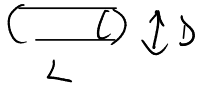
PROPAGAZIONE DELLE INCERTEZZE.

Anche propagazione degli "errori" → meglio "incertezze"

Es.:



$D = (1.35 \pm 0.05)\ cm$ → $A = \frac{\pi}{4} D^2 = 1.43188\ cm^2$

$L = (5.15 \pm 0.05)\ cm$ $\quad \Delta A = ?$

$$\begin{cases} A_{max} = \frac{\pi}{4}(D+\Delta D)^2 = \frac{\pi}{4}(D^2 + 2D\Delta D + \Delta D^2) \\ A_{min} = \frac{\pi}{4}(D-\Delta D)^2 = \frac{\pi}{4}(D^2 - 2D\Delta D + \Delta D^2) \end{cases} \Rightarrow \Delta A = \frac{A_{max} - A_{min}}{2}$$

$$= \frac{1}{\cancel{2}} \frac{\pi}{4} \cancel{2} \cdot 2D\Delta D = \frac{\pi}{4} 2D\Delta D$$

$$\frac{\Delta A}{A} = \frac{\cancel{\frac{\pi}{4}} 2 \cancel{D} \Delta D}{\cancel{\frac{\pi}{4}} D^{\cancel{2}}} = 2\frac{\Delta D}{D}$$

Calcolo:

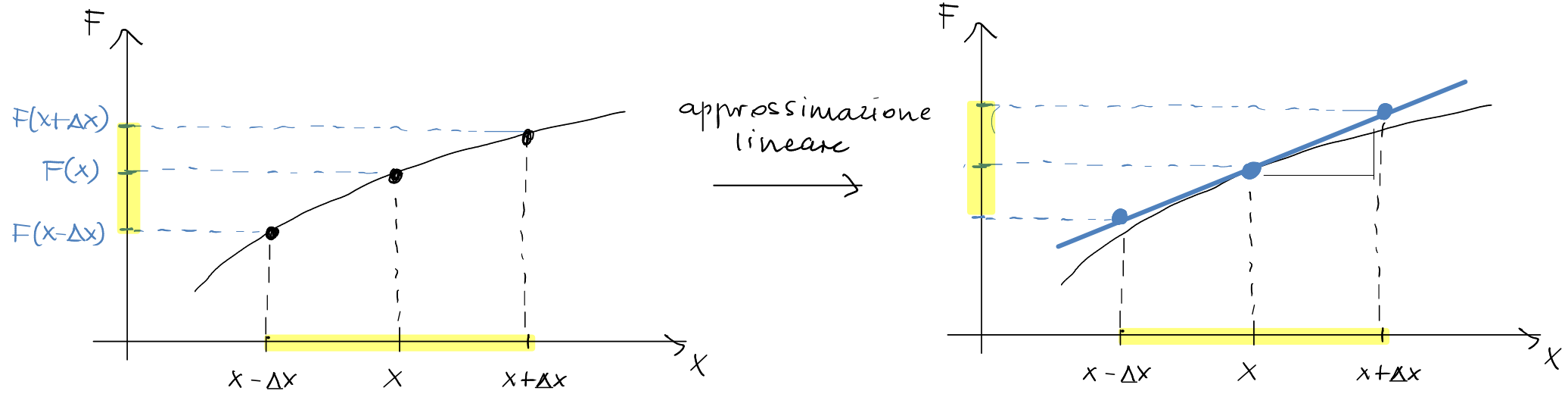
$$\frac{\Delta D}{D} = \frac{0.05}{1.35} = 3.7\% = 0.037 \Rightarrow \frac{\Delta A}{A} = 2 \times 0.037 = 7.4\% = 0.074$$

$$\Delta A = 0.074 \times 1.43188\ cm^2 = 0.107391\ cm^2 \rightarrow A \pm \Delta A = (1.43 \pm 0.11)\ cm^2$$

Regola pratica: leggi di potenza o prodotti per fattori numerici esatti → conservo il numero di cifre significative

Propagazione dell'incertezza: funzioni di 1 variabile

Misuro $X \pm \Delta X$ → Grandezza derivata $F(X)$ → Incertezza su F: $\Delta F = ?$ [ES: $F(D) = \frac{\pi}{4} D^2$]



$$\lim_{\Delta X \to 0} \frac{\Delta F}{\Delta X} = \frac{dF}{dX} \rightarrow F(X') = F(X) + \frac{dF}{dX} \cdot (X' - X) \quad \text{retta tangente}$$

$$\begin{cases} F(X+\Delta X) = F(X) + \frac{dF}{dX}\Delta X + O(\Delta X^2) \approx F(X) + \frac{dF}{dX}\Delta X \\ F(X-\Delta X) \quad \text{---} \quad \approx F(X) - \frac{dF}{dX}\Delta X \end{cases} \rightarrow \Delta F = \frac{|F(X+\Delta X) - F(X-\Delta X)|}{2} \approx \left|\frac{dF}{dX}\right| \Delta X$$

ES: $F(X) = aX^2 \rightarrow \Delta F = |2aX|\,\Delta X \Rightarrow \frac{\Delta F}{|F|} = \frac{|2aX|}{|aX^2|}\Delta X = 2\frac{\Delta X}{|X|}$

Leggi di potenza: $F(x) = aX^{\alpha} \rightarrow \frac{\Delta F}{|F|} = \alpha \frac{\Delta X}{|X|}$

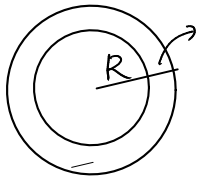
Approssimazioni lineari utili : $|x| \ll 1$

1) $(1+x)^{\alpha} \approx 1+\alpha x \rightarrow \frac{1}{1+x} \approx 1-x$

2) $\exp(x) \approx 1+x$

3) $\ln(1+x) \approx x$

$\rightarrow$ Esempio: volume calotta sferica per $\frac{\Delta R}{R} \ll 1$



$$\Delta V = \frac{4}{3}\pi(R+\Delta R)^3 - \frac{4}{3}\pi R^3 =$$

$$= \frac{4}{3}\pi R^3 \underbrace{\left(1+\frac{\Delta R}{R}\right)^3} - \frac{4}{3}\pi R^3 \approx$$

$$= \frac{4}{3}\pi R^3 \left[\underbrace{\left(1+3\frac{\Delta R}{R}\right)} - 1\right]$$

$$\Delta V \approx \frac{4}{\cancel{3}}\pi R^{\cancel{3}\,2} \cdot \cancel{3}\frac{\Delta R}{\cancel{R}} = 4\pi R^2 \Delta R$$

Propagazione delle incertezze: funzione di 2 variabili

$V = A \cdot L = \frac{\pi}{4} D^2 \cdot L$

Grandezze $X \pm \Delta X$ e $Y \pm \Delta Y \rightarrow F(X,Y) \rightarrow \Delta F = ?$

Casi particolari:

+ e − : somma incertezze assolute

1) Somma: $F = X + Y$

$$\begin{cases} F_{max} = (X+\Delta X) + (Y+\Delta Y) = (X+Y) + (\Delta X + \Delta Y) \\ F_{min} = (X-\Delta X) + (Y-\Delta Y) = (X+Y) - (\Delta X + \Delta Y) \end{cases} \Rightarrow \Delta F = \frac{|F_{max} - F_{min}|}{2} = \Delta X + \Delta Y$$

2) Differenza: $F = X - Y$

$$\begin{cases} F_{max} = (X+\Delta X) - (Y-\Delta Y) = (X-Y) + (\Delta X + \Delta Y) \\ F_{min} = (X-\Delta X) - (Y+\Delta Y) = (X-Y) - (\Delta X + \Delta Y) \end{cases} \Rightarrow \Delta F = \Delta X + \Delta Y \ ⚠ \Rightarrow \frac{\Delta F}{|F|} \text{ può crollare}$$

3) Prodotto: $F = X \cdot Y \quad X, Y > 0$

$$\begin{cases} F_{max} = (X+\Delta X)(Y+\Delta Y) = XY + X\Delta Y + Y\Delta X + \Delta X \Delta Y \\ F_{min} = (X-\Delta X)(Y-\Delta Y) = XY - X\Delta Y - Y\Delta X + \Delta X \Delta Y \end{cases} \Rightarrow \Delta F = X\Delta Y + Y\Delta X \Rightarrow \frac{\Delta F}{|F|} = \frac{\Delta X}{|X|} + \frac{\Delta Y}{|Y|}$$

4) Divisione: $F = X/Y$

$$\Rightarrow \frac{\Delta F}{|F|} = \frac{\Delta X}{|X|} + \frac{\Delta Y}{|Y|}$$

Es: $V = \frac{\pi}{4} D^2 L \Rightarrow \frac{\Delta V}{V} = 2\frac{\Delta D}{D} + \frac{\Delta L}{L}$

× e ÷ : somma incertezze relative

5) Legge di potenza: $F = X^a Y^b \Rightarrow \frac{\Delta F}{|F|} = a\frac{\Delta X}{|X|} + b\frac{\Delta Y}{|Y|} \Rightarrow$

$$\frac{\Delta V}{V} = \frac{\Delta A}{A} + \frac{\Delta L}{L} = 0.074 + 0.0097 = 0.084 \Rightarrow \Delta V = 0.619 \text{ cm}^3$$

$$V = (7.4 \pm 0.6) \text{ cm}^3$$

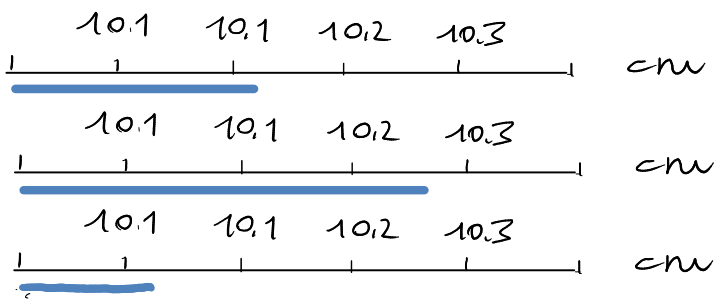
Esercizio: incertezza su $F = x^2 - y^2 \Rightarrow \Delta F = ?$

1) calcolo $\Delta(x^2)$ e $\Delta(y^2)$ $\rightarrow \frac{\Delta(x^2)}{x^2} = 2\frac{\Delta x}{\Delta x}$ ... $\Rightarrow \Delta(x^2)$ e $\Delta(y^2)$

2) calcolo $\Delta F = \Delta(x^2) - \Delta(y^2)$

Applicazione numerica: $x = 1.01$, $y = 1.00$, $\Delta x = 0.01$, $\Delta y = 0.01$ $\Rightarrow$ cifre significative su F?

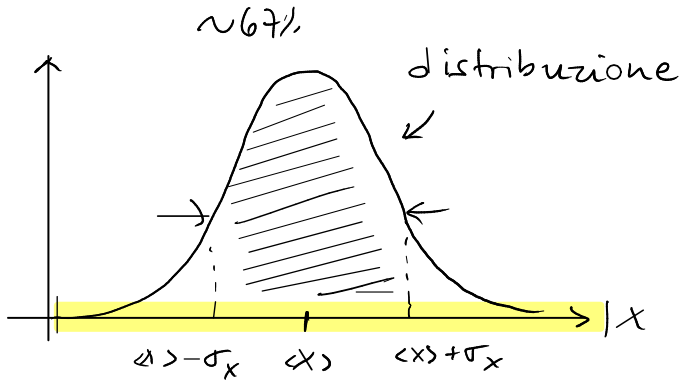
Incertezze statistiche e loro propagazione



$x = (10{,}15 \pm 0.05)$ cm

$x = (10{,}25 \pm 0.05)$ cm

$x = (10{,}15 \pm 0.05)$ cm



$x_1, x_2, \dots x_N \rightarrow$ misure $\longrightarrow \langle x \rangle \pm \sigma_x \leftarrow$ statistica

valore medio: $\langle x \rangle = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$; deviazione standard: $\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x \rangle)^2}$

1) Somma e differenza: $\sigma_F^2 = \sigma_x^2 + \sigma_y^2$

2) prodotto e divisione: $\left(\frac{\sigma_F}{\langle F \rangle}\right)^2 = \left(\frac{\sigma_x}{\langle x \rangle}\right)^2 + \left(\frac{\sigma_y}{\langle y \rangle}\right)^2$ $\rightarrow$ in "quadratura"