IF WE SET $p_x \equiv p_1$; $p_y \equiv p_2$; $p_z \equiv p_3$ $\Rightarrow$

(32) $$H_\perp = \frac{p_1^2}{2m} + \frac{1}{2} m \omega_L^2 q_1^2$$ WHICH IS THE

HAMILTONIAN FOR AN HARMONIC OSCILLATOR.

- PROBLEM 2 CONSIDER A PARTICLE OF MASS m WITH KINETIC ENERGY $T = \frac{1}{2} m v^2$ MOVING IN 1D IN A POTENTIAL $V(x)$. USE THE EULER-LAGRANGE EQ. TO FIND THE EQ. OF MOTION: $d_t \partial_{\dot{x}} L - \partial_x L = 0$.

SOLUTION THE ASSOCIATED LAGRANGIAN L IS $L = \frac{1}{2} m \dot{x}^2 - V(x)$. NEXT IS THE DERIVATIVE OF L RESPECT TO $\dot{x}$ $\rightarrow$ $\frac{d}{dt}\left(\partial_{\dot{x}} L\right)$ $\rightarrow$ DERIVATIVE OF $\partial_x L$.

- $\partial_{\dot{x}} L = \partial_{\dot{x}} \left[\frac{1}{2} m \dot{x}^2 - V(x)\right] = m\dot{x}$ (THIS IS A MOMENTUM TERM).

·· $\frac{d}{dt}\left(\partial_{\dot{x}} L\right) = \dot{m}\dot{x} + m\ddot{x} = m\ddot{x}$ (THIS IS m TIME ACCELERATION)

··· $\partial_x L = \partial_x \left[\frac{1}{2} m \dot{x}^2 - V(x)\right] = -\partial_x V$ $\Rightarrow$

THIS IS A FAMILIAR RESULT IF $\bar{F}$ IS CONSERVATIVE $\Rightarrow$ $\bar{F} = -\bar{\nabla} V$

- PROBLEM 3. CONSIDER A MASS m UNDERGOING SIMPLE HARMONIC MOTION

THE FORCE ON THE PARTICLE IS GIVEN BY $\bar{F}(x) = -kx$. DERMINE THE EQ. OF MOTION USING THE EULER-LAGRANGE EQ.

SOLUTION

$$T = \frac{1}{2} m \dot{x}^2 \; ; \; V = \frac{1}{2} k x^2 \Rightarrow$$

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \longrightarrow \text{APPLAY E-L} \Rightarrow$$

- $\partial_{\dot{x}} L = \partial_{\dot{x}} \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right) = m \dot{x} \Rightarrow \frac{d}{dt} (\partial_{\dot{x}} L) = \frac{d}{dt} (m \dot{x}) = m \ddot{x}$

- $\partial_x L = -kx \Rightarrow d_t \partial_{\dot{x}} L - \partial_x L = 0 \Rightarrow m \ddot{x} = -kx$

THIS IS THE FAMILIAR EQ. OF MOTION OF THE HARMONIC OSCILLATOR

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

WITH $\omega_0^2 = k/m$.

- FROM CLASSICAL DYNAMICS TO QUANTUM MECHANICS OF A CHARGE IN AN EM FIELD

IN SUMMARY THE CLASSICAL HAMILTONIAN FOR A CHARGE PARTICLE IN AN E.M. FIELD IS GIVEN BY (27)

$$H = \frac{1}{2m} \left| \bar{p} + e \bar{A}(\bar{x}, t) \right|^2 + e\varphi(\bar{x}, t) \qquad (33)$$

WHERE WE HAVE USED $-e \equiv q$

USING THIS RESULT WE CAN NOW TURN TO THE Q.M. FORMULATION. ALTHOUGH WE WILL STUDY THE DERIVATION OF THE Q.M. FROM THE CLASSICAL PHYSICS IN THE FUTURE LECTURES, HERE WE ASSUME THAT THE CANONICAL QUANTIZATION PROMOTES THE CONJUGATE VARIABLES TO OPERATORS

$$\boxed{\bar{p} \to \hat{p} = -i\hbar\bar{\nabla},\ x \to \hat{x}} \quad (34)$$

WITH THE COMMUTATION RELATIONS

$$\boxed{[\hat{p}_i, \hat{x}_j] = -i\hbar\,\delta_{ij}} \quad (35)$$

THEREFORE THE HAMILTONIAN OPERATOR $\hat{H}$ IS

$$\boxed{\hat{H} = \frac{1}{2m}\left(\hat{p} + e\bar{A}(\bar{x},t)\right)^2 - e\varphi(\bar{x},t)} \quad (36)$$

EXPANDING THE $\hat{H}$ IN $\bar{A}$ WE CAN IDENTIFY, ADOPTING THE COULOMB GAUGE $\bar{\nabla}\cdot\bar{A}=0$ TWO TYPES OF CONTRIBUTION:

- THE CROSS-TERM (KNOWN AS PARAMAGNETIC TERM)

$$\frac{e}{2m}\left(\hat{p}\cdot\bar{A} + \bar{A}\cdot\hat{p}\right)$$

WHERE BY REPLACING $\hat{p} \to -i\hbar\bar{\nabla}$ $\Rightarrow -\frac{iq\hbar}{2m}\left(\underbrace{\bar{\nabla}\cdot\bar{A}}_{=0} + \bar{A}\bar{\nabla}\right)$

$$\Rightarrow \boxed{-\frac{ie\hbar}{2m}\bar{A}\cdot\bar{\nabla}} \quad (37)$$

AND

- A DIAGONAL TERM $\frac{e^2}{2m}A^2$

ALL TOGETHER WE OBTAIN FOR THE ELECTRON THE OPERATOR

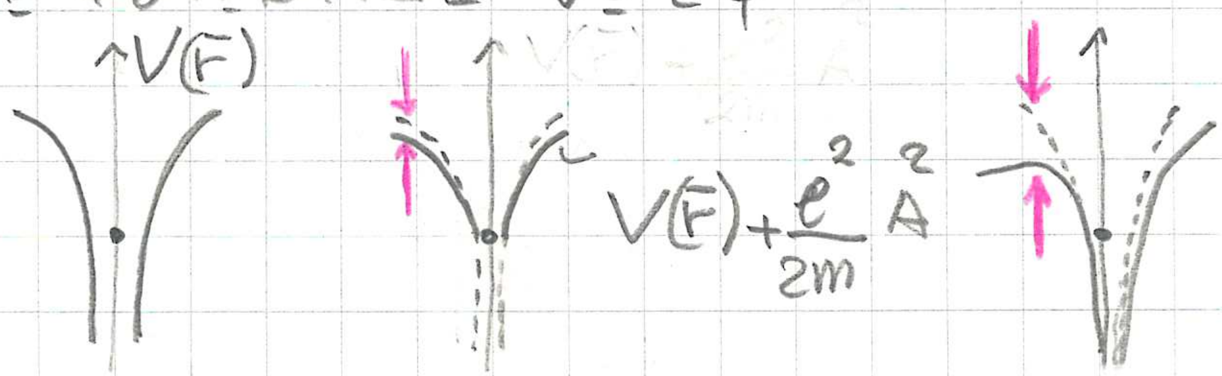
$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{ie\hbar}{m}\bar{A}\cdot\bar{\nabla} + \frac{e^2}{2m}\hat{A}^2 - e\varphi$$

38

THIS IS $\hat{H}_0$ OF THE ATOM UNPERTURBED

THIS IS THE INTERACTION TERM

THE INTERACTION TERM DESCRIBES HOW THE CARGES ARE EXCITED BY THE E.M. FIELD HERE DESCRIBED BY $\bar{A}$ ($\bar{A}$ IS THE CLASSICAL FIELD). THE PARAMAGNETIC TERM MODIFIES THE KINETIC ENERGY OF THE CHARGE. IT IS SEMICLASSICAL FOR THE MOMENTUM IS QUANTIZED BUT NOT THE FIELD. THE DIAMAGNETIC TERM $\left(q^2/2m\right)A^2$ OVERLAPS WITH THE POTENTIAL TERM SO IT WILL INTERACT WITH THE e^- ONLY INDIRECTLY PERTURBING OR MODIFING THE POTENTIAL $V = e\varphi$



POTENTIAL UNPERTURBED — PERTURBED POTENTIAL — MODIFIED POTENTIAL

SO WE NEED ELECTRIC FIELDS COMPARABLE WITH THE COULOMB $\bar{E}$ FIELDS OF THE

ATOM $\left(\sim 10^8 \div 10^{10}\ V/m\right)$ TO MODIFY ITS ATOMIC COULOMB FIELD. $\Rightarrow$ NORMAL LIGHT DO NOT MODIFIES THE COULOMB POTENTIAL OF AN ATOM. ONLY VERY STRONG LASER LIGHT DOES. WE WILL ALSO SEE IN THE NEXT PART THAT IN TERMS OF ABSOLUTE ENERGY FOR A BOUND ELECTRON THE PARAMAGNETIC TERM DOMINATE BY FAR THE DIAMAGNETIC TERM. NOT SO FOR A FREE e^- (OR CHARGE) OR NEUTRON STARS.

LET'S NOW REPEAT THE SAME CALCULATIONS DONE FOR THE CLASSICAL HAMILTONIAN IN A CONSTANT (STATIONARY) UNIFORM $\bar{B}$.

WE CAN USE THE GAUGE (SYMMETRIC GAUGE) $\boxed{\bar{A}(\bar{r}) = -\frac{1}{2}\bar{r}\times\bar{B}}$. THE PARAMAGNETIC COMPONENT OF $\hat{H}$ IS GIVEN BY

$$\frac{iQ\hbar}{m}\bar{A}\cdot\bar{\nabla} = -\frac{iQ\hbar}{2m}(\bar{r}\times\bar{B})\cdot\bar{\nabla} \Rightarrow$$

$$= \frac{iQ\hbar}{2m}(\bar{r}\times\bar{\nabla})\cdot\bar{B} = \boxed{-\frac{Q}{2m}\hat{L}\cdot\bar{B}} \quad \text{WHERE} \quad (39)$$

$\hat{L} = \bar{r}\times(-i\hbar\bar{\nabla})$ IS THE ANGULAR MOMENTUM OPERATOR.

- FOR FIELD $\bar{B} = B\hat{z}$ THE DIAMAGNETIC TERM

$$\frac{Q^2}{2m}\bar{A}^2 = \frac{Q^2}{8m}(\bar{r}\times\bar{B})^2 = \frac{Q^2}{8m}\left[r^2B^2 - (\bar{r}\cdot\bar{B})^2\right] =$$

$$= \frac{Q^2B^2}{8m}(x^2+y^2), \text{ BEING } \bar{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (40)$$

$$\Rightarrow \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \underbrace{\frac{QB}{2m}\hat{L}_z}_{\text{PARAM}} + \underbrace{\frac{Q^2B^2}{8m}(x^2+y^2)}_{\text{DIAM.}} + Q\varphi . \quad (41)$$

IN THE FOLLOWING WE WILL SEE TWO EXAMPLES OF ELECTRON ($q \equiv -e$) MOTION IN A UNIFORM MAGNETIC FIELD $\bar{B} = B\hat{z}$

- ATOMIC HYDROGEN $\Rightarrow Q\varphi(\bar{r}) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \equiv V(\bar{r})$
- FREE ELECTRONS $\Rightarrow \varphi(\bar{r}) = 0$

- ATOMIC HYDROGEN. IN THIS CASE THE DIAMAGNETIC TERM IS NEGLIGIBLE, WHEREAS FOR FREE e^- BOTH TERMS CONTRIBUTE SIGNIFICATLY TO THE DYNAMICS. $\Rightarrow$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{eB}{2m}\hat{L}_z + \frac{e^2B^2}{8m}(x^2+y^2) - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \quad (42)$$

WE CAN ASSUME THAT $(x^2+y^2) \simeq a_0^2$ BEING a_0 THE BOHR RADIUS, AND $\langle L_z\rangle \simeq \hbar$, THE RATIO OF PARAMAGNETIC/DIAMAGNETIC TERMS IS

$$\frac{\text{DIA}\ (e^2/8m_e)(x^2+y^2)B^2}{\text{PARA}\ (e/2m_e)\langle L_z\rangle B} = \frac{e}{4\hbar}a_0^2 B \simeq 10^{-6}$$

$\Rightarrow$ FOR BOUD e^- THE DIAMAGNETIC TERM IS NEGLIGIBLE AS WE SAID BEFORE. WHEN THE PARA TERM IS COMPARED WITH THE COULOMB FIELD ENERGY SCALE WE GET

$$\frac{(e/2m)\hbar B}{m_e c^2\alpha^2/2} = \frac{e\hbar}{(m_e c\alpha)^2}B \simeq 10^{-5}$$

WHERE

$\alpha = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\hbar c} \simeq \frac{1}{137}$ DENOTES THE FINE (43)

STRUCTURE CONSTANT $\Rightarrow$ THE PARAMAGNETIC TERM CAN BE CONSIDERED A PERTURBATION (THIS IS THE ORIGIN OF "PERTURBATIVE" THEORY) $\Rightarrow$

(44) TERM

$$\hat{H} \simeq \frac{\hbar^2}{2m}\nabla^2 + \frac{e}{2m}\bar{B}\cdot\hat{L} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

- OBSERVATION, WE CAN REWRITE $\hat{H}$ AS $\hat{H} = \hat{H}_0 + \hat{H}_{INT}$, WHERE $\hat{H}_0$ IS THE HAMILTONIAN OF THE UNPERTURBED ATOM:

$\frac{\hbar^2}{2m}\nabla^2$ IS THE KINETIC TERM AND

$-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$ IS THE POTENTIAL TERM FOR

AN ELECTRON IN THE CENTRAL COULOMB FIELD. $\hat{H}_{INT}$ IS THE TERM OF THE INTERACTION OF THE BOUND e^- WITH A UNIFORM STATIONARY $\bar{B}$ FIELD.

IN GENERAL, TERM LINEAR IN $\bar{B}$ DEFINES MAGNETIC DIPOLE MOMENT $\bar{\mu} \equiv \hat{H}_{INT.M} = -\bar{\mu}\cdot\bar{B}$.

THIS RESULT SHOWS THAT ORBITAL DEGREES OF FREEDOM OF THE e^- LEAD TO A MAGNETIC MOMENT

$$\mu = -\frac{e}{2m_e}\hat{L} \qquad (44)$$

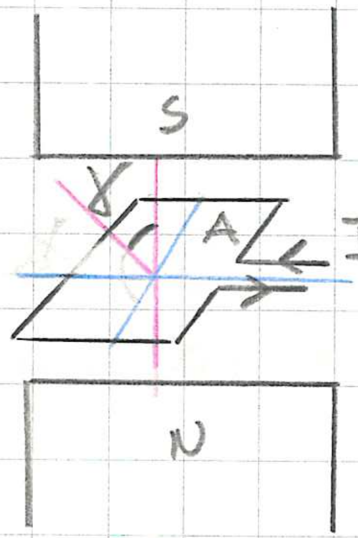
- OBSERVATION IT IS INTERESTING THE COMPARISON WITH THE CLASSICAL E.D.

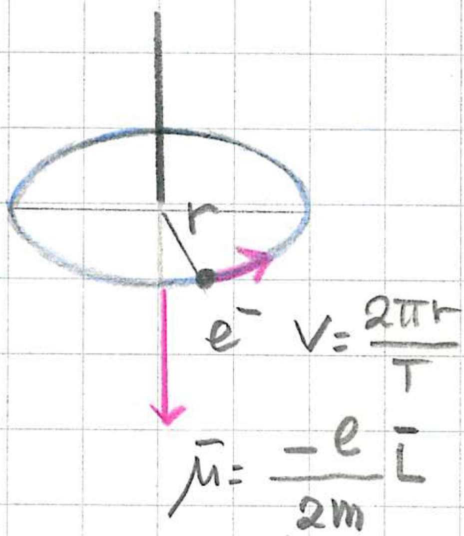
IN THIS CASE THE MAGNETIC MOMENT OF A CONDUCTING LOOP IS DEFINED AS

$|\bar{\mu}| = I \cdot A$ [Am^2] WHERE I IS THE CURRENT AND A = AREA. AS WE KNOW FROM D. GRIFFITH – INTRODUC.... – $\bar{\mu} \perp$ TO THE PLANE OF THE LOOP. WE CAN ALSO DEFINE THE TORQUE $\bar{\tau} = \bar{\mu} \times \bar{B}$. THE MAGNETIC POTENTIAL ENERGY OF THE DIPOLE IS

$$V_{MAG} = -\bar{\mu} \cdot \bar{B} = \int_{\pi/2}^{\alpha} \tau \, d\alpha = -|\bar{\mu}|\,|B| \cos\gamma$$

WHERE γ IS THE ANGLE BETWEEN $\bar{\mu}$ AND $\bar{B}$ FOR





IN ATOMIC AND NUCLEAR PHYSICS THE $\bar{\mu}$ IS OFTEN DEFINED AS THE TORQUE IN A UNIFORM AND STATIONARY $|\bar{H}|$ FIELD (NOT $|\bar{B}|$) $\Rightarrow$

$$\bar{\tau} = \bar{\mu}' \times \bar{H} \qquad |\mu'| = \mu_0 I A$$