How to approximate π?

Use: $\tan\frac{\pi}{4} = 1 \qquad \Rightarrow \qquad \frac{\pi}{4} = \arctan(1) \qquad \Rightarrow \qquad \pi = 4 \cdot \arctan(1)$

Know: Taylor expansion of $\arctan x$:

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

which is convergent at for $-1 < x \leq 1$.

Take $x = 1$:

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\pi = 4\arctan(1) = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Expansion: $$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Approximation by truncated Taylor polynomials:

Up to the x^9-term: $\pi \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) = 3.339682540.$

Up to the x^{19}-term: $\pi \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{19}\right) = 3.041839619$

Up to the x^{100}-term: $\pi \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{99}\right) = 3.121594653$

Up to the x^{1000}-term: $\pi \approx 3.139592656$

Up to the x^{50000}-term: $\pi \approx 3.141552653$

Exact value of π: $\pi = 3.14159265358979323 85....$

- Observation:

The series will converge to the exact value of π

But it converges to π very slowly, unlike the convergence of e.

Question: why is this convergence process that slow?

Key: Taylor series is expanded at 0:

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}x^3 + \cdots \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

which is an approximation of $f(x)$ around $x = 0$.

- So: if x is closer to 0, the convergence of $T(x)$ to $f(x)$ is faster. And if x is further away from 0, the convergence gets slower.

- The convergence interval of Taylor series of $\arctan x$ is $-1 < x \leq 1$.

$x = 1$ is the furthest point in the convergence range. This is why the convergence of at $x = 1$ is quite slow.

Question: Any faster algorithm to approximate π?

Identity: $\pi = 4(\arctan\frac{1}{2} + \arctan\frac{1}{3})$

Expansion: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$

Approximations:

Degree	$\arctan\frac{1}{2}$	$\arctan\frac{1}{3}$	π
up to x^3-term	$\frac{1}{2} - \frac{(1/2)^3}{3} = 0.45833$	$\frac{1}{3} - \frac{(1/3)^3}{3} = 0.32098$	3.11728
up to x^5-term	$\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} = 0.464583$	$\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} = 0.321810$	3.145576

Degree		Approximation	Error
x^{20}-term:	$\pi \approx$	3.141592579...	10^{-7}
x^{50}-term:	$\pi \approx$	3.141592653589793266...	10^{-17}
x^{100}-term:	$\pi \approx$	3.14159265358979323846264338327949...	10^{-32}
Exact value:	$\pi =$	3.14159265358979323846264338327950...	

- This series converges to π much faster than the previous one: $\pi = 4\arctan 1$.

Other identities of π:

$$(1) \qquad \pi = 4\arctan(1)$$

$$(2) \qquad \pi = 4\left(4\arctan\frac{1}{2} + \arctan\frac{1}{3}\right)$$

$$(3) \qquad \pi = 4\left(4\arctan\frac{1}{5} - \arctan\frac{1}{239}\right) \qquad \text{(Marchin's formula)}$$

$$(4) \qquad \pi = 4\left(12\arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443}\right)$$

$$(5) \qquad \pi = \left(12\sum_{n=0}^{\infty}\frac{(-1)^n\cdot(6n)!\cdot(13591409+545140134n)}{(3n)!\cdot(n!)^3\cdot 640320^{3n+3/2}}\right)^{-1} \qquad \text{(Chudnovsky's)}$$

Compare the algorithms of approximating π :

Algorithm	(1)	(2)	(3)	(4)	(5)
Error of approx. by 10 terms	0.2	10^{-4}	10^{-8}	10^{-18}	10^{-156}
Error of approx. by 100 terms	0.02	10^{-32}	10^{-72}	10^{-172}	10^{-1433}

The latest approximation of e and π, and other mathematical constants:

http://www.numberworld.org/

Approximation of $\sqrt{2}$: Newton's method

- For any positive number A, the sequence defined by recurrence relation

$$a_{n+1} = \frac{a_n}{2} + \frac{A}{2a_n}, \text{ with } a_1 = 1$$

will converge to $\sqrt{A}$.

- When $A = 2$: the sequence $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ converges to $\sqrt{2}$.

Approximations: $a_1 = 1$

$$a_2 = \frac{a_1}{2} + \frac{1}{a_1} = \frac{1}{2} + \frac{1}{1} = 1.5$$

$$a_3 = \frac{a_2}{2} + \frac{1}{a_2} = \frac{1.5}{2} + \frac{1}{1.5} = 1.41666...$$

$$a_4 = \frac{a_3}{2} + \frac{1}{a_3} = \frac{1.41666}{2} + \frac{1}{1.41666} = 1.414215686...$$

$$a_5 = 1.4142135623746... \qquad \text{(Error: } 10^{-12}\text{)}$$

$$a_6 = 1.414213562373095048801689623 5... \qquad \text{(Error: } 10^{-25}\text{)}$$

Exact value: $\sqrt{2} = 1.4142135623730950488016887242...$