BASI per il corso

FUNZIONI COMPOSTE

Prendiamo una funzione $f(x)$

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x)$$

es. $f(x) = e^x, \cos x, x^2, \dots$

La sua derivata $f'(x) \equiv \frac{d}{dx} f(x)$ è essa stessa una funz.

$$\frac{df}{dx} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{df}{dx}(x)$$

Posso definire funz. composte, sostituendo all'argomento x una funzione $\xi(t)$ $\quad \xi : \mathbb{R} \to \mathbb{R}$, $t \mapsto \xi(t)$

$$\hookrightarrow \quad g(t) \equiv f(\xi(t)) \qquad g : \mathbb{R} \to \mathbb{R}$$

Facciamo la derivata di g

$$\frac{dg(t)}{dt} \equiv \dot{g}(t) = f'(\xi(t))\, \dot{\xi}(t) \equiv \frac{df}{dx}(\xi(t)) \cdot \dot{\xi}(t)$$

Es: $f(x) = x^2 \quad \xi(t) = e^t \quad \to g(t) = e^{2t}$

$$\to \quad \dot{g}(t) = 2e^{2t}$$

$$= \underbrace{\frac{df}{dx}}_{\frac{df}{dx}(x) = 2x} \cdot \dot{\xi} = 2\xi \cdot \underbrace{\dot{\xi}}_{e^t} = 2e^t \cdot e^t = 2e^{2t}$$

Partiamo da una funz. a due variabili

$$f(x,y) = x^2+y^2$$

→ c'è una mappa $\quad f : \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \mapsto x^2+y^2$$

Le sue derivate parziali sono ancora delle funzioni

$$\frac{\partial f}{\partial x} : \mathbb{R}^2 \to \mathbb{R} \qquad\qquad \frac{\partial f}{\partial y} : \mathbb{R}^2 \to \mathbb{R}$$

$$(x,y) \mapsto \frac{\partial f}{\partial x}(x,y) \qquad\qquad (x,y) \mapsto \frac{\partial f}{\partial y}(x,y)$$

Prendiamo delle funzioni ξ e η

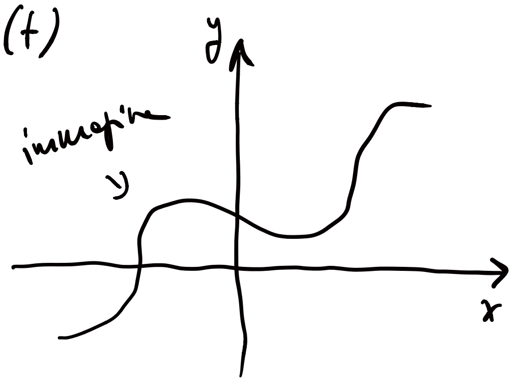
$$\xi : \mathbb{R} \to \mathbb{R} \qquad\qquad \eta : \mathbb{R} \to \mathbb{R}$$

$$t \mapsto \xi(t) \qquad\qquad t \mapsto \eta(t)$$

$$(\xi, \eta) : \mathbb{R} \to \mathbb{R}^2$$

$$t \mapsto (\xi(t), \eta(t))$$

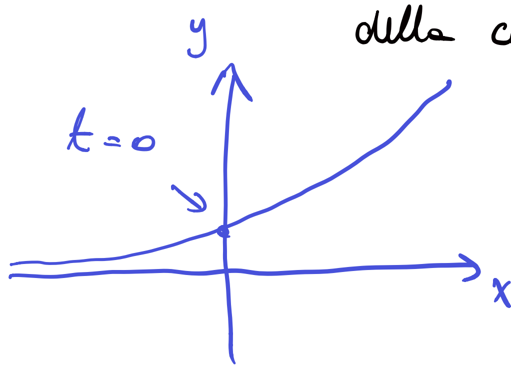
⇝ immagine



t è detto il parametro della curva

Es. $\quad \xi(t) = t$

$\eta(t) = e^t$



Def. la funz. composta

$$g : \mathbb{R} \to \mathbb{R}$$

$$t \mapsto g(t) = f(\xi(t), \eta(t))$$

Es. $\quad g(t) = f(t, e^t) = t^2 + e^{2t}$

Di qta funz. mi può interessare la derivata

$$\dot g(t) = 2t + 2e^{2t}$$

oppure uso la regola di derivazione di funz. comp.

$$\dot g(t) = \frac{d}{dt} f(\xi(t), \eta(t)) =$$

$$= \frac{\partial f}{\partial x}(\xi(t), \eta(t)) \cdot \dot\xi(t) + \frac{\partial f}{\partial y}(\xi(t), \eta(t)) \cdot \dot\eta(t)$$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y \qquad f(x,y) = x^2 + y^2$$

$$= 2\xi\dot\xi + 2y\dot\eta = 2t \cdot 1 + 2e^t \cdot e^t$$

$$= 2t + 2e^{2t}$$

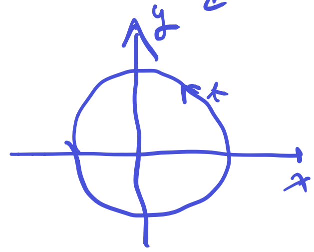
Proviamo a fare un'altra scelta per le funz. $\xi(t), \eta(t)$

$$\xi(t) = \cos t \qquad \eta(t) = \operatorname{sen} t \qquad (f = x^2 + y^2)$$

$$\hookrightarrow g(t) = \xi(t)^2 + \eta(t)^2 = \cos^2 t + \operatorname{sen}^2 t = 1$$

$$\downarrow$$

$$\dot g(t) = 0$$



$$\dot g = \frac{\partial f}{\partial x}\dot\xi + \frac{\partial f}{\partial y}\dot\eta =$$

$$= 2\xi\dot\xi + 2\eta\dot\eta = -2\cos t \operatorname{sen} t + 2 \operatorname{sen} t \cos t = 0$$

Altra possibilità:

$$h : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(x, y, t) \longmapsto h(x, y; t)$$

Es. $h(x,y;t) = xy^2 - \operatorname{senh}^2 t$

Consideriamo una curva in $\mathbb{R}^2$ data da

$$\xi(t) = 1 \qquad \eta(t) = \cosh t$$
$$\dot{\xi} = 0 \qquad \dot{\eta} = \operatorname{senh} t$$

Def. $K : \mathbb{R} \longrightarrow \mathbb{R}$

$$t \longmapsto h(\xi(t), \eta(t); t)$$

$$K(t) = h(\xi(t), \eta(t); t) = \xi(t)\,\eta(t)^2 - \operatorname{senh}^2 t$$
$$= 1 \cdot \cosh^2 t - \operatorname{senh}^2 t = 1$$
$$\rightarrow \dot{K} = 0$$

Calcoliamoci $\dot{K}$ usando la regola delle derivate di funz comp.

$$\dot{K}(t) = \frac{\partial h}{\partial x}(\xi(t), \eta(t); t)\,\dot{\xi}(t) + $$
$$+ \frac{\partial h}{\partial y}(\xi(t), \eta(t); t)\,\dot{\eta}(t) +$$
$$+ \frac{\partial h}{\partial t}(\xi(t), \eta(t); t)$$

$\frac{\partial h}{\partial x} = y^2$

$\frac{\partial h}{\partial y} = 2xy$

$h(x,y;t) = xy^2 - \operatorname{senh}^2 t$

$$= \eta^2 \dot{\xi} + 2\xi\eta\dot{\eta} - 2 \operatorname{senh} t \cosh t =$$

$$= (\cosh t)^2 \cdot 0 + 2 \cdot 1 \cdot \cosh t \operatorname{senh} t - 2 \operatorname{senh} t \cosh t = 0.$$

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SPAZI VETTORIALI

Spazio vettoriale V def. su $\mathbb{R}$ ($\mathbb{C}$, ...)

è un INSIEME di oggetti che chiamiamo VETTORI

$\vec{v}_1, \vec{v}_2, \ldots \in V \qquad \alpha, \beta, \ldots \in \mathbb{R}$

su cui definiamo le operazioni $+$ e "prodotto per un numero":

$\bar{v}_1 + \bar{v}_2 \in V$

$\alpha \bar{v}_1 \in V$

t.c. $\alpha(\bar{v}_1 + \bar{v}_2) = \alpha\bar{v}_1 + \alpha\bar{v}_2$

$\bar{v}_1, \ldots, \bar{v}_m$ sono LINEARMENTE DIPENDENTI se

$\exists$ una combinaz. lineare nulla, cioè se

$\exists\ \alpha_1, \ldots, \alpha_m \in \mathbb{R}$ t.c.

$$\underbrace{\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \ldots + \alpha_m\bar{v}_m}_{\equiv \sum_{i=1}^{m} \alpha_i \bar{v}_i} = 0$$

(I vett. sono lin. indip. se non sono lin. dip)

BASE per V: insieme di vettori lin. indip.

$\{\bar{e}_1, \ldots, \bar{e}_n\}$ t.c.

$\forall\, \bar{v} \in V \quad \exists!\, (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ t.c.

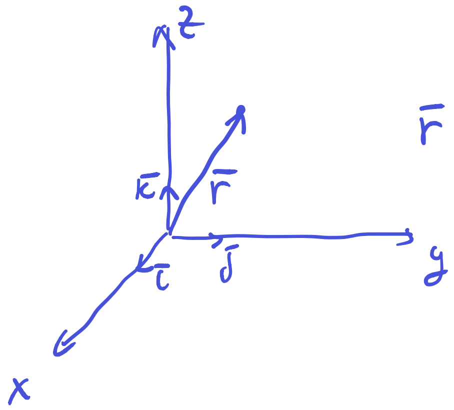
$$\bar{v} = \alpha_1\bar{e}_1 + \alpha_2\bar{e}_2 + \ldots + \alpha_n\bar{e}_n = \sum_{i=1}^{n} \alpha_i \bar{e}_i$$

$$\updownarrow$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

componenti del vettore

Es. $V = \mathbb{R}^3$



$\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

$\updownarrow$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Prodotto scalare

$V \times V \longrightarrow \mathbb{R}$

$\bar{v}_1, \bar{v}_2 \longmapsto \bar{v}_1 \cdot \bar{v}_2 \in \mathbb{R}$

prodotti scalari def. positivi

$\downarrow$

$\rightsquigarrow \|\bar{v}\|^2 \equiv \bar{v} \cdot \bar{v}$

$\rightsquigarrow$ Base ortonormale $\{\bar{e}_1, \dots, \bar{e}_n\}$

$$\bar{e}_i \cdot \bar{e}_j = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$

$$= \delta_{ij}$$

$\bar{v}_1, \bar{v}_2 \in V$

$$\bar{v}_1 = \sum_{i=1}^{m} \alpha_i \bar{e}_i = \sum_{\star=1}^{m} \alpha_{\star} \bar{e}_{\star} = \alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \dots + \alpha_m \bar{e}_m$$

$$\sum_{\star=1}^{m} \alpha_{\star} \bar{e}_{\star} = \sum_{k=1}^{m} \alpha_k \bar{e}_k$$

$$\bar{v}_2 = \sum_{j=1}^{m} \beta_j \bar{e}_j$$

$(a+b)(c+d)$
$= a(c+d) +$
$+ b(c+d)$

$$\bar{v}_1 \cdot \bar{v}_2 = \left(\sum_{i=1}^{m} \alpha_i \bar{e}_i \right) \cdot \left(\sum_{j=1}^{m} \beta_j \bar{e}_j \right) =$$

$$= \sum_{i=1}^{m} \alpha_i \left[\bar{e}_i \cdot \left(\sum_{j=1}^{m} \beta_j \bar{e}_j \right) \right] =$$

$$= \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{m} \beta_j \, \bar{e}_i \cdot \bar{e}_j = \sum_{i=1}^{m} \alpha_i \left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) =$$

$$\sum_{j=1}^{n} \beta_j \delta_{ij} = \beta_1 \delta_{i1} + \beta_2 \delta_{i2} + \beta_3 \delta_{i3} + \dots + \beta_n \delta_{in} =$$

$$= \beta_1 \delta_{i1} + \dots + \underset{j=i}{\beta_i} \delta_{ii} + \dots + \beta_n \delta_{in}$$

$$= 0 + \dots + \beta_i + 0 + \dots + 0 = \beta_i$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_n \beta_n$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$