8/3/2021

$F \in K[x_0, x_1]$ homogeneous of degree d
K algebraically closed field

x_0 | F $\quad F = x_0^{m_0} \underline{G(x_0,x_1)}, \quad x_0 \nmid G$

$x_0 \nmid F$ $\quad F(1,x_1)$ is a polynomial in x_1 of degree d

Then: $F(1,x_1) = c(x_1-\lambda_1) \cdots (x_1-\lambda_d)$

$F(x_0,x_1) = x_0^d F(1, \frac{x_1}{x_0}) =$

$= c x_0^d (\frac{x_1}{x_0} - \lambda_1) \cdots (\frac{x_1}{x_0} - \lambda_d) =$

$= c(x_1 - \lambda_1 x_0) \cdots (x_1 - \lambda_d x_0) =$

$= c(x_1-\lambda_1 x_0)^{m_1} \cdots (x_1 - \lambda_r x_0)^{m_r}$

$m_1 + \cdots + m_r = d$

In general

$F(x_0,x_1) = c\, x_0^{m_0} (x_1 - \lambda_1 x_0)^{m_1} \cdots (x_1 - \lambda_r x_0)^{m_r}$

$m_0 + m_1 + \cdots + m_r = d$

Walker, pag. 52-53

$F(x_0,x_1,x_2)$ square-free, $\deg F = d$

$\forall P \notin V_p(F) =: X$, there exist lines meeting X in d distinct points.

Square-free means

$F = F_1 \cdots F_s$, $\quad F_1, \dots, F_s$ irreducible

2 by 2 distinct

We can choose $P=[0,0,1]$, point at infinity.

A line through P can be parametrized as follows in non-homogeneous coordinates ($x_0=1$):

$$\begin{cases} x_1=\alpha_1 \\ x_2=t \end{cases}$$

$F(1,\alpha_1,t)=0$: the roots of this polynomial wrt t correspond to the points of $V_P(F)\cap L$; its degree is d because $P\notin V_P(F)$.

Assume by contradiction that $\forall \alpha_1\in K$, $F(1,\alpha_1,t)$ has a multiple root: it is also a root of the derivative of $F(1,\alpha_1,t)$.

Therefore the discriminant of $F(1,x_1,t)$, $R(x_1)$ vanishes $\forall x_1\in K$, which is infinite

$\Longrightarrow R(x_1)=0$ the zero polynom.

$\Longrightarrow F(1,x_1,t)$ and its derivative have a common factor $\Longrightarrow$ $F(1,x_1,t)$ has a multiple factor: contradiction.

For hypersurfaces the argument is similar.

$F \in K[x_1, \dots, x_n]$ $F \notin K$ $V(F) \subseteq \mathbb{A}^n_K$ hypersurface

$F = F_1^{m_1} \cdots F_r^{m_r}$ $V(F) = V(F_1 \cdots F_r)$ $F_1 \cdots F_r$ reduced polynom.

" square-free

$F_1 \cdots F_r = 0$ reduced equation of $X = V(F)$

$F \in K[x_0, \dots, x_n]$, F homog., $F \notin K$ $V_p(F) \subseteq \mathbb{P}^n_K$

$F = F_1^{m_1} \cdots F_r^{m_r}$ $F_1, \dots, F_r$ homog. polynomials

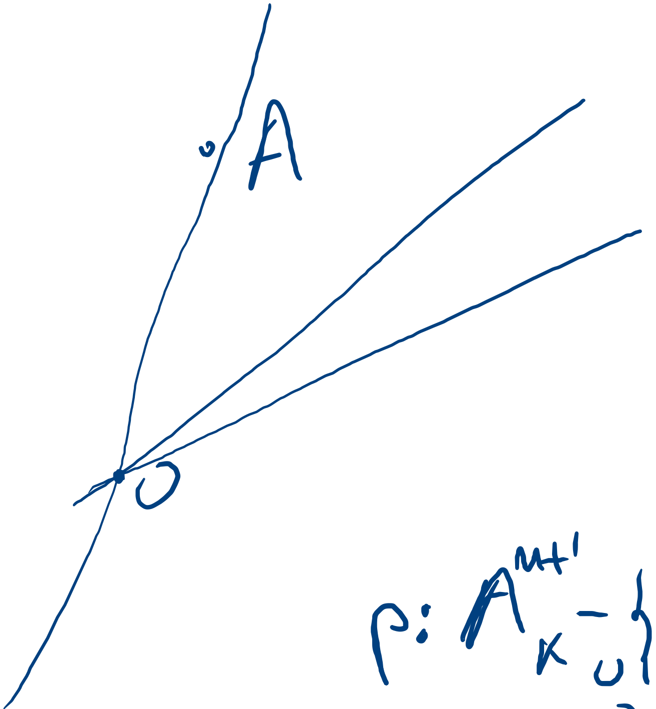
reduced equation of $V_p(F) = X$ is $F_1 \cdots F_r = 0$

degree of X is the degree of its reduced equation

$V(F) \subseteq \mathbb{A}^{n+1}_K$ F homog. $\Rightarrow$ $(a_0, \dots, a_n) \in \mathbb{A}^{n+1}_K$

$F(a_0, \dots, a_n) = 0 \Rightarrow \forall \lambda \in K$ $F(\lambda a_0, \dots, \lambda a_n) = 0$

$A(a_0, \dots, a_n) \in V(F) \Rightarrow$ all points of L: line $\overline{AO}$ are in $V(F)$



$V(F)$ is a cone with vertex $O(0,\dots,0)$

F $\quad V_p(F) \subseteq \mathbb{P}^n_K$

$V(F) \subseteq \mathbb{A}^{n+1}_K$

$p : \mathbb{A}^{n+1}_K - \{0\} \longrightarrow \mathbb{P}^n_K$

$\qquad \cup \qquad\qquad\qquad \cup$

$V(F) - \{0\} \longrightarrow V_p(F)$

$p(V(F) - \{0\}) = V_p(F)$

$p^{-1}(V_p(F)) = V(F) - \{0\}$

$V(F) = C(V_p(F))$

cone of $V_p(F)$

$X = V_p(\underbrace{F_1, \dots, F_r}_{\text{homog.}})$

$V(F_1, \dots, F_r) \subseteq \mathbb{A}^{n+1}$

is a cone of X $\quad C(X)$

Products

1) product of 2 affine spaces $\mathbb{A}^n_K \times \mathbb{A}^m_K \longleftrightarrow \mathbb{A}^{n+m}_K$

$((a_1 \ldots a_n)(b_1 \ldots b_m)) \longleftrightarrow (a_1 \ldots a_n b_1 \ldots b_m)$

natural bijection

$\mathbb{A}^{n+m}$ Zariski topology

$\overset{\parallel}{\mathbb{A}^n \times \mathbb{A}^m}$ product topology of Zariski topologies on $\mathbb{A}^n$ and $\mathbb{A}^m$

Prop. The Zariski topology on $\mathbb{A}^{n+m}$ is strictly finer than the product topology.

Pf: take $X \subseteq \mathbb{A}^n$ closed : $X = V(\alpha)$, $\alpha \subseteq K[x_1 \ldots x_n]$

$Y \subseteq \mathbb{A}^m$ $\qquad Y = V(\beta)$, $\beta \subseteq K[y_1, \ldots, y_m]$

$X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$

Look for an ideal $\gamma \subseteq K[x_1 \ldots x_n, y_1 \ldots y_m]$ st.

$X \times Y = V(\gamma)$ $\qquad \alpha \cup \beta \subseteq K[x_1 \ldots x_n y_1 \ldots y_m]$

$\gamma = \langle \alpha \cup \beta \rangle$

$P \in X \times Y \qquad P = (a_1 \dots a_n, b_1 \dots b_m) \qquad F(a_1 \dots a_n) = 0$

$G(b_1 \dots b_m) = 0 \quad \forall G \in \beta \qquad\qquad \forall F \in \alpha$

$\Rightarrow$ P is a zero of $H \in \gamma \qquad H = \sum H_i F_i + \sum K_j G_j$

$F_i \in \alpha$, $G_j \in \beta$

$P \in V(\gamma) \qquad \gamma \supseteq \alpha, \ \gamma \supseteq \beta$

$P(\underbrace{a_1 \dots a_n}_{V(\alpha)}, \underbrace{b_1 \dots b_m}_{V(\beta)}) \in V(\alpha) \times V(\beta)$

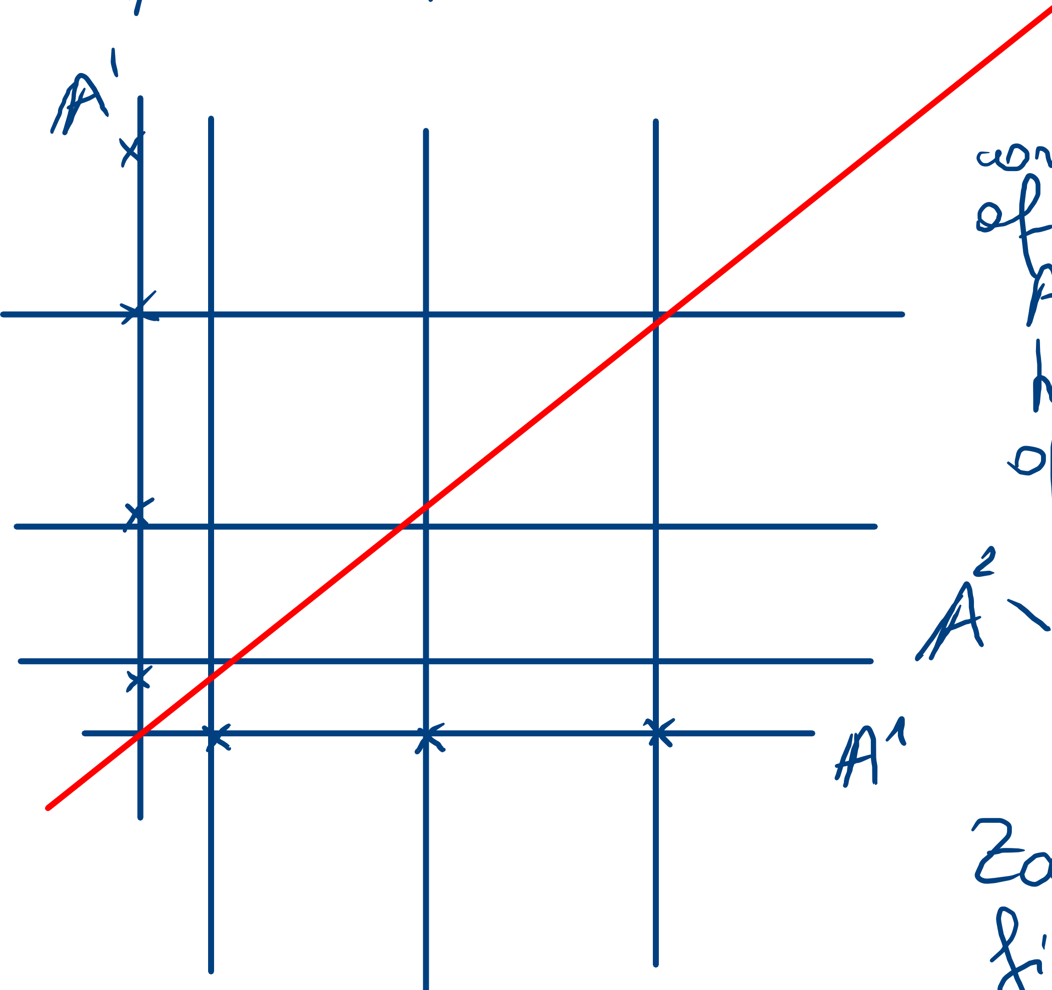
U open in $\mathbb{A}^n$ $\quad U = \mathbb{A}^n - X$, X closed in $\mathbb{A}^n$

V open in $\mathbb{A}^m$ $\quad V = \mathbb{A}^m - Y$, Y closed in $\mathbb{A}^m$ $\Big\}$ Zariski

$U \times V = \mathbb{A}^n \times \mathbb{A}^m \setminus [\underbrace{(\mathbb{A}^n \times Y) \cup (X \times \mathbb{A}^m)}_{Z\text{-closed}}]$

set-theoretical fact $\Rightarrow U \times V$ is Zariski-open

$\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$



complementar of this union of lines is open

All open sets in the product topology are unions of open sets of this form

$\mathbb{A}^2 \setminus V(x-y)$ Zariski open

not union of

Zariski top. is strictly finer

$\mathbb{P}^n \times \mathbb{P}^m$ the same argument of the affine space doesn't apply

$([a_0, \dots, a_n], [b_0, \dots, b_m])$

$[a_0, \dots, a_n, b_0, \dots, b_m] \in \mathbb{P}^{n+m+1}$
not well defined
$[\lambda a_0, \dots, \lambda a_n, \mu b_0, \dots, \mu b_m]$
if $\lambda \neq \mu$, get a different pt

$\mathbb{P}^n \times \mathbb{P}^m$ can be interpreted as a projective algebraic set in a suitable projective space

$n = m = 1$ $\quad \mathbb{P}^1 \times \mathbb{P}^1$ We will define a map σ

$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ s.that 1) σ injective
2) $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ is closed in $\mathbb{P}^3$

This will allow to identify via σ $\mathbb{P}^1 \times \mathbb{P}^1$ with a projective variety

$\mathbb{P}^1$ $\quad [x_0, x_1]$ first copy

$\mathbb{P}^1$ $\quad [y_0, y_1]$ second copy

$([x_0, x_1], [y_0, y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$

Def. $\sigma([x_0,x_1],[y_0,y_1]) = [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$

Good definition: $([\lambda x_0,\lambda x_1],[\mu y_0,\mu y_1]) \xrightarrow{\sigma} [\lambda\mu x_0y_0, \lambda\mu x_0 y_1, \dots]$

We cannot get $(x_0y_0, x_0y_1, x_1y_0, x_1y_1) = (0,0,0,0)$

$\exists i \;\; x_i \neq 0, \; \exists j \;\; y_j \neq 0 \qquad x_iy_j \neq 0$

Proof that σ is injective:

$\sigma([x_0,x_1],[y_0,y_1]) = \sigma([x_0',x_1'],[y_0',y_1'])$

$[x_0y_0, x_0y_1, x_1y_0, x_1y_1] = [x_0'y_0', x_0'y_1', x_1'y_0', x_1'y_1']$

$\Rightarrow \exists \lambda \in K^*$ s.t. $\forall i,j = 0,1 \quad \boxed{x_iy_j = \lambda x_i'y_j'}$

We want to prove that $[x_0,x_1] = [x_0',x_1']$, $[y_0,y_1] = [y_0',y_1']$

Ass. that $y_0 \neq 0 \qquad x_1y_0 = \lambda x_1'y_0' \Rightarrow x_1 = \lambda \frac{y_0'}{y_0} x_1'$

$x_0 = \lambda \frac{y_0'}{y_0} x_0', \qquad x_1 = \lambda \frac{y_0'}{y_0} x_1'$

If $y_0 = 0$, $y_1 \neq 0$ - - - - - .

$\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ is closed in $\mathbb{P}^3$ with coordinates $[z_0, z_1, z_2, z_3]$

We interpret σ as a parametrization of the image:

$[z_0, \dots, z_3] \in \sigma(\mathbb{P}^1 \times \mathbb{P}^1) \iff \exists\, [x_0, x_1] \in \mathbb{P}^1, [y_0, y_1] \in \mathbb{P}^1$ s.t.

$$(*) \quad \begin{cases} z_0 = x_0 y_0 \\ z_1 = x_0 y_1 \\ z_2 = x_1 y_0 \\ z_3 = x_1 y_1 \end{cases}$$

We need equations in $K[z_0, z_1, z_2, z_3]$ which are satisfied precisely from the points of this form

We have ~~to~~ eliminate x_0, x_1, y_0, y_1 from these $(*)$

$z_0 z_3 = z_1 z_2$: this equation is satisfied

$\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq V_P(z_0 z_3 - z_1 z_2)$ hypersurface of deg 2

quadric of rank 4 : smooth quadric

We prove that $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) = V_p(z_0z_3 - z_1z_2)$

Let $[z_0, z_1, z_2, z_3] \in \mathbb{P}^3$ s.t. $\boxed{z_0z_3 = z_1z_2}$

Ass. $z_0 \neq 0 \implies$

$$[z_0, -, -, z_3] = [z_0^2, z_0z_1, z_0z_2, \underline{z_0z_3}]$$

$$\overset{?}{=}$$

$$[x_0y_0, x_0y_1, x_1y_0, \underline{x_1y_1}]$$

given: $x_0 = z_0$

$y_0 = z_0$

$y_1 = z_1$ $\qquad x_1y_1 = z_1z_2$

$x_1 = z_2$

$$\underline{[z_0, \; -, z_3]} = \sigma([z_0, z_2], [z_0, z_1])$$

$z_0 = 0$, and $z_1 \neq 0$

$$[z_0z_1, \underline{z_1}^2, \underset{z_0z_3}{\overset{\|}{z_1z_2}}, \underline{z_1}z_3]$$

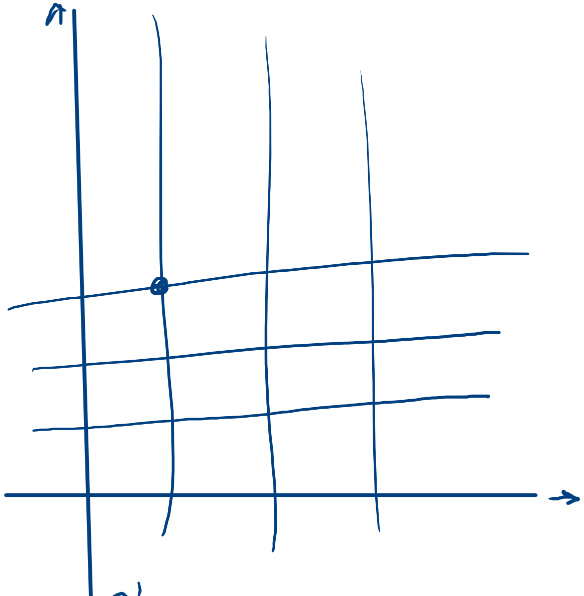
given: $x_0 = z_1$

$y_1 = z_1$

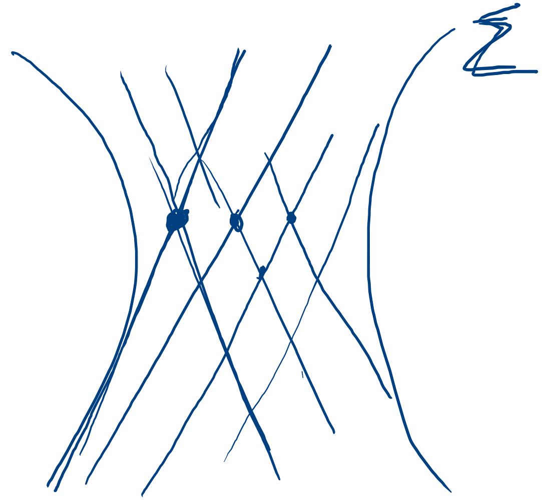
$y_0 = z_0$

$x_1 = z_3$

$$x_0y_0 \quad x_0y_1 \quad x_1y_0 \quad \underline{x_1}y_1$$



$\mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{P}^3$

$\Sigma = \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ $\quad \sigma([a_0,a_1] \times \mathbb{P}^1) = \{[a_0y_0, a_0y_1, a_1y_0, a_1y_1]\}$
a line in $\mathbb{P}^3$

$[a_0', a_1'] \times \mathbb{P}^1$

$\mathbb{P}^1 \times [b_0, b_1]$ $\quad \sigma(\mathbb{P}^1 \times [b_0,b_1]) = \{[x_0b_0, x_0b_1, x_1b_0, x_1b_1]\}$

$\sigma([a_0,a_1][b_0,b_1]) = \sigma([a_0,a_1] \times \mathbb{P}^1) \cap \sigma(\mathbb{P}^1 \times [b_0,b_1])$

σ Segre map, $\Sigma = \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ Segre quadric

from Corrado Segre

$\sigma : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N \qquad N = (n+1)(m+1) - 1$