Buona Primavera!

Buon Dantedì!

$$\int_\gamma f(z)\,dz = \int_\gamma f(\gamma(t))\cdot\gamma'(t)\,dt = \int_I (u x' - v y')\,dt + i\int_I (u y' + v x')\,dt$$

$\gamma : I \to \mathbb{C}$ $\qquad f \simeq (u, v)$

Se $\gamma = \gamma_1 + \gamma_2$ $\qquad \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$

Se $\gamma_1 \simeq \gamma_2$ $\qquad \gamma_2(t) = \gamma_1(h(t))$ $\qquad h : I_2 \to I_1$ $\qquad \gamma_2 : I_2 \to \mathbb{C}$ $\quad \gamma_1 : I_1 \to \mathbb{C}$

$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$ se γ_1 e γ_2 hanno lo stesso verso

$$= -\int_{\gamma_1} f(z)\,dz$$ se hanno verso opposto

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$$

$$\int_\gamma (\alpha f + \beta g)\,dz = \alpha\int_\gamma f\,dz + \beta\int_\gamma g\,dz$$

Stime del modulo $\left[f: \mathbb{R} \to \mathbb{R} \quad \left| \int_I f(x)\,dx \right| \le \int_I |f(x)|\,dx \right]$

$$\left| \int_\gamma f(z)\,dz \right| \underset{?}{\le} \cancel{\int_\gamma |f(z)|\,dz}$$

$\in \mathbb{C}$

$$\tilde{f}(x,y) = f(x+iy)$$

$$\le \int_\gamma |f(z)|\,|dz| = \int_\gamma |\tilde{f}(z)|\,ds = \int_I |f(\gamma(t))| \cdot |\gamma'(t)|\,dt$$

Dim

$f \rightsquigarrow \tilde{f} = (u,v) \qquad f(x+iy) = u(x,y) + i\,v(x,y)$

$\gamma(t) = x(t) + i\,y(t) \simeq (x(t), y(t))$

Lemma sulle norme dell'integrale

Sia $h = (h_1, h_2) : [a,b] \to \mathbb{R}^2 \implies \int_a^b h(t)\,dt = \left(\int_a^b h_1\,dt, \int_a^b h_2\,dt \right)$

$$\left\| \int_a^b h\,dt \right\| \le \int_a^b \|h(t)\|\,dt.$$

Dim

$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b (ux' - vy')\,dt + i \int_a^b (uy' + vx')\,dt \right| =$$

$$= \left\| \left(\int_a^b (ux' - vy')\,dt, \int_a^b (uy' + vx')\,dt \right) \right\|_{\mathbb{R}^2} \le \int_a^b \| (\underbrace{ux' - vy', \quad uy' + vx'}) \|_{\mathbb{R}^2}\,dt$$

$$= \int_a^b \sqrt{(ux' - vy')^2 + (uy' + vx')^2}\,dt = \int_a^b \sqrt{u^2 x'^2 - \cancel{2ux'vy'} + v^2 y'^2 + u^2 y'^2 + \cancel{2uy'vx'} + v^2 x'^2}\,dt$$

$$\underbrace{(u^2 + v^2) x'^2} + \underbrace{(u^2 + v^2) y'^2} = (u^2 + v^2) \cdot (x'^2 + y'^2)$$

$$= \int_a^b \|(u,v)\| \cdot \|\gamma'\|\,dt = \int_\gamma |f(z)|\,ds$$

Passaggio del limite sotto il segno integrale $\quad \gamma : [a,b] \to \mathbb{C}$

(*) $\lim_{n \to +\infty} f_n = f \quad \overset{?}{\Longrightarrow} \quad \lim_{n \to +\infty} \int_\gamma f_n(z)\,dz \overset{?}{=} \int_\gamma f(z)\,dz$

Supponiamo che esista $g : A \subseteq \mathbb{C} \to \mathbb{R}$ con $\int_\gamma g(z)\,dz$

cioè la funzione $g(\gamma(t)) \cdot \gamma'(t)$ è integrabile su $[a,b]$

e $|f_n(z)| \leq g(z) \quad \forall z \in A$. Allora vale (*)

per il teorema di convergenza dominata di Lebesgue

$$|f_n(\gamma(t)) \cdot \gamma'(t)| \leq \underbrace{g(\gamma(t)) \cdot |\gamma'(t)|}_{\text{è integrabile}} \quad //$$

Funzione primitivabile: $f: A \subseteq \mathbb{C} \to \mathbb{C}$ si dice primitivabile se esiste una primitiva $F: A \to \mathbb{C}$ tale che $F'(z) = f(z)$ $\forall z \in A$.

OSS: se A è connesso, F una primitiva di f. Allora G è primitiva di f se e solo se $F - G =$ costante.

Formula di Torricelli-Barrow: sia f primitivabile e sia F una primitiva di f, sia f continua; allora

$$\int_\gamma f(z)\,dz = \int_a^b \underbrace{F'(\gamma(t)) \cdot \gamma'(t)}_{= \frac{d}{dt}(F \circ \gamma)(t)}\,dt = F(\gamma(b)) - F(\gamma(a))$$

$\gamma: [a,b] \to \mathbb{C}$

In particolare se γ è un circuito $\gamma(a) = \gamma(b)$, si ha

$$\int_\gamma f(z)\,dz = 0 \quad //$$

$f(z) = \frac{1}{z}$ è primitivabile su $\mathbb{C} \setminus \{0\}$?

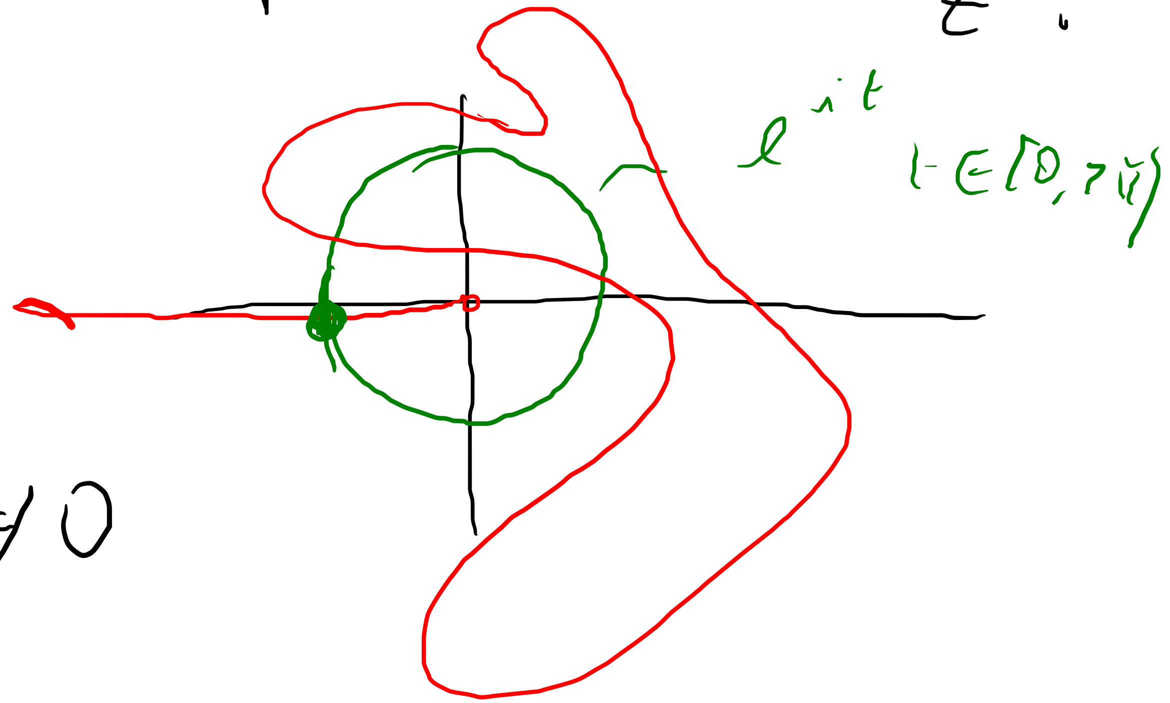
NO!

Ricordo che $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$

$\operatorname{Log} z$ è una primitiva di $\frac{1}{z}$!

$\log z : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

<u>ma</u> non è derivabile su $]-\infty, 0[$!!



$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot i \cancel{e^{it}}\, dt = \;\; = 2\pi i \neq 0$$

$\gamma(t) = e^{it}$

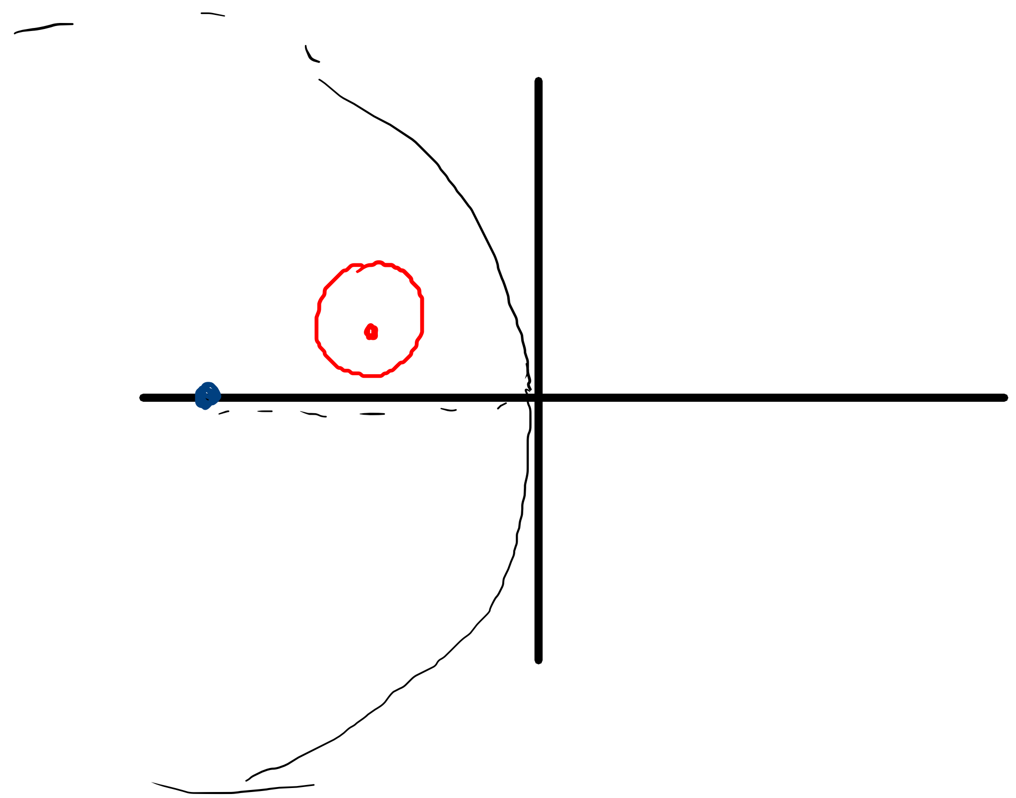
$$\int_\gamma \frac{1}{z}\, dz$$

f è localmente primitivabile, cioè per ogni $z_0 \in A$ esiste un intorno U di z_0 tale che $f|_U$ è primitivabile.

Es $f(z) = \frac{1}{z}$ se $z_0 \notin]-\infty, 0]$

prendo $B(z_0, \rho) \subset \mathbb{C} \setminus]-\infty, 0]$

Allora $\operatorname{Log} z : B(z_0, \rho) \to \mathbb{C}$ è derivabile e si ha $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$



Se $z_0 \in]-\infty, 0[$, considero $B(z_0, |z_0|)$

definisco $F(z) = \log|z| + i\vartheta$ con $0 < \vartheta < 2\pi$

(è una determinazione non principale del logaritmo)

$F(z)$ è derivabile su $\mathbb{C} \setminus [0, +\infty[$ e $F'(z) = \frac{1}{z}$

Il Teorema di Cauchy

$A \subseteq \mathbb{C}$ aperto connesso, $f : A \to \mathbb{C}$ olomorfa ; $\gamma : [a,b] \to \mathbb{C}$ circuito

con $\Gamma \cup \Gamma_{int} \subset A$

Γ sostegno $\gamma([a,b])$ — Γ_{int} parte interna al circuito.

Allora si ha $\int_\gamma f(z)\, dz = 0$

⚠ $f(z) = \frac{1}{z}$ olomorfa $\int_\gamma \frac{1}{z}\, dz = 2\pi i \neq 0$!

$\gamma(t) = e^{it}$ $t \in [0, 2\pi]$

Qui $\Gamma \subset A = \mathbb{C} \setminus \{0\}$

però $\Gamma_{int} \not\subset A$!

Dim (caso facilitato in cui f è C^1)

$f(x+iy) = u(x,y) + i\,v(x,y)$ $\qquad \gamma(t) = x(t) + i\,y(t)$

Formula di Gauss Green $\quad g(x,y) = (X(x,y), Y(x,y)) : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \quad C^1$

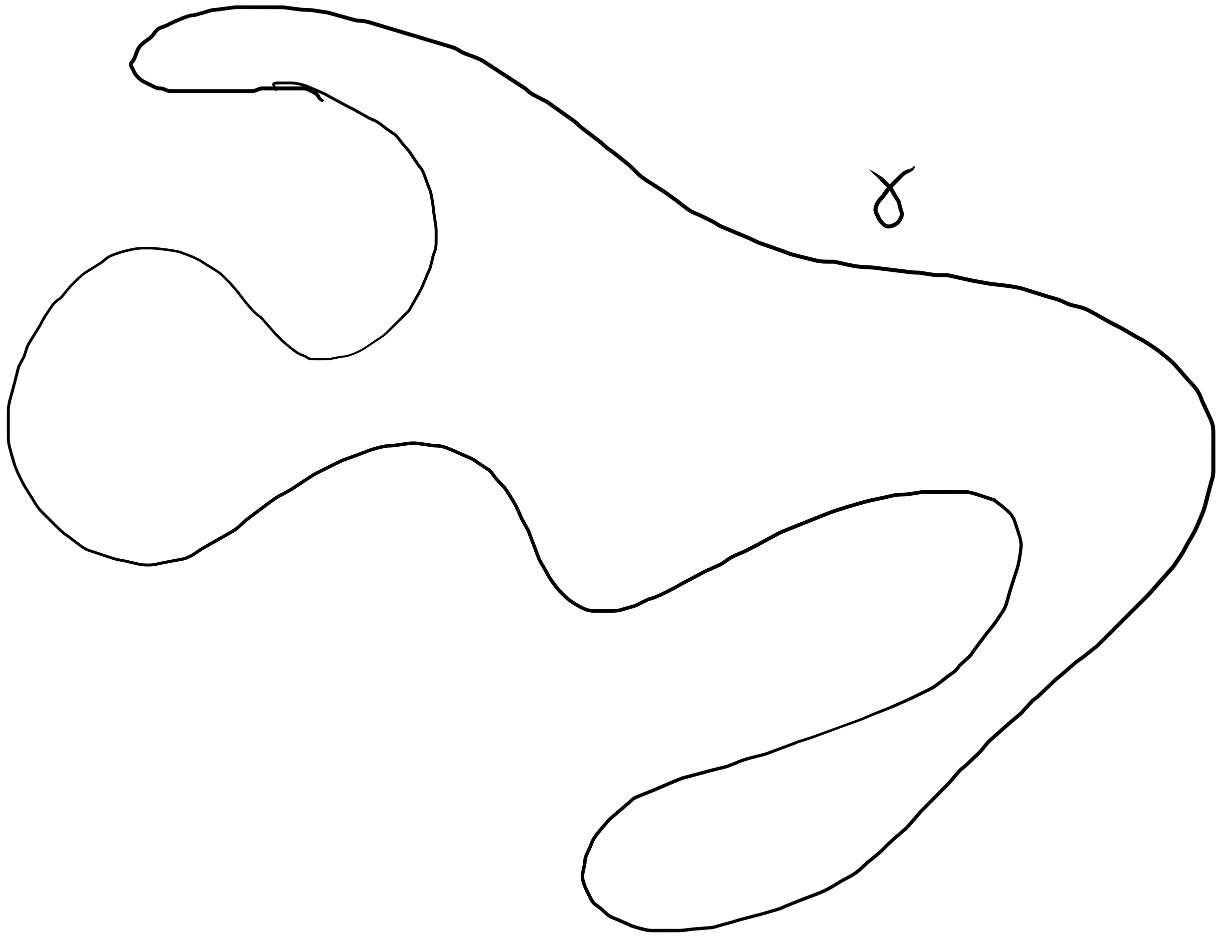
$$\iint_\Omega \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy = \int_{\partial\Omega^+} \langle g, \tau \rangle\, ds = \int_a^b \underbrace{\langle g(\gamma(t)), \gamma'(t) \rangle\, dt}_{X\,dx + Y\,dy}$$

$$\int_\gamma f\,dz = \int_a^b \underbrace{(u x' - v y')}_{\langle (u,-v), (x',y') \rangle} dt + i \int_a^b \underbrace{(u y' + v x')}_{\langle (v,u), (x',y') \rangle} dt = \int_{\gamma = \partial\Gamma_{int}^+} \langle (\underset{X}{u}, \underset{Y}{-v}), \tau \rangle\, ds + i \int_\gamma \langle (v,u), \tau \rangle\, ds$$

$$\underset{G.G.}{=} \iint_{\Gamma_{int}} \underbrace{\left(-\frac{\partial(v)}{\partial x} - \frac{\partial u}{\partial y}\right)}_{=0} dx\,dy + i \iint_{\Gamma_{int}} \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0} dx\,dy = 0$$

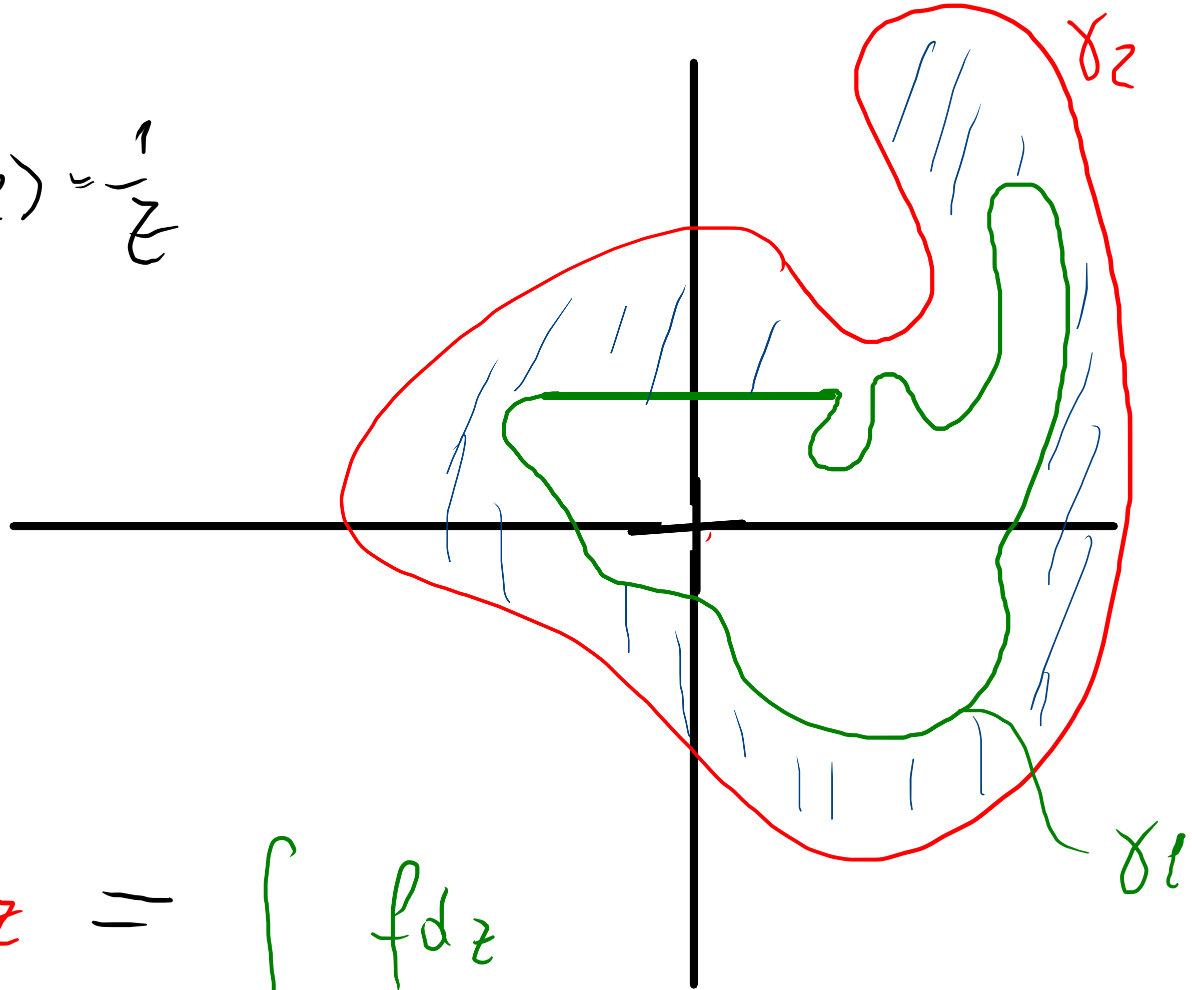
$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ per le condizioni di Cauchy-Riemann

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$



$$\int_\gamma f\, dz = 0$$

$$f(z) = \frac{1}{z}$$



$$\int_{\gamma_2} f\, dz = \int_{\gamma_1} f\, dz$$

Il Teorema dei due circuiti

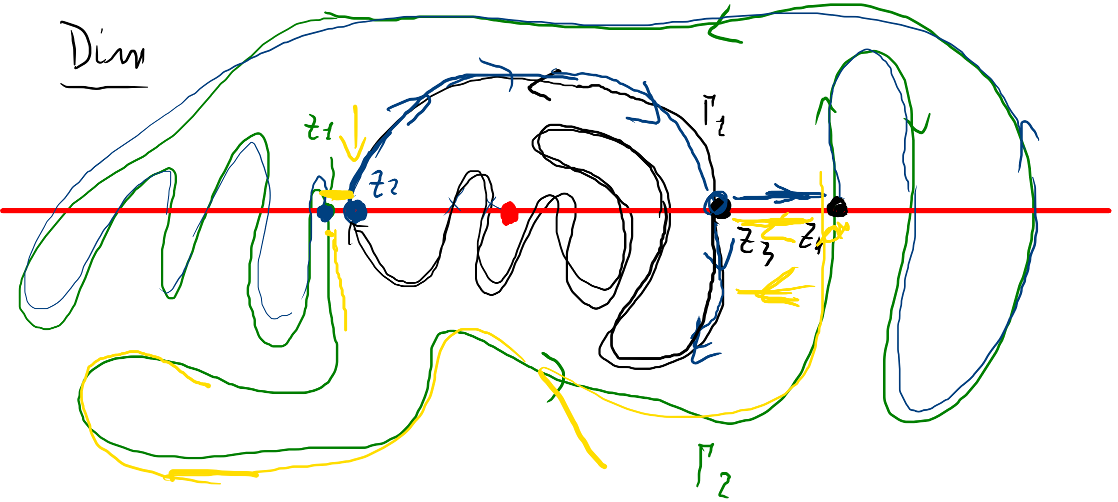
$A \subseteq \mathbb{C}$ aperto connesso; $f: A \to \mathbb{C}$ olomorfa

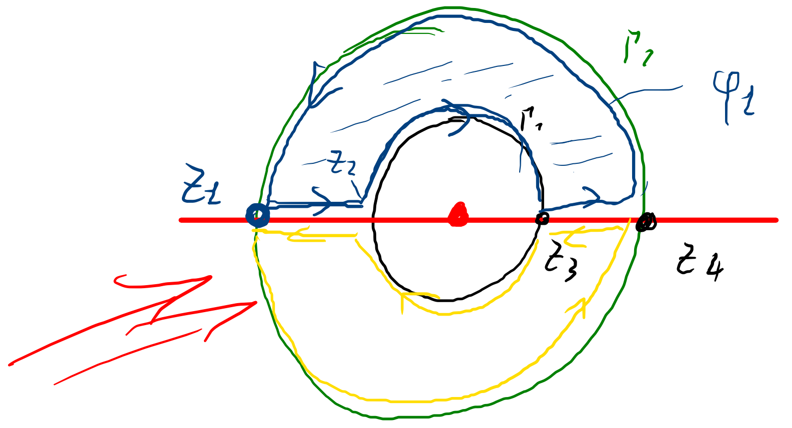
Siano γ_1, γ_2 circuiti $\Gamma_1 \subset \Gamma_{2\,int}$

e $\Gamma_{2\,int} \setminus \Gamma_{1\,int} \subset A$.

Allora $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$

Dim





Definiamo $\varphi_1 \quad -\gamma_1 \,(\text{da } z_2 \text{ a } z_3) + [z_3, z_4] + \gamma_2 \,(\text{da } z_4 \text{ a } z_1) + [z_1, z_2]$

$\varphi_2 \quad \gamma_2 \,(\text{da } z_1 \text{ a } z_4) - [z_3, z_4] - \gamma_1 \,(\text{da } z_3 \text{ a } z_2) - [z_1, z_2]$

Si noti che $\Phi_1 \cup \Phi_{1int} \subset A \quad \Phi_2 \cup \Phi_{2int} \subset A$; quindi per il Teorema di Cauchy si ha $\int_{\varphi_1} f(z)\,dz + \int_{\varphi_2} f(z)\,dz = 0$

per l'additività dell'integrale

$$\left[-\int_{\gamma_1(\text{da } z_2 \text{ a } z_3)} f\,dz + \cancel{\int_{z_3}^{z_4} f\,dz} + \int_{\gamma_2(\text{da } z_4 \text{ a } z_1)} f\,dz + \cancel{\int_{z_1}^{z_2} f\,dz}\right] + \left[\int_{\gamma_2(\text{da } z_1 \text{ a } z_4)} f\,dz + \right.$$

$$\left. - \cancel{\int_{z_3}^{z_4} f\,dz} - \int_{\gamma_1(\text{da } z_3 \text{ a } z_2)} f\,dz - \cancel{\int_{z_1}^{z_2} f\,dz}\right] = 0$$

$$\Rightarrow -\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz = 0 \quad \Rightarrow \boxed{\int_{\gamma_2} f\,dz = \int_{\gamma_1} f\,dz}$$

Formule integrali di Cauchy

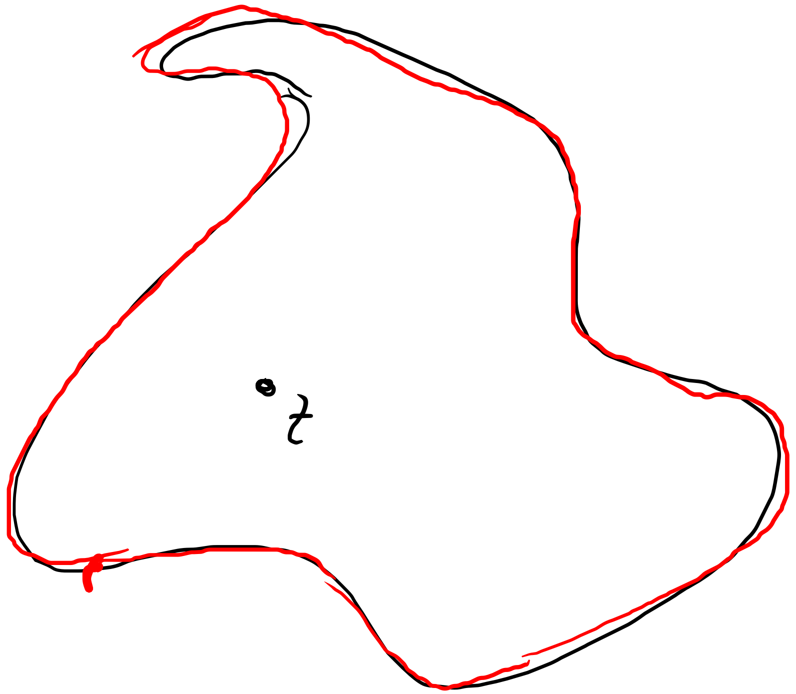
$A \subseteq \mathbb{C}$ aperto connesso $\quad f: A \to \mathbb{C}$ olomorfa

γ circuito con $\Gamma \cup \Gamma_{int} \subset A$

Allora $\forall z \in \Gamma_{int}$ si ha

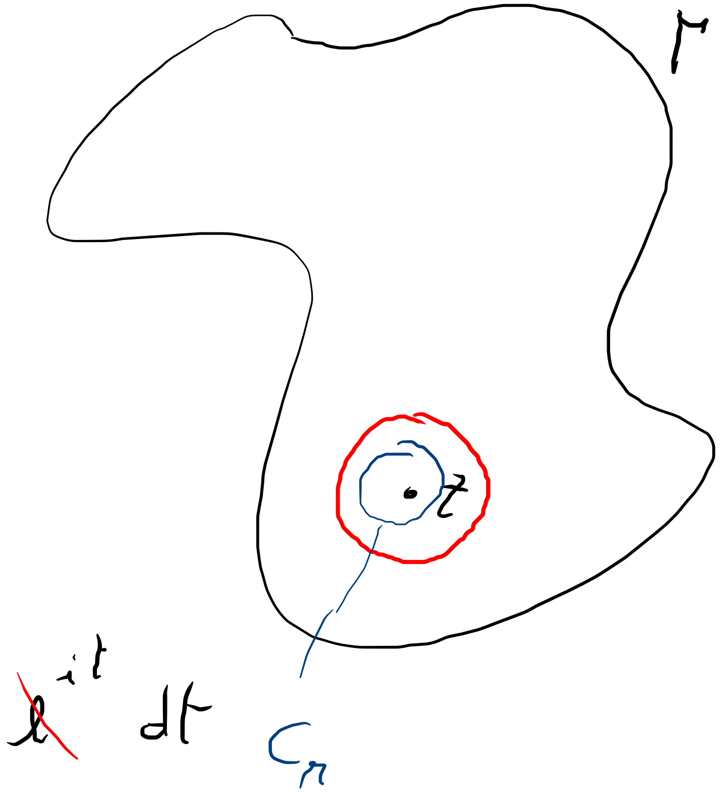
$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z}\, d\xi}$$

$\xi \in \Gamma$



$f(z)$

Dim Sia $z \in \Gamma_{int}$



Sia $B(z, r_0) \subset \Gamma_{int}$

Poniamo $C_r(t) = z + r e^{it}$

$r \in]0, r_0]$ $\quad$ $t \in [0, 2\pi]$

$$\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \underset{\substack{\uparrow \\ \text{T. dei} \\ \text{2 circuiti}}}{=} \int_{C_r} \frac{f(\xi)}{\xi - z}\, d\xi = \int_0^{2\pi} \frac{f(z + r e^{it})}{\cancel{z + r e^{it} - z}}\, \cancel{r}\, i\, \cancel{e^{it}}\, dt$$

è olomorfa tra Γ e C_r

$$= i \int_0^{2\pi} f(z + r e^{it})\, dt \quad \forall r \in]0, r_0]$$

$$\lim_{r \to 0} \longrightarrow i \int_0^{2\pi} f(z)\, dt$$

$$h(t, r) = f(z + r e^{it}) \qquad h : [0, 2\pi] \times [0, r_0] \to \mathbb{C}$$

è continua

$\Rightarrow$ è continua la funzione $\boxed{H(r) = \int_0^{2\pi} h(t, r)\, dt}$

$$\lim_{r \to 0} H(r) = H(0)$$

Si ha $\forall r \in]0, r_0]$

$$\int_{C_r} \frac{f(\xi)}{\xi - z}\, d\xi = \boxed{i \int_0^{2\pi} f(z + r e^{it})\, dt} = 2\pi i\, f(z)$$

$$\parallel$$

$$\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

↑ non dipende da r!

Teorema della media

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + r e^{it})\, dt}$$

"media"

$$\Rightarrow \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \lim_{r \to 0} i \int_0^{2\pi} f(z + r e^{it})\, dt = i\, f(z) \cdot 2\pi$$

$$\Rightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = f(z) \quad //$$