10/3/2021

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$([x_0,x_1],[y_0,y_1]) \longrightarrow [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$$

- σ is injective
- $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) = V_p(z_0z_3 - z_1z_2)$

quadric surface in $\mathbb{P}^3$
"
degree 2 homogeneous

A hom. pol. of deg 2 : a quadratic form on K^4 $\longleftrightarrow$ a bilinear form on K^4 symmetric

Fixed a basis $\mathcal{B}$ in K^4 $\longrightarrow$ matrix 4×4 symmetric

$$F(x_0, \dots, x_3) = \sum_{i,j} a_{ij} x_i x_j \qquad a_{ij} = a_{ji}$$

The coeff. x_ix_j in $2a_{ij}$

$A = (a_{ij})$

change basis $\rightsquigarrow$ $B = {}^tMAM$,
M = matrix of change of basis

$\operatorname{rk} B = \operatorname{rk} A$: rk of F = rk of the quadric $V_p(F)$

If $\operatorname{char} K \neq 2$ ($2 \neq 0$ in K) :
$2 = 1+1$

$\exists$ a basis in K^4 s.t. the matrix of the quadr. form is diagonal.

$A \rightsquigarrow B = {}^tMAM$ diagonal

For a suitable choice of coord.
any quadric has an equation of the form

$$a_0 x_0^2 + a_1 x_1^2 + \text{———} + a_n x_n^2 = 0$$

or more precisely

$$a_0 x_0^2 + \text{———} + a_{r-1} x_{r-1}^2 = 0$$

$r = \mathrm{rk}$ of the quadric

$a_0, -, a_{r-1} \neq 0$

If K is algebraically closed, it is possible to find coordinates s.t.

$$x_0^2 + x_1^2 + \text{—} - + x_{r-1}^2 = 0$$

If $K = \mathbb{R}$, $x_0^2 + \text{—} + x_p^2 - x_{p+1}^2 - \cdots - x_{r-1}^2 = 0$

$$\boxed{z_0 z_3 - z_1 z_2 = 0}$$

$$\mathrm{rk} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} = 4$$

$$\boxed{K = \mathbb{C}} \quad \boxed{z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0} \leftarrow$$

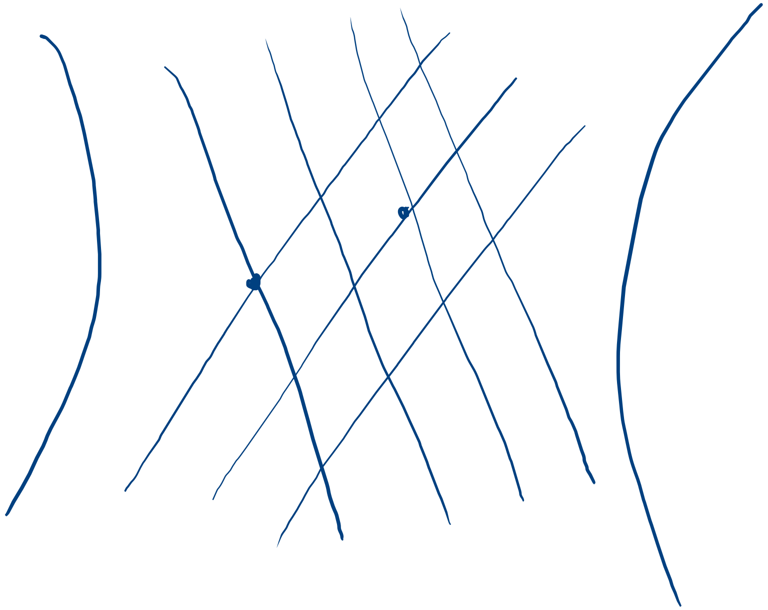
$K = \mathbb{R}$ eigenvalues 1 $m_a(1) = 2$

-1 $m_a(-1) = 2$

$K = \mathbb{C}$ $\quad z_0^2 + z_1^2 - z_2^2 - z_3^2 = 0$

If rk is $r+1$, $\mathbb{P}^n$
cone with vertex a proj subspace
of dim $(n+1) - (r+1) - 1$

$n - r - 1$



$j_0 : \mathbb{A}^n \longrightarrow \mathbb{P}^n$ $\qquad$ $\mathbb{A}^n \hookrightarrow U_0$

$U_0 = \mathbb{P}^n \setminus \{x_0 = 0\}$

$\{x_0 = 0\}$ $\qquad$ $V_p(x_0) = H_0$

U_0 open in the Zariski topology

$j_0 : (a_1, \dots, a_n) \longrightarrow [1, a_1, \dots, a_n]$

$\varphi_0 : U_0 \longrightarrow \mathbb{A}^n$

$[x_0, \dots, x_n] \longrightarrow \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$, $x_0 \neq 0$

$\forall i = 0, \dots, n$ $\qquad$ $j_i : \mathbb{A}^n \longrightarrow U_i = \mathbb{P}^n \setminus V_p(x_i) = \mathbb{P}^n \setminus H_i$

$\varphi_i : \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$

$\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$

because each point has at least one coord. $\neq 0$

$\Longrightarrow$ open covering of $\mathbb{P}^n$ with open subsets in <u>bijection</u> with $\mathbb{A}^n$

$i=0$ $\quad j_0 : \mathbb{A}^n \longrightarrow U_0$ $\qquad$ $\mathbb{A}^n$ Zariski topology

U_0 induced topology from Zariski top. of $\mathbb{P}^n$

Homogeneization and de-homogeneization of polynomials

1) dehomogeneization w. respect to x_0 $\quad {}^a : K[x_0, \dots, x_n] \longrightarrow K[y_1, \dots, y_n]$

$$F(x_0, \dots, x_n) \longrightarrow {}^aF(y_1 \dots y_n) \overset{\text{def}}{=} F(1, y_1, \dots, y_n)$$

^{a} is a ring homom.. If F is homog, in gen. ${}^aF$ is not.

Ex. $F = x_0x_3 - x_1x_2 \longrightarrow {}^aF = y_3 - y_1y_2$

$$F = x_0x_3 - x_0^2 \longrightarrow {}^aF = y_3 - 1 \quad : \deg = 1$$

$$= x_0(x_3 - x_0)$$

$\deg({}^aF) \leq \deg F$; equality holds $\Longleftrightarrow x_0 \nmid F$

2) homogeneization w. respect to x_0

$$^h : K[Y_1, \dots, Y_n] \longrightarrow K[x_0, x_1, \dots, x_n]$$

$$\underline{G(Y_1, \dots, Y_n)} \longrightarrow {}^hG(x_0, \dots, x_n) \overset{\text{def.}}{=} x_0^d G\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

where $d = \deg G$

A monomial in G $\quad Y_1^{i_1} \cdots Y_n^{i_n} \quad r = i_1 + \dots + i_n \le d$

$$x_0^d \left(\frac{x_1}{x_0}\right)^{i_1} \cdots \left(\frac{x_n}{x_0}\right)^{i_n} = x_0^{d-r} x_1^{i_1} \cdots x_n^{i_n} \quad \text{mon. of degree}$$

$\Rightarrow {}^hG$ is homogeneous of deg d

^{h} is not a homomorphism: sum is not preserved

$${}^h(G + G') \neq {}^hG + {}^hG' \quad \text{in general}$$

1) $\deg G \neq \deg G'$

2) $\deg G = \deg G'$ $\qquad G = Y_1 + Y_2Y_3 + Y_3^3, \quad G' = 2 - Y_1 - Y_3^3$

$$G + G' = 2 + Y_2Y_3 \qquad {}^h(G+G') = 2x_0^2 + x_2x_3$$

$${}^hG = x_0^2\underline{x_1} + x_0x_2x_3 + \underline{x_3^3}, \quad {}^hG' = 2x_0^3 - x_0^2\underline{x_1} - \underline{x_3^3}$$

$$\begin{aligned}{}^hG + {}^hG' = x_0 x_2 x_3 + 2x_0^3 \\ = x_0\, {}^h(G+G')\end{aligned}$$

Compare ^{a} and ^{h}

$$G(y_1 \ldots y_n) \xrightarrow{h} {}^hG(x_0 \ldots x_n) \xrightarrow{a} {}^a({}^hG)(y_1 \ldots y_n) = G(y_1 \ldots y_n)$$

$${}^hG(x_0 \ldots x_n) = x_0^d G\left(\frac{x_1}{x_0} \ldots \frac{x_n}{x_0}\right)$$

$${}^a({}^hG) = G$$

$$F(x_0 \ldots x_n) \longrightarrow {}^aF(y_1 \ldots y_n) \longrightarrow {}^h({}^aF)(x_0 \ldots x_n) \neq F(x_0 \ldots x_n)$$

is F is not homog.

Ans. F homog. : ${}^h({}^aF) \neq F$ in general

$$F = x_0^r \underbrace{F'}_{\text{not divisible by } x_0} \text{ homog.} \qquad {}^aF = {}^aF' \qquad {}^h({}^aF) = {}^h({}^aF') = F'$$

$$\boxed{F = x_0^r \, {}^h({}^aF)}$$

$j_0 : \mathbb{A}^n \longrightarrow U_0 \subseteq \mathbb{P}^n$ <u>Thm</u> j_0 is a homeomorphism ; $\varphi_0 : U_0 \longrightarrow \mathbb{A}^n$

<u>Pf</u>. 1) $X \subseteq U_0$ closed : we prove that $\varphi_0(X)$ is closed in $\mathbb{A}^n$
2) $\underline{Y \subseteq \mathbb{A}^n}$ Zariski closed : " " $j_0(Y)$ - - - U_0

1) X closed in U_0 : $X = \underline{U_0 \cap V_p(I)}$, I homog. ideal of $K[x_0 \dots x_n]$

$\underline{\varphi_0(X) \overset{?}{=} V(\alpha)}$ $\alpha \subseteq K[y_1 \dots y_n]$ Put $\alpha = {}^a I$ = image of I in ${}^a : K[x_0 \dots x_n] \to K[y_1 \dots y_n]$

$I \longrightarrow {}^a I$ ideal because ^{a} is a ring homom.

$\alpha = \{ {}^a F \mid F \in I \}$

$P \in U_0$ $P[\underset{\neq 0}{x_0} \dots x_n] = [1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}] = j_0\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$

$\varphi_0(P) = \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ $P \in V_p(I)$: $\forall F \in I$ $F(P) = 0$
P is a proj. zero of F

$F\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = 0 = {}^a F\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = {}^a F(\varphi_0(P))$

$\varphi_0(P) \in V({}^a I) = V(\alpha)$: $\varphi_0(X) \subseteq V(\alpha)$

$V(\alpha) \subseteq \varphi_0\big(U_0 \cap V_P(I)\big)$ $\qquad Q(y_1 \dots y_n) \in V(\alpha)$: if $G \in \alpha$
$G(y_1 \dots y_n) = 0$

$\varphi_0(Q) = [1, y_1, \dots, y_n] \in U_0$

F : homog. polyn. in I $\qquad F(1, y_1 \dots y_n) = {}^aF(y_1 \dots y_n) = 0$
${}^aF \in \alpha = {}^aI$

I homog. : it is generated by its homog. polynomials

$\Longrightarrow \varphi_0(Q) \in V_P(I)$.

2) Y closed in A^n : $Y = V(\beta)$, $\beta \subseteq K[Y_1 \dots Y_n]$

$j_0(Y) \subseteq U_0$: we look for X closed in P^n s.t.

$j_0(Y) = X \cap U_0$ $\qquad h : K[x_0 \dots x_n] \longrightarrow K[Y_1 \dots Y_n]$

Def. ${}^h\beta = \langle {}^hG \mid G \in \beta \rangle$ homog. ideal

$j_0(Y) = V_P({}^h\beta) \cap U_0$

$$j_0(Y) = j_0(V(\beta)) \subseteq U_0 \cap V_p({}^h\beta)$$

$Q \in V(\beta)$

$\shortparallel$

$(y_1 \ldots y_n)$

$j_0(Q) = [1\, y_1 \ldots y_n]$ Take one of the generators of ${}^h\beta$

${}^hG$, $G \in \beta$ ${}^hG(1, y_1, \ldots, y_n) = G(y_1 \ldots y_n) = 0$

$\Longrightarrow j_0(Q) \in V_p({}^h\beta)$ because $Q \in V(\beta)$.

$P \in U_0 \cap V_p({}^h\beta)$ $P[\underset{\neq 0}{x_0} \ldots x_n] = \left[1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right] = j_0\Big(\underbrace{\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}}\Big)$

$G \in \beta$: $G\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$

${}^hG(\underset{\shortparallel}{x_0}, \ldots, x_n) = 0$ $P \in V_p({}^h\beta)$

${}^hG\left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = {}^a({}^hG)\left(\frac{x_1}{x_0} \ldots \frac{x_n}{x_0}\right)$

$= G\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = 0$

$$\mathbb{A}^n \xleftrightarrow{\sim} U_0$$

$X \subseteq \mathbb{A}^n$ $\quad X = V(F_1, \ldots, F_r) = V(\alpha)$

$K[x_1 \ldots x_n]$

Different ideals define the same hypersurface

$X = V(F) = V(F_1 \ldots F_r)$ $\quad \alpha = (F)$

$F = F_1^{m_1} \ldots F_r^{m_r}$ $\quad \langle F_1 \ldots F_r \rangle$

$P(0,0) \in \mathbb{A}^2$ $\quad P = V(x,y) = V(x^2, y) = V(x^2, xy, y^2) = V(x,y)^2 =$

$K[x,y]$

$\langle x,y \rangle^d = \langle x^d, x^{d-1}y, \ldots, y^d \rangle$

$m = \langle x,y \rangle$ $\quad \underline{m} \supset m^2 \supset m^3 \supset \ldots$

$Y \subseteq \mathbb{A}^n_K$ any set $\quad$ Def. the ideal of Y

$I(Y) \overset{def}{=} \{F \in K[x_1 \ldots x_n] \mid F(P) = 0 \ \forall P \in Y\}$: it is an ideal

$F, F' \in I(Y)$ $\quad (F - F')(P) = F(P) - F'(P) = 0 - 0 = 0$ $\quad P \in Y$

$F \in I(Y), H \in K[x_1 \ldots x_n]$ $\quad (HF)(P) = H(P)\underbrace{F(P)}_{=0} = 0$

Properties: $Y \subseteq Y' \implies I(Y') \subseteq I(Y)$

$I(Y \cup Y') = I(Y) \cap I(Y')$

$F \in I(Y \cup Y') \implies F \in I(Y),\ F \in I(Y') \implies I(Y \cup Y') \subseteq I(Y) \cap I(Y')$

$F \in I(Y) \cap I(Y') : \quad P \in Y \cup Y' \quad \begin{cases} P \in Y & F \in I(Y) \quad F(P) = 0 \\ P \in Y' & F \in I(Y') \quad F(P) = 0 \end{cases}$

$I(Y \cap Y') \supseteq I(Y) + I(Y') : \quad Y \cap Y' \subseteq Y \quad I(Y) \subseteq I(Y \cap Y')$

$\subseteq Y' \quad I(Y') \subseteq$

In general no equality

$\mathbb{P}^n \quad Z \subseteq \mathbb{P}^n \quad \langle \{F \in k[x_0 \dots x_n], \text{homogeneous} \mid F(P) = 0,\ \forall P \in Z\} \rangle$

homogeneous ideal of $Z \quad I_h(Z)$

$X \subseteq \mathbb{A}^n$ $\quad \underset{\overset{\shortparallel}{V(\alpha)}}{X}$ algebraic set $\quad I(X)$: what is the relation between α and $I(X)$?

$\alpha \subseteq K[x_1 \dots x_n]$ $\quad \sqrt{\alpha}$ radical of α : is an ideal

$\sqrt{\alpha} = \{F \in K[x_1 \dots x_n] \mid \exists\, r \geq 1 \text{ s.t. } F^r \in \alpha\}$

$\alpha \subseteq \sqrt{\alpha}$; $\quad \sqrt{\alpha} = \underset{\overset{\shortparallel}{\langle 1 \rangle}}{K[x_1 \dots x_n]} \Longleftrightarrow \alpha = K[x_1 \dots x_n]$

$\Longleftarrow$

$\Longrightarrow$

Prop. $Y \subseteq \mathbb{A}^n$ any subset, $I(Y)$ is a radical ideal i.e.

$I(Y) = \sqrt{I(Y)}$

Pf. $I(Y) \subseteq \sqrt{I(Y)}$; $\quad F \in \sqrt{I(Y)}$: $\quad F^r \in I(Y)$: $\forall P \in Y$
$r \geq 1$ $\quad \underset{\overset{\shortparallel}{F(P)^r}}{F^r(P)} = 0 \in K \Rightarrow F(P) = 0$

$\Rightarrow F \in I(Y)$ $\qquad \sqrt{I(Y)} \subseteq I(Y)$

1) $\alpha \subseteq k[x_1 \dots x_n]$: $I(V(\alpha)) \supseteq \alpha$ this is true by def.

$F \in \alpha$, $P \in V(\alpha)$: $\underline{F(P) = 0}$

2) $Y \subseteq \mathbb{A}^n$: $V(\underline{I(Y)}) \supseteq Y$: by def.

$P \in I$, $F \in I(Y)$: $F(P) = 0$

3) $Y \subseteq \mathbb{A}^n$ $\boxed{V(I(Y)) = \overline{Y}}$ closure of Y in the Zariski top.

$Y \subseteq V(I(Y))$: take closures $\overline{Y} \subseteq \overline{V(I(Y))} = V(I(Y))$

Conversely: want to prove that $V(I(Y)) \subseteq \overline{Y}$

$V(I(Y))$ is a closed set containing Y; we want to check that, if V is closed and contains Y, then it also contains $V(I(Y))$

$V = V(\beta) \supseteq Y \Rightarrow \underset{\beta \subseteq}{I(V(\beta))} \subseteq \underline{I(Y)} \Rightarrow \underset{V}{\underset{\parallel}{V(\beta)}} \supseteq V(I(Y))$

In particular: if $X \subseteq \mathbb{A}^n$ is algebraic aff. variety

$\boxed{X = V(I(X))}$

$V \circ I$ this is the identity if applied to algebraic sets

$I \circ V$