{c} \times \bar{B}\right), \quad \bar{\nabla} \cdot \bar{B} = 0; \quad \bar{\nabla} \times \bar{E} = -\frac{1}{c}\partial_t \bar{B}$$

$\Rightarrow \bar{\nabla} \times \bar{E} + \frac{1}{c}\partial_t \bar{B} = \emptyset$. SINCE $\bar{B} = \bar{\nabla} \times \bar{A} \Rightarrow$

$\bar{\nabla} \times \left(\bar{E} + \frac{1}{c}\partial_t \bar{A}\right) = 0 \Rightarrow \left(\bar{E} + \frac{1}{c}\partial_t \bar{A}\right)$ IS A

CONSERVATIVE FIELD $\Rightarrow \bar{E} + \frac{1}{c}\partial_t \bar{A} = -\bar{\nabla}\varphi$

- QUESTION: WHY WE USE HERE φ AND NOT $V(q, t)$?

$$\bar{E} = -\bar{\nabla}\varphi - \frac{1}{c}\partial_t \bar{A} \Rightarrow \boxed{\bar{F} = Q\left[-\bar{\nabla}\varphi - \frac{1}{c}\left(\partial_t \bar{A} - \bar{v} \times (\bar{\nabla} \times \bar{A}\right)\right]} \quad (3)$$

THIS IS THE LORENTZ FORCE EXPRESSED IN TERM OF POTENTIALS UNDER THE APPROXIMATION OF DISTANT SOURCES.

WE APPLY NOW THE GENERAL VECTOR RELATION $\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a} \cdot \bar{c}) - (\bar{a} \cdot \bar{b})\bar{c} \Rightarrow$

$$\bar{v} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\bar{v} \cdot \bar{A}) - (\bar{v} \cdot \bar{\nabla})\bar{A}.$$

$$\boxed{\bar{F} = Q\left\{-\bar{\nabla}\varphi - \frac{1}{c}\left[\partial_t \bar{A} + (\bar{v}\cdot\bar{\nabla})\bar{A} - \bar{\nabla}(\bar{v}\cdot\bar{A})\right]\right\}}$$

THE TOTAL TIME DERIVATIVE OF $\bar{A}(\bar{r},t)$

$$\frac{d}{dt}\bar{A}(\bar{r},t) = \partial_t \bar{A}(\bar{r},t) + \underbrace{\sum_i v_i \partial_{x_i} \bar{A}(\bar{r},t)}_{\Downarrow}$$

$$(\bar{v}\cdot\bar{\nabla})\bar{A}(\bar{r},t)$$

IF WE WRITE $\bar{\nabla}(\bar{v}\cdot\bar{A}) = (\bar{v}\cdot\bar{\nabla})\bar{A}$ (THE OPERATOR $\bar{\nabla}$ AND $\cdot$ ARE LINEAR AND THEY COMMUTE) $\Rightarrow$

$$\boxed{\bar{F} = Q\left(-\bar{\nabla}\varphi - \frac{1}{c}\frac{d}{dt}(\bar{A}) + \frac{1}{c}\bar{\nabla}(\bar{v}\cdot\bar{A})\right)} \quad (4)$$

- LAGRANGIAN FORMALISM

LET'S TRY TO ADD A VECTOR POTENTIAL TERM $U_{\bar{A}}(\bar{r},\bar{v},t) = -\frac{Q}{c}\bar{v}\cdot\bar{A}$ TO THE LAGRANGIAN

$$L = \underbrace{\frac{1}{2}mv^2 - Q\varphi(\bar{r},t)}_{\mathrm{I}} + \underbrace{\frac{Q}{c}\bar{v}\cdot\bar{A}}_{\mathrm{II}} \quad (5)$$

WE NOW APPLY THE EULER-LAGRANGIAN EQ. OF MOTION ON THE PART I, WE GET $m\frac{d\bar{v}}{dt} + Q\bar{\nabla}\varphi = 0$. IF WE APPLY THE E-L EQ. TO PART II $\frac{d}{dt}\left(\bar{\nabla}_{\bar{v}} U_{\bar{A}}\right) - \bar{\nabla}U_{\bar{A}} =$

$= \frac{Q}{c}\frac{d\bar{A}}{dt} - \frac{Q}{c}\bar{\nabla}(\bar{v}\cdot\bar{A}) = 0$. IF WE IDENTIFY

ALL TOGETHER, THE E-L EQ. OF MOTION

APPLIED ON THE LAGRANGIAN (5) GIVES

$$m\frac{d\bar{V}}{dt} + Q\bar{\nabla}\varphi + \frac{Q}{c}\frac{d\bar{A}}{dt} - \frac{Q}{c}\bar{\nabla}(\bar{V}\cdot\bar{A}) = 0$$

SETTING $m\frac{d\bar{V}}{dt} = \bar{F}$ GIVEN BY THE NEWTON LAW AND SOLVING FOR $\bar{F}$ $\Rightarrow$

$$(6) \quad \boxed{\bar{F} = Q\left(-\bar{\nabla}\varphi - \frac{1}{c}\frac{d\bar{A}}{dt} + \frac{1}{c}\bar{\nabla}(\bar{V}\cdot\bar{A})\right)}$$

WHICH IS THE LORENTZ FORCE IN POTENTIAL TERMS, IN GAUSS UNITS.

THE PHYSICAL MEANING OF (6) IS MORE CLEAR IF WE START FROM (1)

$L = L_{PAR} + L_{INT} + L_{FIELD}$. FOR SEMPLICITY WE CONSIDER ONLY ONE PARTICLE, OMITTING THE INDEX i. $L = L_{PAR} + L_{INT}$.

$$(7) \quad L = \underbrace{\frac{d}{dt}\left(m\dot{x} + QA_x(\bar{r})\right)}_{L_{PART.}} - \underbrace{\partial_x\left(Q\bar{A}(\bar{r})\cdot\dot{\bar{r}} - Q\varphi(\bar{r})\right)}_{L_{INTER.}}$$

REMEMBERING THAT $\frac{df}{dt} = \partial_t f + \dot{\bar{r}}\cdot\bar{\nabla}f$

WE OBTAIN

$$m\ddot{x} = Q\left[-\partial_t A_x - \partial_x\varphi\right] + Q\left[\dot{y}(\partial_x A_y - \partial_y A_x) + \right.$$

$$(8) \quad \left. - \dot{z}(\partial_z A_x - \partial_x A_z)\right) \equiv F_x \text{ LORENTZ FORCE}$$

WHICH IS THE X COMPONENT OF THE LORENTZ FORCE $m\ddot{r} = Q\left[\bar{E} + (\bar{V}\times B)\right]$

THIS CALCULUS CAN BE EXTENDED TO THE RELATIVISTIC CASE.

IN THIS CASE THE VARIATION OF THE MOMENTUM IS $\frac{d}{dt}(\gamma m \dot{x}) = \frac{dp_x}{dt}$ WHERE p_x IS THE X COMPONENT 1 OF $p^\mu = (p^0, p^1, p^2, p^3) \Rightarrow$ 3D

$$\frac{d}{dt}(\gamma m \dot{\bar{r}}) = \frac{d\bar{p}}{dt} = Q(\bar{E} + \bar{v} \times \bar{B}) \quad (9)$$

$\bar{p}$ SPACE COMPONENT OF p^μ.

- QUESTION: WRITE THE SAME EQ IN GAUSS UNITS.

- CONTINUUM DISTRIBUTION OF CHARGES AND CURRENTS

LET'S START REMINDING THE L_{INT} FOR DISCRETE CHARGES AND CURRENTS.

$$L_{INT} = \sum_i \left(Q_i \bar{A}(\bar{r}_i) \cdot \dot{\bar{r}}_i - Q_i \varphi(\bar{r}_i) \right) \Rightarrow$$

FOR A CONTINUUM IT BECOMES

$$(10) \quad L_{INT} = \int_\tau [\bar{J} \cdot \bar{A} - \rho \varphi] d^3\bar{r} \quad (d^3\bar{r} \equiv d\tau)$$

- PROBLEM: SHOW THAT THE INTEGRANDING FUNCTION IS A LORENTZ SCALAR, WHICH IS $\bar{J} \cdot \bar{A} - \rho\varphi = -J_\mu A^\mu$

WE CAN NOW CALCULATE THE LAGRANGIAN FOR A FREE E.M. FIELD. WE CAN START CONSIDERING THE FIELD ENERGY