Chapter 1

Affine algebraic sets and Zariski topology.

1.1 Introduction

The aim of this course is to introduce the notion of algebraic variety in the classical sense, over a field K.

Roughly speaking, algebraic varieties are sets of solutions of a system of algebraic equations, i.e. equations given by polynomials. The natural space where to look at these solutions seems to be the affine space, but one realizes that the projective ambient is more convenient. On one hand the projective space extends the affine space and includes it naturally, on the other hand the projective ambient allows to prove more general and complete results.

After introducing the notions of affine and projective varieties, we will study the notion of dimension. Then we will introduce two kinds of transformations of algebraic varieties: regular and rational maps. They give rise to two types of equivalence or isomorphism: biregular isomorphism and birational equivalence, and therefore to two classification problems.

In this course we will see many examples of varieties, and of regular and rational maps. In particular we will see some classes of varieties related to the notion of tensor (without symmetries, symmetric, skew-symmetric); they are much studied because of many recent applications in fields as control theory, signal transmission, etc. We will see also examples of rational and unirational varieties, hopefully this will give a taste of the modern classification problems. We will then study the notions of tangent space, and of smoothness.

Classical algebraic geometry is the basis and gives the motivations for modern algebraic geometry: from schemes, introduced by Grothendieck in the sixties of last century, to the stacks, due to Mumford and Artin. All these notions are strongly based on commutative algebra, i.e. the theory of commutative rings, in particular polynomial rings and their quotients, local rings, and homological algebra.

The reference books I've chosen, all of which have become classics, have different flavours: the book [S] of Šafarevič is complete and precise, and contains almost all algebraic notions needed; Harris' book [jH] has a more geometric flavour, proofs are not complete but there are many many examples and ideas; Harthshorne's book [rH], the "Bible" of algebraic geometry since its appearance, treats classical varieties quickly in the first chapter, then moves to modern language, but always with an eye to classical problems. Further bibliographic references will be recommended later, for particular topics or as in-depth reading.

Notation. The ideal generated by a set S will be denoted by $\langle S\rangle$.

1.2 Affine and projective spaces.

In this first section, we begin by fixing the ambient in which we will work: the affine and the projective space over any field K. In particular we recall some basic facts about the projective space.

Let K be a field. For us the *affine space* of dimension n over K will simply be the set K^n : on it, the additive group of K^n acts naturally by translation. The affine space will be denoted by $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$. So the points of $\mathbb{A}^n_K$ are $n-$tuples $(a_1,\dots,a_n)$, where $a_i\in K$ for $i=1,\dots,n$.

Let V be a $K-$vector space of dimension $n+1$. Let $V^*=V\setminus\{0\}$ be the subset of non–zero vectors. The following relation in V^* is an equivalence relation (relation of proportionality): $v\sim v'$ if and only if $\exists\lambda\neq 0,\lambda\in K$, such that $v'=\lambda v$.

The quotient set $V^*/\!\sim$ is called the *projective space* associated to V and is denoted by $\mathbb{P}(V)$. The points of $\mathbb{P}(V)$ are the lines in V (through the origin) deprived of the origin. In particular, $\mathbb{P}(K^{n+1})$ is denoted by $\mathbb{P}^n_K$ (or simply $\mathbb{P}^n$) and called the *numerical projective n-space*. By definition, the dimension of $\mathbb{P}(V)$ is equal to $\dim V-1$.

There is a canonical surjection $p: V^*\to\mathbb{P}(V)$ which maps a vector v to its equivalence class $[v]$. If $(x_0,\dots,x_n)\in(K^{n+1})^*$, we will denote the corresponding point of

$\mathbb{P}^n$ by $[x_0, \ldots, x_n]$. Another notation, used for instance in [S], is $(x_0 : \ldots : x_n)$. So $[x_0, \ldots, x_n]$=$[x'_0, \ldots, x'_n]$ if and only if $\exists \lambda \in K^*$ such that $x'_0 = \lambda x_0, \ldots, x'_n = \lambda x_n$.

If we fix a basis $e_0, \ldots, e_n$ of V, then there is an associated system of *homogeneous coordinates* in V, in the following way: if $v = x_0 e_0 + \ldots + x_n e_n$, then $x_0, \ldots, x_n$ are called homogeneous coordinates of the corresponding point P =$[v]$= $p(v)$ in $\mathbb{P}(V)$. We also write $P[x_0, \ldots, x_n]$. Note that homogeneous coordinates of a point P are not uniquely determined by P, but are defined only up to multiplication by a non–zero constant. If $\dim V = n + 1$, a system of homogeneous coordinates allows to define a *bijection*

$$\mathbb{P}(V) \longrightarrow \mathbb{P}^n$$

$$P = [v] \longrightarrow [x_0, \ldots, x_n]$$

where $v = x_0 e_0 + \ldots + x_n e_n$.

The points $E_0[1, 0, \ldots, 0], \ldots, E_n[0, 0, \ldots, 1]$ are called fundamental points, and $U[1, \ldots, 1]$ unit point of the given system of coordinates.

A *projective* (or *linear*) *subspace* of $\mathbb{P}(V)$ is a subset of the form $\mathbb{P}(W)$, where $W \subset V$ is a vector subspace of V.

If W, U are vector subspaces of V, the following *Grassmann relation* holds:

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

From this relation, observing that $\mathbb{P}(U \cap W) = \mathbb{P}(U) \cap \mathbb{P}(W)$, we get in $\mathbb{P}(V)$:

$$\dim \mathbb{P}(U) + \dim \mathbb{P}(W) = \dim(\mathbb{P}(U) \cap \mathbb{P}(W)) + \dim \mathbb{P}(U + W).$$

Note that $\mathbb{P}(U+W)$ is the minimal linear subspace of $\mathbb{P}(V)$ containing both $\mathbb{P}(U)$ and $\mathbb{P}(W)$: it is denoted $\mathbb{P}(U) + \mathbb{P}(W)$.

Example 1.2.1. *Let $V = K^3$, $\mathbb{P}(V) = \mathbb{P}^2$, $U, W \subset K^3$ subspaces of dimension 2. Then $\mathbb{P}(U), \mathbb{P}(V)$ are lines in the projective plane. There are two cases:*

(i) $U = W = U + W = U \cap W$;

(ii) $U \neq W$, $\dim U \cap W = 1$, $U + W = K^3$.

In case (i) the two lines in $\mathbb{P}^2$ coincide; in case (ii) $\mathbb{P}(U) \cap \mathbb{P}(W) = \mathbb{P}(U \cap W) = [v]$, if $v \neq 0$ is a vector generating $U \cap W$. Observe that never $\mathbb{P}(U) \cap \mathbb{P}(W) = \emptyset$.

What are the possible reciprocal positions of two lines in $\mathbb{P}^3$? Of two planes? Of a line and a plane?

Let $T \subset \mathbb{P}(V)$ be a non–empty set. The linear span $\langle T \rangle$ of T is the intersection of the projective subspaces of $\mathbb{P}(V)$ containing T, i.e. the minimum subspace containing T.

For example, assume that $T = \{P_1, \dots, P_t\}$ is a finite set, and that $v_1, \dots, v_t$ are vectors such that $P_1 = [v_1], \dots, P_t = [v_t]$. Then $\langle P_1, \dots, P_t \rangle = \mathbb{P}(W)$, where W is the vector subspace of V generated by $v_1, \dots, v_t$.

So $\dim\langle P_1, \dots, P_t \rangle \leq t-1$ and equality holds if and only if $v_1, \dots, v_t$ are linearly independent; in this case, also the points $P_1, \dots, P_t$ are called *linearly independent*. In particular, if $t = 2$, two points are linearly independent if they generate a line; if $t = 3$, three points are linearly independent if they generate a plane, etc. It is clear that, if $P_1, \dots, P_t$ are linearly independent, then $t \leq n+1$, and any subset of $\{P_1, \dots, P_t\}$ is formed by linearly independent points.

Definition 1.2.2 (Points in general position in $\mathbb{P}^n$)**.** $P_1, \dots, P_t$ are said to be *in general position* if either $t \leq n+1$ and they are linearly independent, or $t > n+1$ and they are $n+1$ by $n+1$ linearly independent.

Proposition 1.2.3. *The fundamental points $E_0, \dots, E_n$ and the unit point U of a system of homogeneous coordinates on $\mathbb{P}^n$ are $n+2$ points in general position. Conversely, if $P_0, \dots, P_n, P_{n+1}$ are $n+2$ points in general position, then there exists a system of homogeneous coordinates in which $P_0, \dots, P_n$ are the fundamental points and P_{n+1} is the unit point.*

Proof. The proof is linear algebra. If $e_0, \dots, e_n$ is a basis, then clearly the $n+1$ vectors $e_0, \dots, \hat{e_i}, \dots, e_n, e_0 + \dots + e_n$ are linearly independent: this proves the first claim. To prove the second claim, we fix vectors $v_0, \dots, v_{n+1}$ such that $P_i = [v_i]$ for all i. So $v_0, \dots, v_n$ is a basis and there exist $\lambda_0, \dots, \lambda_n$ in K such that $v_{n+1} = \lambda_0 v_0 + \dots + \lambda_n v_n$. The assumption of general position easily implies that $\lambda_0, \dots, \lambda_n$ are all different from 0, hence $\lambda_0 v_0, \dots, \lambda_n v_n$ is a new basis such $[\lambda_i v_i] = P_i$ and P_{n+1} is the corresponding unit point. $\square$

1.3 Embedding of the affine space in the projective space

Let a system of homogeneous coordinates be fixed in $\mathbb{P}^n$. We introduce the subspaces $H_0 = \langle E_1, \dots, E_n \rangle, H_1 = \langle E_0, E_2, \dots, E_n \rangle, \dots, H_n = \langle E_0, \dots, E_{n-1} \rangle$: they are $n+1$ hyperplanes in $\mathbb{P}^n$ (subspaces of codimension 1). Note that H_i is defined by the equation $x_i = 0$. These hyperplanes are called the *fundamental hyperplanes*.

Let $U_i = \mathbb{P}^n \setminus H_i = \{P[x_0, \dots, x_n] \mid x_i \neq 0\}$. Note that $\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n$, because no point in $\mathbb{P}^n$ has all coordinates equal to zero.

There is a map $\varphi_0 : U_0 \longrightarrow \mathbb{A}^n (= K^n)$ defined by

$$\varphi_0([x_0, \dots, x_n]) = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}).$$

φ_0 is bijective and the inverse map is $j_0 : \mathbb{A}^n \longrightarrow U_0$ such that $j_0(y_1, \dots, y_n) = [1, y_1, \dots, y_n]$. So φ_0 and j_0 establish a bijection between the affine space $\mathbb{A}^n$ and the subset U_0 of the projective space $\mathbb{P}^n$. Similarly, there are maps φ_i and j_i for any $i = 1, \dots, n$, that establish bijections between $\mathbb{A}^n$ and U_i. So $\mathbb{P}^n$ is covered by $n+1$ subsets, each one in natural bijection with $\mathbb{A}^n$.

There is a natural way of thinking of $\mathbb{P}^n$ as a completion of $\mathbb{A}^n$; this is done by identifying $\mathbb{A}^n$ with U_i via φ_i, and by interpreting the points of $H_i (= \mathbb{P}^n \setminus U_i)$ as *points at infinity* of $\mathbb{A}^n$, or directions in $\mathbb{A}^n$. We do this explicity for $i = 0$. First of all we identify $\mathbb{A}^n$ with U_0 via φ_0 and j_0. So if $P[a_0, \dots, a_n] \in \mathbb{P}^n$, either $a_0 \neq 0$ and $P \in \mathbb{A}^n$, or $a_0 = 0$ and $P[0, a_1, \dots, a_n] \notin \mathbb{A}^n$. Then we consider in $\mathbb{A}^n$ the line L, passing through $O(0, \dots, 0)$ and of direction given by the vector $(a_1, \dots, a_n)$. The following are parametric equations of L:

$$\begin{cases} x_1 = & a_1 t \\ x_2 = & a_2 t \\ \dots & \dots \\ x_n = & a_n t \end{cases}$$

with $t \in K$. The points of L are identified (via j_0) with the points of U_0 with homogeneous coordinates $x_0, \dots, x_n$ given by:

$$\begin{cases} x_0 = & 1 \\ x_1 = & a_1 t \\ x_2 = & a_2 t \\ \dots & \dots \end{cases}$$

or equivalently, if $t \neq 0$, by:

$$\begin{cases} x_0 = & \frac{1}{t} \\ x_1 = & a_1 \\ x_2 = & a_2 \\ \dots & \dots \end{cases}$$

Now, roughly speaking, if t *tends to infinity*, this point "tends" to $P[0, a_1, \dots, a_n]$. Clearly this is not a rigorous argument, but just a hint to the intuition.

In this way $\mathbb{P}^n$ can be interpreted as $\mathbb{A}^n$ with the points at infinity added, each point at infinity corresponding to one direction in $\mathbb{A}^n$.

Exercises 1.3.1. Let V be a vector space of finite dimension over a field K. Let $\check{V}$ denote the dual of V, i.e. the space of linear forms (or functionals) on V. Prove that $\mathbb{P}(\check{V})$ can be put in bijection with the set of the hyperplanes of $\mathbb{P}(V)$ (hint: the kernel of a non-zero linear form on V is a subvector space of V of codimension one). $\mathbb{P}(\check{V})$ is the *dual projective space.*

1.4 Algebraic sets.

Roughly speaking, algebraic subsets of the affine or of the projective space are sets of solutions of systems of algebraic equations, i.e. common roots of sets of polynomials.

Examples of algebraic sets are: linear subspaces of both the affine and the projective space, plane algebraic curves, quadrics, graphs of polynomials functions, ...

Algebraic geometry is the branch of mathematics which studies algebraic sets (and their generalizations). Our first aim is *to give a formal definition of algebraic sets in the affine space.*

1.4.1 Affine algebraic sets

Let $K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field K. If $P(a_1, \ldots, a_n) \in \mathbb{A}^n$, and $F = F(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$, we can consider the value of F at P, i.e. $F(P) = F(a_1, \ldots, a_n) \in K$. We say that P is a *zero of* F if $F(P) = 0$.

For example the points $P_1(1, 0)$, $P_2(-1, 0)$, $P_3(0, 1)$ are zeros of $F = x_1^2 + x_2^2 - 1$ over any field. If $G = x_1^2 + x_2^2 + 1$ then G has no zeros in $\mathbb{A}^2_{\mathbb{R}}$, but does have zeros in $\mathbb{A}^2_{\mathbb{C}}$.

Definition 1.4.1. A subset X of $\mathbb{A}^n_K$ is an *affine algebraic set*, or an *affine variety*, if X is the set of common zeros of a family of polynomials of $K[x_1, \ldots, x_n]$.

Remark. In some texts the term "variety" is reserved to the affine algebraic sets which are *irreducible.* The notion of irreducible algebraic set will be introduced in Chapter 6.

X is an affine algebraic set means that there exists a subset $S \subset K[x_1, \ldots, x_n]$ such that

$$X = \{P \in \mathbb{A}^n \mid F(P) = 0 \ \forall \ F \in S\}.$$

In this case X is called the zero set of S and is denoted by $V(S)$ (or in some books $Z(S)$, e.g. this is the notation of Hartshorne's book [rH]). In particular, if $S = \{F\}$, then $V(S)$ will be denoted simply by $V(F)$.

Example 1.4.2. *1. $S = K[x_1, \ldots, x_n]$: then $V(S) = \emptyset$, because S contains non–zero constants.*

2. $S = \{0\}$: *then* $V(S) = \mathbb{A}^n$.

3. $S = \{xy - 1\}$: *then* $V(xy - 1)$ *is a hyperbola in the affine plane.*

4. *If* $S \subset T$, *then* $V(S) \supset V(T)$.

5. $V(F_1, \ldots, F_r) = V(F_1) \cap \ldots \cap V(F_r)$.

Let $S \subset K[x_1, \ldots, x_n]$ be a set of polynomials, let $\alpha := \langle S \rangle$ be the ideal generated by S. Recall that $\alpha = \{$finite sums of products of the form HF where $F \in S,\ H \in K[x_1, \ldots, x_n]\}$.

Proposition 1.4.3. *Let* $\alpha = \langle S \rangle$. *Then* $V(S) = V(\alpha)$.

Proof. From $S \subset \alpha$ it follows that $V(S) \supset V(\alpha)$.

Conversely, if $P \in V(S)$, let $G = \sum_i H_i F_i$ be a polynomial of α ($F_i \in S\ \forall\ i$). Then $G(P) = (\sum H_i F_i)(P) = \sum H_i(P) F_i(P) = 0$. □

Proposition 1.4.3 is important in view of the following:

Theorem 1.4.4 (Hilbert's Basis Theorem)**.** *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

Proof. Assume by contradiction that $R[x]$ is not Noetherian. Let $I \subset R[x]$ be a non-finitely generated ideal. Let $f_1 \in I$ be a non-zero polynomial of minimum degree. We define by induction a sequence $\{f_k\}_{k \in \mathbb{N}}$ of polynomials as follows: if f_k ($k \geq 1$) has already been chosen, let f_{k+1} be a polynomial of minimum degree in $I \setminus \langle f_1, \ldots, f_k \rangle$. Let n_k be the degree of f_k, and let a_k be its leading coefficient. Note that, due to the choice of f_k, the chain of the degrees is increasing: $n_1 \leq n_2 \leq \ldots$.

We will prove now that $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \ldots$ is a chain of ideals, that does not become stationary: this will give the required contradiction. Indeed, if $\langle a_1, \ldots, a_r \rangle = \langle a_1, \ldots, a_r, a_{r+1} \rangle$ for some r, then $a_{r+1} = \sum_{i=1}^r b_i a_i$, for suitable $b_i \in R$. In this case, we consider the polynomial $g := f_{r+1} - \sum_{i=1}^r b_i x^{n_{r+1} - n_i} f_i$: g belongs to I, but $g \notin \langle f_1, \ldots, f_r \rangle$, and its degree is strictly lower than the degree of f_{r+1}: contradiction. □

Corollary 1.4.5. *Any affine algebraic set $X \subset \mathbb{A}^n$ is the zero set of a finite number of polynomials, i.e. there exist $F_1, \ldots, F_r \in K[x_1, \ldots, x_n]$ such that $X = V(F_1, \ldots, F_r)$.*

Note that $V(F_1, \ldots, F_r) = V(F_1) \cap \ldots \cap V(F_r)$, so every algebraic set is a finite intersection of algebraic sets of the form $V(F)$, i.e. zeros of a unique polynomial F. If $F = 0$, then $V(0) = \mathbb{A}^n$; if $F = c \in K \setminus \{0\}$, then $V(c) = \emptyset$; if *deg* $F > 0$, then $V(F)$ is called a *hypersurface*.

1.4.2 The Zariski topology on the affine space

Proposition 1.4.6. *The affine algebraic sets of $\mathbb{A}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology.*

Proof. It is enough to check that finite unions and arbitrary intersections of algebraic sets are again algebraic sets.

Let $V(\alpha), V(\beta)$ be two algebraic sets, with α, β ideals of $K[x_1, \dots, x_n]$. We recall that the product ideal of α and β is

$$\alpha\beta = \{ \sum_{\text{finite}} a_i b_i \mid a_i \in \alpha, b_i \in \beta \}.$$

Then $V(\alpha) \cup V(\beta) = V(\alpha \cap \beta) = V(\alpha\beta)$. Indeed: $\alpha\beta \subset \alpha \cap \beta$ so $V(\alpha \cap \beta) \subset V(\alpha\beta)$, and both $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \subset \beta$ so $V(\alpha) \cup V(\beta) \subset V(\alpha \cap \beta)$. Assume now that $P \in V(\alpha\beta)$ and $P \notin V(\alpha)$: hence $\exists F \in \alpha$ such that $F(P) \neq 0$; on the other hand, if $G \in \beta$ then $FG \in \alpha\beta$ so $(FG)(P) = 0 = F(P)G(P)$, which implies $G(P) = 0$.

Let $V(\alpha_i), i \in I$, be a family of algebraic sets, $\alpha_i \subset K[x_1, \dots, x_n]$. Then $\cap_{i \in I} V(\alpha_i) = V(\sum_{i \in I} \alpha_i)$, where $\sum_{i \in I} \alpha_i$ is the sum ideal of α_i's. Indeed $\alpha_i \subset \sum_{i \in I} \alpha_i \ \forall i$, hence $V(\sum_i \alpha_i) \subset V(\alpha_i) \ \forall i$ and $V(\sum_i \alpha_i) \subset \cap_i V(\alpha_i)$. Conversely, if $P \in V(\alpha_i) \ \forall i$, and $F \in \sum_i \alpha_i$, then $F = \sum_i F_i$; therefore $F(P) = \sum F_i(P) = 0$. □

Example 1.4.7. 0. Every point of $\mathbb{A}^n$ is closed in the Zariski topology, indeed $A = (a_1, \dots, a_n) = V(x_1 - a_1, \dots, x_n - a_n)$.

1. The Zariski topology on the affine line $\mathbb{A}^1$.

Let us recall that the polynomial ring $K[x]$ in one variable is a PID (principal ideal domain), so every ideal $I \subset K[x]$ is of the form $I = \langle F \rangle$. Hence every closed subset of $\mathbb{A}^1$ is of the form $X = V(F)$, the set of zeros of a unique polynomial $F(x)$. If $F = 0$, then $V(F) = \mathbb{A}^1$, if $F = c \in K^*$, then $V(F) = \emptyset$, if $\deg F = d > 0$, then F can be decomposed in linear factors in the polynomial ring over the algebraic closure of K; it follows that $V(F)$ has at most d points.

We conclude that the closed sets in the Zariski topology of $\mathbb{A}^1$ are: $\mathbb{A}^1$, $\emptyset$ and the finite sets.

2. If $K = \mathbb{R}$ or $\mathbb{C}$, then the Zariski topology and the Euclidean topology on $\mathbb{A}^n_K$ can be compared, and it results that the Zariski topology is coarser. Indeed every open set in the Zariski topology is open also in the usual topology. Let $X = V(F_1, \dots, F_r)$ be a closed set in the Zariski topology, and $U := \mathbb{A}^n \setminus X$; if $P \in U$, then $\exists\ F_i$ such that $F_i(P) \neq 0$, so there exists an open neighbourhood of P in the usual topology in which F_i does not vanish.

Conversely, there exist closed sets in the usual topology which are not Zariski closed, for example the balls. The first case, of an interval in the real affine line, follows from part 1.

1.4.3 Projective algebraic sets

We want to define now the projective algebraic sets, or projective varieties, in $\mathbb{P}^n$.

The idea is the same as in the affine space: a projective variety is the set of solutions of a system of polynomial equations. The difference is that a point in the projective space does not have a well defined set of coordinates: homogeneous coordinates are defined only up to proportionality. So it may happen that, given a polynomial F and a point $P \in \mathbb{P}^n$ with homogeneous coordinates $[x_0, \dots, x_n]$, the n-tuple $x_0, \dots, x_n$ is a zero of F, but other proportional n-tuples of the form $[\lambda x_0, \dots, \lambda x_n]$ are not.

To give a good definition, we have to consider only homogeneous polynomials, because for them the problem does not occur. Otherwise, to say that a point $p \in \mathbb{P}^n$ is a zero of a polynomial F, we must ask that it annihilates F for each choice of its homogeneous coordinates.

Let's now formalize what I have anticipated.

Let $K[x_0, x_1, \dots, x_n]$ be the polynomial ring in $n+1$ variables. If we fix a polynomial $G(x_0, x_1, \dots, x_n) \in K[x_0, x_1, \dots, x_n]$ and a point $P[a_0, a_1, \dots, a_n] \in \mathbb{P}^n$, then, in general,

$$G(a_0, \dots, a_n) \neq G(\lambda a_0, \dots, \lambda a_n),$$

so the value of G at P cannot be defined.

Example 1.4.8. *Let* $G = x_1 + x_0x_1 + x_2^2$, $P[0,1,2] = [0,2,4] \in \mathbb{P}^2_{\mathbb{R}}$. *So* $G(0,1,2) = 1 + 4 \neq G(0,2,4) = 2 + 16$. *But if* $Q = [1,0,0] = [\lambda, 0, 0]$, *then* $G(1,0,0) = G(\lambda, 0, 0) = 0$ *for each* λ.

Definition 1.4.9 (Homogeneous polynomials)**.** Let $G \in K[x_0, x_1, \dots, x_n]$: G is *homogeneous of degree* d, or G is *a form* of degree d, if G is a linear combination of monomials of degree d.

Lemma 1.4.10. *If G is homogeneous of degree d, $G \in K[x_0, x_1, \dots, x_n]$, and t is a new variable, then* $G(tx_0, \dots, tx_n) = t^d G(x_0, \dots, x_n)$.

Proof. It is enough to prove the equality for monomials, i.e. for

$$G = ax_0^{i_0}x_1^{i_1}\dots x_n^{i_n} \text{ with } i_0 + i_1 + \dots + i_n = d:$$

$G(tx_0, \dots, tx_n) = a(tx_0)^{i_0}(tx_1)^{i_1}\dots(tx_n)^{i_n} = at^{i_0+i_1+\dots+i_n}x_0^{i_0}x_1^{i_1}\dots x_n^{i_n} =$
$= t^d G(x_0, \dots, x_n).$ □

Definition 1.4.11. Let G be a homogeneous polynomial of $K[x_0, x_1, \dots, x_n]$. A point $P[a_0, \dots, a_n] \in \mathbb{P}^n$ is a *zero of G* if $G(a_0, \dots, a_n) = 0$. In this case we write $G(P) = 0$.

Note that by Lemma 1.4.10 if $G(a_0, \dots, a_n) = 0$, then

$$G(\lambda a_0, \dots, \lambda a_n) = \lambda^{\deg G} G(a_0, \dots, a_n) = 0$$

for every choice of $\lambda \in K^*$. (Remind: K^* denotes $K \setminus \{0\}$.)

Definition 1.4.12. A subset Z of $\mathbb{P}^n$ is a *projective algebraic set*, or a *projective variety*, if Z is the set of common zeros of a set of homogeneous polynomials of $K[x_0, x_1, \dots, x_n]$.

If $T \subset K[x_0, x_1, \dots, x_n]$ is any subset formed by homogeneous polynomials, then the corresponding algebraic set will be denoted by $V_P(T)$.

1.5 Graded rings and homogeneous ideals

We want now to give an interpretation of projective varieties as sets of zeros of ideals, as we did in the affine case, see Proposition 1.4.3. But of course the ideal generated by a family of homogeneous polynomials contains also polynomials that are not homogeneous.

Let $\alpha = \langle T \rangle$ be the ideal generated by the polynomials of T, all assumed to be homogeneous. For any $F \in \alpha$, there is en expression $F = \sum_i H_i F_i$, $F_i \in T$.

So if $P[a_0, \dots, a_n] \in V_P(T)$, then

$$F(a_0, \dots, a_n) = \sum H_i(a_0, \dots, a_n) F_i(a_0, \dots, a_n) = 0$$

for any choice of coordinates of P, regardless if F is homogeneous or not. We say that P is a *projective zero* of F.

We want to formalize this situation in the context of the *graded rings*, of which the polynomial rings are a prototype. In particular in a graded ring there will be a situation similar to the following one: if F is a polynomial, then F can be written *in a unique way* as a sum of homogeneous polynomials, called the homogeneous components of F: $F = F_0 + F_1 + \dots + F_d$, where, for any index i, either the degree of F_i is equal to i, or $F_i = 0$.

We give the following definition:

Definition 1.5.1. Let A be a ring (as usual assumed to be commutative with unit). A is called *a graded ring over* $\mathbb{Z}$ if there exists a family of additive subgroups of A, $\{A_i\}_{i\in\mathbb{Z}}$, such that:

(i) $A = \oplus_{i\in\mathbb{Z}} A_i$; and

(ii) $A_i A_j \subset A_{i+j}$ for any pair of indices.

The elements of A_i are called *homogeneous of degree* i, and A_i is the homogeneous component of A of degree i. Condition (i) regards the additive structure of A; it means that any element a of A has a unique finite expression $a = \Sigma_{i\in\mathbb{Z}} a_i$, finite sum of homogeneous elements. Condition (ii) regards the multiplicative structure: a product of homogeneous elements is homogeneous of degree the sum of the degrees. Notice that 0 belongs to all homogeneous components of A.

The standard example of graded ring is the polynomial ring with coefficients in a ring R. R is the homogeneous component of degree 0, the variables have all degree 1. In this case the homogeneous components of negative degrees are all zero.

Proposition 1.5.2 (Proposition - Definition of homogeneous ideal)**.** *Let $I \subset A$ be an ideal of a graded ring. I is called* **homogeneous** *if the following equivalent conditions are fulfilled:*

(i) I is generated by homogeneous elements (this means: there is a system of generators formed by homogeneous elements);

(ii) $I = \oplus_{k\in\mathbb{Z}}(I \cap A_k)$, i.e. if $F = \Sigma_{k\in\mathbb{Z}} F_k \in I$, then all homogeneous components F_k of F belong to I.

Proof of the equivalence. "(ii)⇒(i)": given a system of generators of I, write each of them as sum of its homogeneous components: $F_i = \Sigma_{k\in\mathbb{Z}} F_{ik}$. Then a set of homogeneous generators of I is formed by all the elements F_{ik}.

"(i)⇒(ii)": let I be generated by a family of homogeneous elements $\{G_\alpha\}$, with $\deg G_\alpha = d_\alpha$. If $F \in I$, then F is a combination of the elements G_α with suitable coefficients H_α; write each H_α as sum of its homogeneous components: $H_\alpha = \Sigma H_{\alpha k}$. Note that the product $H_{\alpha k} G_\alpha$ is homogeneous of degree $k+d_\alpha$. By the unicity of the expression of F as sum of homogeneous elements, it follows that all of them are combinations of the generators $\{G_\alpha\}$ and therefore they belong to I. □

Let $I \subset K[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal. Note that, by the noetherianity, I admits a finite set of homogeneous generators.

Let $P[a_0, \dots, a_n] \in \mathbb{P}^n$. If $F \in I$, $F = F_0 + \cdots + F_d$, then $F_0 \in I, \dots, F_d \in I$. We say that *$P$ is a zero of I* if P is a projective zero of any polynomial of I or, equivalently, of any homogeneous polynomial of I. This also means that P is a zero of any homogeneous polynomial of a set generating I. The set of zeros of I will be denoted $V_P(I)$: all projective algebraic subsets of $\mathbb{P}^n$ are of this form.

As in the affine case, the projective algebraic subsets of $\mathbb{P}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology of $\mathbb{P}^n$. This time the empty set can be expressed as $V_P(1)$, as well as $V_P(x_0, \dots, x_n)$: indeed the n-tuple $[0, ..., 0]$ is not a point of $\mathbb{P}^n$. As for the other axioms of closed sets, the idea is always the same: the equations of the intersection of a family of algebraic sets are the union of all the equations, while the union of two algebraic sets X and Y is defined by all the possible products of two equations, one of X and the other of Y.

From the point of view of ideals, it is useful to make the following remark, whose proof follows from Proposition 1.5.2. Let I, J be homogeneous ideals of $K[x_0, x_1, \dots, x_n]$. Then $I + J$, IJ and $I \cap J$ are homogeneous ideals. Indeed both I and J are generated by homogeneous polynomials, $I + J$ is generated by the union of all of them, IJ is generated by products of two of them, one in I and the other in J, so in both cases by homogeneous polynomials. For $I \cap J$ it is enough to use Proposition 1.5.2 (ii).

Note that also all subsets of $\mathbb{A}^n$ and $\mathbb{P}^n$ have a structure of topological space, with the induced topology, which is still called the Zariski topology.

Exercises 1.5.3.

1. Let $F \in K[x_1, \dots, x_n]$ be a non–constant polynomial. The set $\mathbb{A}^n \setminus V(F)$ will be denoted $\mathbb{A}^n_F$. Prove that $\{\mathbb{A}^n_F \mid F \in K[x_1, \dots, x_n] \setminus K\}$ is a topology basis for the Zariski topology.

2. Let $B \subset \mathbb{R}^n$ be a ball. Prove that B is not Zariski closed.

3. Prove that the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $t \to (t, t^2, t^3)$ is a homeomorphism between $\mathbb{A}^1$ and its image, for the Zariski topology.

4. Let $X \subset \mathbb{A}^2_{\mathbb{R}}$ be the graph of the map $\mathbb{R} \to \mathbb{R}$ such that $x \to \sin x$. Is X closed in the Zariski topology? (hint: intersect X with a line....)